# Infinite energy solutions of the surface quasi-geostrophic equation.

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**Abstract**

We study the formation of singularities of a 1-D non linear and non local equation. We show that this equation provides solutions of the surface quasi-geostrophic equation with infinite energy. The existence of self-similar solutions and the blow-up for classical solutions are shown.

## 1 Introduction.

In this paper we study the existence of particular solutions of the surface quasi-geostrophic equation (SQG), i.e.

$$\begin{aligned} \partial_t\theta + u\cdot\nabla\theta &= 0, \qquad (1.1)\\ \theta(x,0) &= \theta^0(x), \end{aligned}$$

where

$$\begin{aligned} \theta &: \mathbb{R}^2\times\mathbb{R}^+\to\mathbb{R} \\ \theta &= \Lambda\Psi \quad \text{and} \qquad (1.2)\\ u &= \nabla^{\perp}\Psi = (-\partial_{x_2}\Psi, \partial_{x_1}\Psi) \qquad (1.3) \end{aligned}$$

with $\Lambda = (-\Delta)^{1/2}$.

Specifically, we shall analyze the case in which the stream function, $\Psi$, is given by the expression

$$\Psi(x_1,x_2,t) = -x_2 Hf(x_1,t), \qquad (1.4)$$

where is the Hilbert transform, i.e.

$$Hf(x) = \frac{1}{\pi}P.V.\int\frac{f(y)}{x-y}dy.$$

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The SQG system (1.1) is a model of geophysical origin which was proposed by P. Constantin, A. Majda and E. Tabak [7] as a model of the 3D-Euler equation. Numerical experiments, carried out by those authors, showed evidence of fast growth of the gradient of the active scalar when the geometry of the level sets contain a hyperbolic saddle. Later, further numerical studies were performed in [20] and [8] suggesting a double exponential growth in time. An analytical study in [11] showed that a *simple hyperbolic saddle breakdown* can not occur in finite time. In fact, the angle of the saddle is bounded below by a double exponential in time and a quadruple exponential upper bound was obtained for the growth of the derivatives of the active scalar. Subsequently, this bound was improved, for a formation of a *semi-uniform sharp front* in [12], by a double exponential. In [13], under certain assumptions on the local geometry of the level sets, the same bound is obtained. Recently, there has been different approaches to understand the growth of the derivatives: in [4] it is shown an a priori estimate from below for the Sobolev norms, a study of the spectrum of the linearized SQG is performed in [16] and the existence of the unstable eigenvalues is proven, and in [18] they prove that the 0 solution is strongly unstable in $H^{11}$.

With the choice (1.4) of the stream function (see below in section 2), the solutions of (1.1) can be written as

$$\theta(x_1, x_2, t) = x_2 f_{x_1}(x_1, t), \tag{1.5}$$

where $f_x$ satisfies the following one dimensional equation

$$\begin{aligned} \partial_t f_x + Hf f_{xx} &= Hf_x f_x, \\ f_x(x,0) &= f_x^0(x). \end{aligned} \tag{1.6}$$

In this way, for an odd initial data, the geometry of the level set of the active scalar contain a hyperbolic saddle in a neighborhood of zero. Nevertheless, the angle of opening of the saddle is not observed to go to zero in time. Similar stagnation-point solutions were considered for 2D Navier-Stokes equation in [5].

We will consider that the unknown quantity is the function $f_x$ and we will define $f$ by the expression

$$f(x) = \int_{-\infty}^{x} f_x(y)dy.$$

Then, if we take $f_x$ with zero mean and with a suitable decay at infinity, we have

$$Hf(x) \equiv H\left(\int_{-\infty}^{x} f_x(y)dy\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \log(|x-y|) f_x(y)dy.$$

At this point it is important to stress that equation (1.6) is mean preserving. In order to verify this property it is enough to recall the orthogonality character of the Hilbert transform.

A more general version of (1.6) was proposed, in a different context, by H. Okamoto, T. Sakajo and M. Wunsch in [23]

$$\partial_t f_x + aHf f_{xx} = Hf_x f_x, \tag{1.7}$$

where $a$ is real parameter. It was motivated by the work of P. Constantin, P. Lax and A. Majda ([6]) and the work of De Gregorio ([14] and [15]) where the equation is presented as a 1D-model of the 3D vorticity Euler equation.

Indeed, we can write the 3D Euler equation as

$$\partial_t w + (u \cdot \nabla) w = D w, \tag{1.8}$$

where

$$u(w) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times w(y) dy$$

and

$$D(w) = \frac{1}{2}(\nabla u + \nabla u^\top).$$

Thus, $D$ is a singular integral operator and it is easy to check that equation (1.7) and (1.8) are of the same order. The natural question, behind equation (1.7), is if a transport term (preserving the structure of the Euler equation) can cancel the singularities of the model ($a = 0$) in [6]. See [14], [15] and [22] for a discussion on the role played by the convection term.

