KU LEUVEN

Rigidity theory for von Neumann algebras

Joint Mathematics Colloquium (ICMAT-UAM-UC3M-UCM)

22 September 2022

Stefaan Vaes – KU Leuven

Operator algebras

We consider $*$-subalgebras $M \subset B(H)$, where the $*$-operation is the Hermitian adjoint.

- **Operator norm:** for $T \in B(H)$, we put $\|T\| = \sup\{\|T\xi\| \mid \xi \in H, \|\xi\| \leq 1\}$.

  C*-algebras: norm closed $*$-subalgebras of $B(H)$.

- **Weak topology:** $T_i \to T$ if and only if $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$.

  Von Neumann algebras: weakly closed $*$-subalgebras of $B(H)$.

$\rightsquigarrow$ Intimate connections to group theory, dynamical systems, quantum information theory, representation theory, ...

Commutative operator algebras

- Unital commutative C*-algebras are of the form $C(X)$ where X is compact Hausdorff.

  $\rightsquigarrow$ algebraic topology, K-theory, continuous dynamics, geometric group theory.

- Commutative von Neumann algebras are of the form $L^\infty(X,\mu)$ where (X,μ) is a standard probability space.

  $\rightsquigarrow$ ergodic theory, measurable dynamics, measurable group theory.

Discrete groups and operator algebras

Let G be a countable (discrete) group.

- The **group algebra** $\mathbb{C}[G]$ is the vector space (over $\mathbb{C}$) with basis G and the natural product.
- Left regular unitary representation $\lambda : G \to \mathcal{U}(\ell^2(G)) : (\lambda_g \xi)(h) = \xi(g^{-1}h)$.
- Take the norm closure: (reduced) **group C*-algebra** $C_r^*(G)$.
- Take the weak closure: **group von Neumann algebra** $L(G)$.

We have $G \subset \mathbb{C}[G] \subset C_r^*(G) \subset L(G)$.

At each inclusion, information gets lost $\rightsquigarrow$ natural rigidity questions.

KU LEUVEN

Open problems

- Kaplansky's conjectures for torsion-free groups G.
  - Unit conjecture: the only invertibles in $\mathbb{C}[G]$ are multiples of group elements λ_g.
  - Idempotent conjecture: 0 and 1 are the only idempotents in $\mathbb{C}[G]$.
  - Kadison-Kaplansky: 0 and 1 are the only idempotents in $C_r^*(G)$.
- Free group factor problem: is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ if $n \neq m$?
- Connes' rigidity conjecture: $L(\mathrm{PSL}(n,\mathbb{Z})) \not\cong L(\mathrm{PSL}(m,\mathbb{Z}))$ if $3 \leq n < m$.
- Stronger form: if G has property (T) and $\pi : L(G) \to L(\Gamma)$ is a $*$-isomorphism, then $G \cong \Gamma$ and π is essentially given by such an isomorphism.

$\rightsquigarrow$ Structure and classification of operator algebras is highly nontrivial.

II_1 factors

Fundamental work of Murray – von Neumann, 1943

- Weakly closed $*$-subalgebras $M \subset B(H)$. Nowadays called **von Neumann algebras**.
- **Bicommutant theorem:** $M' = \{T \in B(H) \mid ST = TS$ for all $S \in M\}$ and $M = M''$.
- **Factor:** $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.
- Factors come in **different types:** I, II_1, II_∞ and III, with $B(H)$ being of type I.
- **II_1 factors:** factors with a tracial state $\tau : M \to \mathbb{C}$.

$\rightsquigarrow$ $L(G)$ is a II_1 factor for every icc group G with trace $\tau(T) = \langle T\delta_e, \delta_e \rangle$.

$\rightsquigarrow$ Continuous dimension $\tau(p) \in [0,1]$.

$\rightsquigarrow$ One of the most fascinating mathematical structures, sometimes extremely rich in symmetries, sometimes extremely rigid.

The hyperfinite II_1 factor: a first isomorphism theorem

Murray-von Neumann: a factor M is **approximately finite dimensional (AFD)** if there exists an increasing sequence $A_n \subset M$ of finite dimensional $*$-subalgebras with $\bigcup_n A_n$ weakly dense in M.

Canonical construction of a hyperfinite II_1 factor: $R = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$.

Theorem (Murray-von Neumann, 1943)

- All hyperfinite II_1 factors are isomorphic!
- Every II_1 factor M contains copies of $R \hookrightarrow M$.

"The possibility exists that any factor in the case II_1 is isomorphic to a sub-ring of any other such factor."

Digression: amenability for groups

Definition (von Neumann, 1929)

A countable group G is amenable if there exists a finitely additive probability measure m on the subsets of G such that $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in G$ and $\mathcal{U} \subset G$.

$\rightsquigarrow$ The m is never explicit (if G is infinite), but approximations of m are explicit.

- (Banach - Tarski, 1924) It is possible to partition the ball of radius one into finitely many subsets, move these subsets by rotations and translations,

  and obtain two balls of radius one !

