JAE School of Mathematics 2021: Coarse geometry, Groups and Operator Algebras

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Exercises and problems*

Chapter 1: Metric spaces and quasi-isometries

1.1) Sketch a Cayley graph of the cyclic group C_4 and of the Free group $\mathbb{F}_2 = \langle a, b, a^{-1}, b^{-1}\rangle$; (the generators of $\mathbb{F}_2$ satisfy no relations except the obvious one $aa^{-1} = e$ etc.)

1.2) Let (X, d_X) and (Y, d_Y) be metric spaces. Show that if $f\colon X \to Y$ is a quasi-isometry (i.e., a (c, b)-quasi-isometric embedding with M-quasidense image), then there is an (a, a)-quasi-isometric embedding with a-quasi-dense image. (I.e., we can deal just with one constant).

1.3)
a) A metric space (F, d) is of finite diameter if $\sup\{d(x, x') \mid x, x' \in F\} < \infty$. Show that all non-empty finite diameter metric spaces are quasi-isometric.
b) Let (X, d), (F, d) be a metric spaces where F has finite diameter. Show that the embedding $X \to X \times F$ and the projection onto the first component $X \times F \to F$ are quasi-isometries.
c) Consider $\mathbb{R}$ and $\mathbb{R}^2$ with the Euclidean metric. Show that $f\colon \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, 0)$ is not a quasi-isometry.

1.4)* Proof that $f\colon X \to Y$ is a quasi-isometry iff f is a quasi-isometric embedding with quasi-dense image (from the lecture).

1.5) (**Scaling sets and boundaries with quasi-isometries**). Let (X, d_X), (Y, d_Y) be a metric spaces with bounded geometry and $f\colon X \to Y$ a (C, C)-quasi-isometry.

a) Let $A \subset X$ and $B \subset Y$. Show that $A \subset \left(f^{-1} \circ f\right)(A)$ and $\left(f \circ f^{-1}\right)(B) \subset B$. Give an example where the inclusions are proper.

b) Show that for $x_0 \in X$, $y_0 \in Y$ and $R > 0$

$$f\Big(B_R(x_0)\Big) \subset B_{CR+C}\Big(f(x_0)\Big) \quad \text{and} \quad f^{-1}\Big(B_R(y_0)\Big) \subset B_{2CR+C}(x_1)\ , \text{for some } x_1 \in f^{-1}\Big(B_R(y_0)\Big)\ .$$

Moreover, if $F \subset Y$ show also that there is an $M > 0$ with

$$|f^{-1}(F)| \le M|F|\ .$$

c) If $A \subset X$ show that if $x \in \partial_R(A)$ then $d(f(x), f(A)) \le CR + C$. Is it true that

$$f\left(\partial_R(A)\right) \subset \partial_{CR+C}(f(A))\ ?$$

d) Let $F \subset Y$ and put

$$\widehat{F} := f^{-1}(B_C(F))\ , \quad \text{where} \quad B_C(F) = \mathop{\cup}_{y\in F} B_C(x)$$

is the C-fatening of F in the target space Y. Show that there is an $M > 0$ with

$$|\widehat{F}| \ge \frac{1}{M} \cdot |F| \quad \text{and that} \quad f\left(\partial_R(\widehat{F})\right) \subset \partial_{CR+2C}\,(F)\ .$$

Why do we need the C-fatening of F?

Chapter 2: Amenability and property A

2.1) Show that the group G is amenable if and only if for every $g \in G$ there exists a sequence of non-empty finite Følner sets $\{F_n\}_{n\in\mathbb{N}} \subset G$ such that

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0 ,$$

where Δ denotes the symmetric difference of the corresponding sets.

2.2)* Use Følner sets to show that if $G = \langle S \rangle$, $|S| < \infty$, is an amenable group, then any subgroup $H \leq G$ is also amenable. (The proof using von Neumann's characterization in terms of invariant means is almost trivial).