In [23] the authors show local existence of classical solutions for (1.7) with $f_x^0 \in H^1(\mathbb{T})$ and they present a blow up criteria: *The local solution of (1.6) can be extend to time $T$ if*

$$\int_0^T ||Hf_x||_{L^\infty} dt < \infty.$$

In addition they carry out a numerical analysis (see also [21]) and conjecture that solutions may blow-up for $-1 \leq a < 1$ and global existence otherwise.

The case $a = -1$ has been proposed as a 1-D model of the SQG equation (see [9]) and as a 1-D model of the vortex sheet problem (see [1] and [19]). For this case, local existence is proven in [19], singularities in finite time are shown for even, compact support and positive classical solutions in [9] and for a more general positive initial data in [10]. Exact self-similar solutions were constructed in [2].

The main results of this paper are organized as follows: In section 2 we will show that the solutions of equation (1.6) provides solutions of the SQG equation. In section 3 we will construct self-similar solutions for equation (3.1) for any value of the parameter $a$. The existence of such solutions for the SQG equation has been studied by D. Chae in [3]. He showed that self-similar solutions do not exist with the form

$$\begin{aligned} u(x,t) &= \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}} U\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right), \\ \theta(x,t) &= \frac{1}{(T-t)^{\beta}} \Phi\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right), \\ \alpha,\, \beta \in \mathbb{R}, & \qquad \alpha \neq -1, \end{aligned}$$

if the profile $\Phi$ is in the class $L^{p_1} \cap L^{p_2}$, with $0 < p_1 < p_2 \leq \infty$, and if the profile $U \in C([0,T); C^1(\mathbb{R}^2; \mathbb{R}^2))$ generates a $C^1$ diffeomorphism from $\mathbb{R}^2$ onto itself. Nevertheless, this theorem can not be applied to solutions with the form (1.4). Finally, in section 4, we will prove blow up for classical solutions of (3.1) with $a < 0$.

## 2 SQG solutions with infinite energy.

In order to obtain the evolution equation for the function, $f(x,t)$, we will use the following representation of the operator $\Lambda$

$$\Lambda\Psi(x_1,x_2)=\frac{1}{2\pi}\,P.V.\int_{\mathbb{R}^2}\frac{\Psi(x_1,x_2)-\Psi(y_1,y_2)}{|x-y|^3}dy. \tag{2.1}$$

Then, introducing (1.4) in (2.1) we have

$$\begin{aligned}\Lambda\Psi(x_1,x_2)&=-\frac{1}{2\pi}\,P.V.\int_{\mathbb{R}^2}\frac{x_2Hf(x_1)-y_2Hf(y_1)}{|x-y|^3}dy\\
&=-\frac{1}{2\pi}P.V.\int_{\mathbb{R}^2}\frac{x_2\left(Hf(x_1)-Hf(y_1)\right)+\eta Hf(y_1)}{((x_1-y_1)^2+\eta^2)^{\frac{3}{2}}}dy_1d\eta\\
&=-\frac{1}{2\pi}\,x_2\,P.V.\int_{\mathbb{R}}Hf(x_1)-Hf(y_1)\left(P.V.\int_{\mathbb{R}}\frac{1}{\left((x_1-y_1)^2+\eta^2\right)^{\frac{3}{2}}}d\eta\right)dy_1\\
&\quad-\frac{1}{2\pi}\,P.V.\int_{\mathbb{R}}Hf(y_1)\left(P.V.\int_{\mathbb{R}}\frac{\eta}{((x_1-y_1)^2+\eta^2)^{\frac{3}{2}}}d\eta\right)dy_1\\
&=-\frac{1}{\pi}\,x_2P.V.\int_{\mathbb{R}}\frac{Hf(x_1)-Hf(y_1)}{(x_1-y_1)^2}dy_1=-x_2\partial_{x_1}H(Hf(x_1))=x_2\partial_{x_1}f(x_1).\end{aligned}$$

Therefore, from (1.2) follows

$$\theta(x_1,x_2,t)=\Lambda\Psi(x_1,x_2,t)=x_2\partial_{x_1}f(x_1,t). \tag{2.2}$$

Combining (2.2) with (1.3) and with the equation (1.1) yields

$$x_2\left(\partial_t\left(\partial_{x_1}f\right)(x_1,t)+Hf(x_1,t)\partial^2_{x_1}f(x_1,t)-\partial_{x_1}Hf(x_1,t)\partial_{x_1}f(x_1,t)\right)=0.$$

Thus, the solutions of (1.6) provides solutions of the equation (1.1) with infinite energy.