- **Reason:** group of motions of $\mathbb{R}^3$ is not amenable (as a discrete group).
- (Tarski, 1929) There is no paradoxical decomposition of the unit disk.
- **Reason:** group of motions of $\mathbb{R}^2$ is amenable (as a discrete group).

Examples

The following groups are **amenable.**

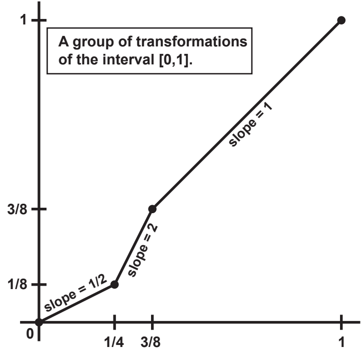
- Finite groups.
- Abelian groups.
- Stable under subgroups, direct limits and extensions.

The following groups are **non-amenable.**

- The free groups $\mathbb{F}_n$.
- Groups containing $\mathbb{F}_2$.
- Also other examples (von Neumann - Day problem).

Open problem :

Is the Thompson group amenable ?



Amenability for von Neumann algebras

Definition (von Neumann, 1929)

A countable group G is amenable if there exists a finitely additive probability measure m on the subsets of G such that $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in G$ and $\mathcal{U} \subset G$.

$\rightsquigarrow$ Equivalently: there exists a G-invariant state $\omega : \ell^\infty(G) \to \mathbb{C}$.

Hakeda-Tomiyama: a von Neumann algebra $M \subset B(H)$ is amenable if there exists a conditional expectation $P : B(H) \to M$.

$\rightsquigarrow$ $L(G)$ is amenable iff G is amenable.

Theorem (Connes, 1976)

Every amenable von Neumann algebra is AFD ! In particular, all amenable II_1 factors are isomorphic with R ! For every amenable icc group $L(G) \cong R$!

Classification of amenable factors

Type III factor $M \rightsquigarrow$ (Connes, Takesaki) associated ergodic flow $\mathbb{R} \curvearrowright (Z, \eta)$.

Definition (Connes)

A type III factor M is of

- type III_λ if the flow is periodic: $\mathbb{R} \curvearrowright \mathbb{R}/(\log \lambda)\mathbb{Z}$,
- type III_1 if the flow is trivial: $Z = \{\star\}$,
- type III_0 if the flow is properly ergodic.

Classification of amenable factors

- (Connes) For each of the following types, there is a unique amenable factor: type II_1, type II_∞, type III_λ with $0 < \lambda < 1$.
- (Connes, Krieger) Amenable factors of type III_0 are classified by the associated flow.
- (Haagerup) There is a unique amenable III_1 factor.

W*-superrigidity for group von Neumann algebras

Recall: for every amenable icc group G, we have $L(G) \cong R$.

Theorem (Ioana-Popa-V, 2010)

There are countable groups $\mathcal{G}$ such that $L(\mathcal{G})$ entirely remembers $\mathcal{G}$:

if Λ is an arbitrary countable group with $L(\mathcal{G}) \cong L(\Lambda)$, then $\mathcal{G} \cong \Lambda$.

$\rightsquigarrow$ These groups are of the form $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$:

Given an action $\Gamma \curvearrowright I$, consider the action of Γ by automorphisms of the direct sum $(\mathbb{Z}/2\mathbb{Z})^{(I)}$, and make a semidirect product.

Theorem (Berbec-V, 2012)

The same is true for $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$, where $\Gamma \times \Gamma$ acts on Γ by left and right multiplication, for many groups Γ, including free groups and free products $\Gamma = \Gamma_1 * \Gamma_2$.

Operator algebras and dynamical systems

Let G be a countable group.

Continuous dynamics and C*-algebras

An action $G \curvearrowright X$ of G by homeomorphisms of a compact Hausdorff space X gives rise to the C*-algebra $C(X \rtimes G)$.

Measurable dynamics and von Neumann algebras

An action $G \curvearrowright (X, \mu)$ of G by measure preserving transformations of (X, μ) gives rise to a von Neumann algebra $L(X \rtimes G)$.

- These operator algebras contain $C(X)$, resp. $L^\infty(X)$, as subalgebras.
- They contain G as unitary elements $(u_g)_{g \in G}$.
- They encode the group action: $u_g \, F \, u_g^* = \alpha_g(F)$ where $(\alpha_g(F))(x) = F(g^{-1} \cdot x)$.

KU LEUVEN

Superrigidity: Popa's deformation/rigidity theory

Consider the **Bernoulli action** $G \curvearrowright (X,\mu) = \prod_{g\in G}(X_0,\mu_0) : (g\cdot x)_h = x_{g^{-1}h}$.

- $M = L(X \rtimes G)$ is a II_1 factor.
- Whenever G is amenable, we have $M \cong R$.

W*-superrigidity theorem (Popa, Ioana, V, 2003-2010)

If G has property (T), e.g. $G = \mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$,

or if $G = G_1 \times G_2$ is a non-amenable direct product group, e.g. $G = \mathbb{F}_2 \times \mathbb{Z}$,

then $L(X \rtimes G)$ remembers the group G and its action $G \curvearrowright (X,\mu)$.