2.3) Show that for groups metric space amenability (using outer boundaries of Følner sets) is equivalent to amenability of groups (using symmetric differences of Følner sets).

2.4)* Let (X, d) be a bounded geometry metric space and consider $Y \subset X$. Show that if X has property A, then Y has also property A. (Hint: Proposition 4.2.5 in Nowak-Yu's book).

Chapter 3: Uniform Roe algebras

3.1) Consider a simple graph $G = (V, E)$ and denote the degree of each vertex v by $\deg(v)$. Define the (combinatorial) Laplacian associated to G by

$$\Delta \colon \ell_2(V) \to \ell_2(V) \quad \text{with} \quad \Big(\Delta f\Big)(v) := \deg(v) f(v) - \sum_{u\sim v} f(u) ,$$

where $v \sim u$ means that both vertices are adjacent. Compute the propagation of Δ. Does this number depend on the fact that G is finite (i.e., $|V| < \infty$) ? Is it relevant that the graph is locally finite (i.e., $\sup\{\deg(v) \mid v \in V\}$)?

3.2) A partial isometry $W \in \mathcal{B}(\mathcal{H})$ is defined by the condition that W^*W (hence also WW^*) is an (orthogonal) projection. In particular, if $W^*W = \mathbb{1}$ we call W an isometry.

a) Show that a set of matrix units E_{ij} $(i, j = 1, \dots, n)$ for M_n, are partial isometries.

b) Let (X, d) be a metric space and denote by $t \colon A \to B$ a partial translation, i.e., a bijection between the subsets $A, B \subset X$. Consider the operator

$$V_t \colon \ell_2(X) \to \ell_2(X) \quad \text{given by} \quad V_t \delta_x = \begin{cases} \delta_{t(x)} & \text{if} \quad x \in A \\ 0 & \text{otherwise} \end{cases} .$$

Show that V_t is a partial bijection. When is V_t of bounded propagation? Give an example of an isometry with infinite propagation. (Use for example $X = \mathbb{N}$ and $\mathcal{H} = \ell_2$.)

3.3)* Consider the metric space (X, d) with bounded geometry and property A. For $R > 0$ and $\varepsilon > 0$ let $S > 0$ and $\xi \colon X \to \mathcal{H}$ witness property A in the Higson-Roe characterization. Define the linear operator T on $\ell_2(X)$ by the following matrix elements

$$T_{xy} = \langle \xi_x \, , \, \xi_y \rangle_{\mathcal{H}} \, , \quad x, y \in X \, .$$

Show that T is a bounded operator with bounded propagation.

3.4) Let G a countable group, $F \subset G$ a finite subset and denote by P the orthogonal projection onto the finite subspace $\ell_(F) \leq \ell_(G)$. Let $L\colon G \to \mathcal{B}(\ell_2(G))$ the left regular representation of G (which is a unitary representation) and identify $M_F(\mathbb{C}) \cong P\mathcal{B}(\ell_2(G))P$ with corresponding matrix units $E_{g,h}$, $g, h \in F$.

a) Show that for $s \in G$

$$E_{g,g} L_s E_{h,h} = E_{g,s^{-1}g} \quad \text{and} \quad PL_sP = \sum_{g \in F \cap sF} E_{g,s^{-1}g} \,.$$

b) Let $C^*_r(G) \subset \mathcal{B}(\ell_2(G))$ be the (reduced) group C*-algebra. Proof that the compression map

$$\varphi\colon C^*_r(G) \to M_F(\mathbb{C}) \quad \text{given by} \quad \varphi(T) = PTP$$

is a unital completely postive map.

c) Show that, also,

$$\psi\colon M_F(\mathbb{C}) \to C^*_r(G) \quad \text{given by} \quad \varphi(E_{g,h}) = \frac{1}{|F|} L_g L_h^*$$

(and extended by linearity) is a unital completely postive map.