## 3 Self-Similar solutions for any $a$.

The aim of this section is to prove the existence of self-similar solutions of the equation (1.7) but since the lack of regularity of this type of solutions we will work with the equation for $f$, instead of $f_x$, which is given by

$$\begin{aligned}\partial_tf+aHff_x&=(1+a)\int_{-\infty}^{x}Hf_x(y)f_x(y)dy\\
f(x,0)&=f^0(x).\end{aligned} \tag{3.1}$$

The theorem we will prove is the following:

**Theorem 3.1** *Let*

$$G(x) = \sqrt{(1-x^2)_+},$$

*where and $f_+$ is the positive part of the function $f$. Then, the function*

$$f(x,t) = -\frac{1}{t^{(1+a)}} G(t^a x)$$

*is a self-similar solution of the equation (3.1).*

The proof of this theorem is based in the next lemma:

**Lemma 3.2** *The Hilbert transform of the function*

$$G(x) = \sqrt{(1-x^2)_+}$$

*is given by the expression*

$$HG(x) = \begin{cases} x - \sqrt{x^2-1} & \text{if } x > 1 \\ x & \text{if } |x| < 1 \\ x + \sqrt{x^2-1} & \text{if } x < -1 \end{cases}$$

**Remark 3.3** *A more general statement is obtained in [17]. Here we will give a simplifier proof for lemma 3.2.*

Proof. Consider the complex function

$$F(z) = \sqrt{1-z^2} + iz, \quad z = x + iy,$$

where the square root is defined by

$$\sqrt{z} \equiv |z|^{\frac{1}{2}} \exp^{\frac{i}{2}\arg(z)} \quad \text{with} \quad -\pi < arg(z) \leq \pi.$$

Then the following properties of $F$ can be checked:

1. $F(z)$ is an analytic function for $y > 0$.
2. $F(z)$ vanishes at infinity.
3. The restriction of $F(z)$ to the real axis is given by the expression

$$\lim_{y \to 0^+} F(z) = \begin{cases} i(x - \sqrt{x^2-1}) & \text{if } x > 1 \\ \sqrt{1-x^2} + ix & \text{if } |x| < 1 \\ i(x + \sqrt{x^2-1}) & \text{if } x < -1 \end{cases}$$

Then, since the restriction of $F(z)$ has to be of the form

$$\lim_{y \to 0^+} F(z) = f(x) + iHf(x),$$

the lemma 3.2 is proven.

By introducing the ansatz, $f(x,t) = \frac{1}{t^{(1+a)}}\Phi(t^a x)$, in the equation (3.1) we obtain

$$a\Phi'(\xi)\left(H\Phi + \xi\right) = (1+a)\int_{-\infty}^{\xi} \Phi'(y)\left(H\Phi'(\eta)+1\right)d\eta. \tag{3.2}$$

Using lemma 3.2 we have that the function, $\Phi(\xi) = -G(\xi)$, satisfies equation (3.2) for any $a$.

An important consequence of this self-similar solutions is the following corollary:

**Corollary 3.4** *The function $f(x,t) = \frac{1}{(1-t)^{1+a}}G\left((1-t)^a x\right)$ is a solution of equation (3.1) with the following behavior at time $t=1$: i) For $a > -1$ there is blow-up. This means, $f(0,t)$ goes to infinity in fininte time. ii) For $a = -1$ the solution collapse in a point.*

In order to prove this corollary it is enough to observe that the equation is time translations invariant and that changing the time direction is the same that changing the sign of the initial data.

# 4 Blow up for classical solutions with $a < 0$.

In this section we will present a proof of blow up of classical solutions for the equation (3.1) with $a < 0$. We will say that a solution $f(x,t)$ of equation (3.1) "blows up in finite time" if there exits $0 < T < \infty$ such that either $f$ is not in $C^\infty(\mathbb{R}\times[0,T])$ or $Hf_x(x,t)$ is unbounded on $\mathbb{R}\times[0,T]$.

**Theorem 4.1** *Let $f_x^0 \in C_c^\infty(\mathbb{R})$ an odd function such that $Hf_x^0(0) > 0$. Then the solution of (3.1) with $a < 0$ blows up in finite time.*

Proof. We will proceed by a contradiction argument. Let us suppose that there exist a solution of (3.1), $f_x \in C^1\left([0,T], C^\infty(\mathbb{R})\right)$ for all $T < \infty$ with $f_x^0$ as in the theorem. Then, $f_x$ satisfies the following properties:

1. $f_x(\cdot,t)$ is odd.
2. $f_x(\cdot,t)$ is of compact support.