More precisely: if $L(X \rtimes G) \cong L(Y \rtimes \Gamma)$ for any other free ergodic measure preserving action $\Gamma \curvearrowright (Y,\eta)$, then $G \cong \Gamma$ and the actions are conjugate (isomorphic).

Free groups

Big open problem: is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$?

Theorem (Popa-V, 2011)

Whenever $n \neq m$, we have that $L(X \rtimes \mathbb{F}_n) \not\cong L(Y \rtimes \mathbb{F}_m)$,

for arbitrary free, ergodic, pmp actions of the free groups.

- Our main result is the **uniqueness of the Cartan subalgebra**:

  if $L(X \rtimes \mathbb{F}_n) \cong L(Y \rtimes \mathbb{F}_m)$, there exists an isomorphism π with $\pi(L^\infty(X)) = L^\infty(Y)$.
- Such a π induces an **orbit equivalence:** a measurable bijection $\Delta : X \to Y$ such that $\Delta(\mathbb{F}_n \cdot x) = \mathbb{F}_m \cdot \Delta(x)$ for a.e. $x \in X$.
- **(Gaboriau)** The L^2-Betti numbers of a group are invariant under orbit equivalence.

  We have $\beta_1^{(2)}(\mathbb{F}_n) = n - 1$.

L^2-Betti numbers of groups

- **Atiyah, Cheeger-Gromov, Lück:** $\beta_n^{(2)}(G) = \dim_{L(G)} H^n(G, \ell^2(G))$.

  $\rightsquigarrow$ $L(G)$-modules have a relative dimension, such that $\dim_{L(G)}(\ell^2(G)^{\oplus n}) = n$.

- **Gaboriau:** invariant under orbit equivalence.

Conjecture (Popa, Ioana, Peterson)

If $L(X \rtimes G) \cong L(Y \rtimes \Gamma)$ for some free, ergodic, pmp actions, then $\beta_n^{(2)}(G) = \beta_n^{(2)}(\Gamma)$.

Big dream (many authors)

Define some kind of L^2-Betti numbers for II_1 factors.

Prove that $\beta_1^{(2)}(L(\mathbb{F}_n)) = n - 1$.

KU LEUVEN

Embeddings of II_1 factors

Recall Murray & von Neumann: *"The possibility exists that any factor in the case II_1 is isomorphic to a sub-ring of any other such factor."*

Notation: $N \hookrightarrow M$ if there exists an embedding of N into M.

Recall: $R = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$ and $R \hookrightarrow M$ for every II_1 factor M.

Amenability is inherited by subalgebras

If $N \hookrightarrow R$, then N must be amenable. Thus, $L(\mathbb{F}_2) \not\hookrightarrow R$.

Haagerup property is inherited by subalgebras (Connes-Jones, 1983)

If Λ has Kazhdan's property (T) and if Γ has the Haagerup property, then $L(\Lambda) \not\hookrightarrow L(\Gamma)$.

For example, $L(\mathrm{SL}(3,\mathbb{Z})) \not\hookrightarrow L(\mathbb{F}_2)$.

Embeddings of II_1 factors

The Cowling-Haagerup increases under embeddings (Cowling-Haagerup, 1988)

If Γ_n is a lattice in $\mathrm{Sp}(n,1)$, then $L(\Gamma_n) \not\hookrightarrow L(\Gamma_m)$ for $n > m$.

$\rightsquigarrow$ These are qualitative results. More rigid objects do not embed in less rigid objects.

Open problems.

- **Conjecture.** If $n > m$, then $L(\mathrm{PSL}(n,\mathbb{Z})) \not\hookrightarrow L(\mathrm{PSL}(m,\mathbb{Z}))$.
- Does $L(\mathbb{F}_2) \hookrightarrow M$ for any nonamenable II_1 factor M ?
  (von Neumann – Day problem for II_1 factors)
- Which II_1 factors M embed into $L(\mathbb{F}_2)$?

Embeddings of Bernoulli crossed products

Theorem (Popa-V, 2021)

Let $\Gamma = \mathbb{F}_n$ be a free group and (A_0, τ) amenable (e.g. abelian).

We build: $M(A_0, \tau) = (A_0, \tau)^{\otimes \Gamma} \rtimes (\Gamma \times \Gamma)$.

Then, $M(B_0, \tau) \hookrightarrow M(A_0, \tau)$ if and only if the initial data embed: $(B_0, \tau) \hookrightarrow (A_0, \tau)$.

- With $A_0 = \mathbb{C}^2$ and $\tau(x, y) = ax + (1 - a)y$: mutually non embeddable $(M_a)_{a \in (0,1/2)}$.
- With $A_0 = L^\infty([0, a] \cup \{1\})$ and $\tau =$ Lebesgue on $[0, a]$ and atom $1 - a$ at 1, we get $M_a \hookrightarrow M_b$ iff $a \leq b$.

  $\rightsquigarrow$ A **chain of II_1 factors** $(M_a)_{a \in [0,1]}$.
- With $A_0 = R \oplus R$ and varying τ : all mutually embeddable, but not isomorphic.