The first property is evident. In order to check the second property, we define the trajectories

$$\begin{aligned} \frac{d\,X(x,t)}{dt} &= aHf(X(x,t),t), \\ X(x,0) &= x. \end{aligned}$$

Then the function $f_x(X(x,t),t)$ satisfies the equation

$$\begin{aligned} \frac{d\,f_x(X(x,t),t)}{dt} &= Hf_x(X(x,t),t)f_x(X(x,t),t), \\ f_x(X(x,0),0) &= f_x^0(x) \end{aligned}$$

and therefore

$$f_x(X(x,t),t) = \exp\left(\int_0^t Hf_x(X(x,\tau),\tau)d\tau\right) f_x^0(x).$$

Taking the Hilbert transform on (3.1) yields

$$\partial_t Hf_x(x,t) + aH(Hff_{xx})(x,t) = \frac{1}{2}\left((Hf_x(x,t)^2 - f_x(x,t)^2\right).$$

By evaluating this equation in $x = 0$ we obtain

$$\frac{d\Lambda f(0,t)}{dt} = \frac{1}{2}(\Lambda f(0,t))^2 - aH(Hff_{xx})(0,t). \tag{4.1}$$

Thus, if we prove that $H(Hff_{xx})(0,t)$ is bigger than 0, we obtain a contradiction since $a < 0$. Therefore, in order to prove theorem (4.1) we just have to show the following lemma:

**Lemma 4.2** *Let $f \in C_c^\infty(\mathbb{R})$ an even function. Then*

$$H(Hff_{xx})(0) \geq 0.$$

Proof. We will use the Fourier transform:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \quad \text{and} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk.$$

Then, we can write

$$H(Hff_{xx})(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H\widehat{(Hff_{xx})}(k)dk$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} -i\text{sign}\,(k)\int_{-\infty}^{\infty}\widehat{Hf}(k-\eta)\widehat{f_{xx}}(\eta)d\eta dk$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\text{sign}\,(k)|\eta|}{k-\eta}\widehat{\Lambda f}(k-\eta)\widehat{\Lambda f}(\eta)d\eta dk$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\text{sign}\,(\xi+\eta)|\eta|}{\xi}\widehat{\Lambda f}(\xi)\widehat{\Lambda f}(\eta)d\eta d\xi.$$

Since $\Lambda f$ is a real even function, we have that $\widehat{\Lambda f}$ is also a real even function. Therefore we can write the previous expression in the following way

$$H(Hff_{xx})(0) = \frac{2}{(2\pi)^2}\int_0^{\infty}\int_0^{\infty}(1+\text{sign}\,(\xi-\eta))\frac{\eta}{\xi}\widehat{\Lambda f}(\eta)\widehat{\Lambda f}(\xi)d\eta d\xi$$

$$= \frac{4}{(2\pi)^2}\int_0^{\infty}\int_0^{\xi}\frac{\eta}{\xi}\widehat{\Lambda f}(\eta)\widehat{\Lambda f}(\xi)d\eta d\xi = \frac{4}{(2\pi)^2}\int_0^{\infty}\int_0^1 \alpha\xi\widehat{\Lambda f}(\alpha\xi)\xi\widehat{\Lambda f}(\xi)d\alpha\frac{d\xi}{\xi}$$

Defining the function, $g(x) = x\widehat{\Lambda f}(x)$, and the dilatation $g_\alpha(x) = g(\alpha x)$ we have that

$$H(Hff_{xx})(0) = \frac{4}{(2\pi)^2}\int_0^1\left(\int_0^{\infty} g(\xi)g_\alpha(\xi)\frac{d\xi}{\xi}\right)d\alpha.$$

Now we recall the definition of the Mellin transform:

**Definition 4.3** *Let $g$ be a real function such that the integral*

$$\int_0^\infty |g(x)|\frac{dx}{x} < \infty.$$

*Then, we define the Mellin transform, $Mg(\lambda)$, of $g(x)$ by the expression*

$$Mg(\lambda) = \int_0^\infty x^{i\lambda} g(x)\frac{dx}{x}.$$

This operator has the following properties:

1. The Mellin transform of a dilatation is given by

$$Mg_\alpha(\lambda) = \alpha^{-i\lambda} Mg(\lambda).$$

2. The Parseval identity

$$\int_0^\infty f(x)g(x)\frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^\infty Mf(\lambda)\overline{Mg}(\lambda)d\lambda.$$

Therefore,

$$H(Hff_{xx})(0) = \frac{4}{(2\pi)^3}\int_0^1\int_{-\infty}^\infty \alpha^{-i\lambda}|Mg|^2(\lambda)d\lambda d\alpha.$$

Since $|Mg|(\cdot)$ is an even function we can conclude that

$$\begin{aligned} H(Hff_{xx})(0) &= \frac{8}{(2\pi)^3}\int_{-\infty}^\infty |Mg|^2(\lambda)\left(\Re\int_0^1 \alpha^{-i\lambda}d\alpha\right)d\lambda \\ &= \frac{8}{(2\pi)^3}\int_{-\infty}^\infty \frac{|Mg|^2(\lambda)}{1+\lambda^2}d\lambda \geq 0. \end{aligned}$$

The lemma (4.2) is proven.

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