Analysis on Graphs

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These lecture notes summarise a course given at the ICMAT in July 2017 (Escuela JAE de Matemáticas 2017). We give an overview on analytic methods applied to graphs, such as analysing the Laplacians on discrete graphs and their spectrum. We will emphasise on results relating analysis (resp. the spectrum) of a graph with its geometry; and especially on results allowing to recover graph information from the spectrum or the Laplacian and related operators.

Contents

Introduction

What is "Analysis on graphs"?

- Interplay between graph theory ("geometry") and matrices resp. operators on graphs (eigenvalues, spectrum, "analysis") — e.g. try to recover or even characterise topological or geometrical properties of the graph by its spectrum.

- Perron-Frobenius theorem: statement about first eigenvector, one can recover weights of a graph (e.g. recover the "most important vertex", Google used this in their first implementations of ranking results
- Cheeger inequality: a relation between eigenvalues (of a Laplacian on the graph) and the connectedness of the graph (isoperimetric constants) — useful in partitioning of a graph
- Periodic graphs: calculation of spectrum of an infinite graph reduced to calculations on finite graphs —

Disclaimer

The result presented here is a choice guided by the author's interest, and by no means meant to be complete.

Literature on Analysis on Graphs

Spectral Graph Theory

A recent book is [BH12]. A bit older are the books [Chu97, CDS95, CDGT88] and [CRS97]. A bit different perspective is [BLS07]. There are also recent lecture notes [Mah16] having recent developments such as graph partitioning and applications to data science in mind.

In French, you also have [CdV98].

References

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... a very incomplete list ...

↓ Lecture 1, 2017-07-03 ↓

1 Graphs, Laplacians and all that (notational stuff)

1.1 Some definitions from Graph Theory

1.1 Definition (Discrete graphs)**.**

(a) A *(discrete) graph* G is given by the datea (V, E, ∂), where

 (i) V resp. E are finite or countable disjoint sets (the set of *vertices* resp. *edges*);

 (ii) $\partial\colon E \longrightarrow V \times V$ is the incidence map: $\partial e = (\partial_- e, \partial_+ e)$ is the pair of initial and terminal vertex of the edge $e \in E$.

In particular, $\partial_\pm e$ fixes an orientation of an edge e.

We sometimes also write $G = (V(G), E(G), \partial^G)$.

(b) For a vertex $v \in V$ we set

$$E_v^\pm := \{\, e \in E \mid \partial_\pm e = v \,\} \tag{1.1}$$

for the set of edges ending $(+)$ resp. starting $(-)$ in v. Moreover, set $E_v = E_v^+ \cup E_v^-$ for the set of all edges adjacent with v.

(c) We call

$$\deg v := \deg_G v := |E_v|$$

the *degree* of a vertex $v \in V$ in G.

(d) For subsets $A, B \subset V$ set

$$E_\pm(A, B) := \{\, e \in E \mid \partial_\mp e \in A, \partial_\pm e \in B \,\}$$

for the set of edges from A to B $(+)$ resp. B to A. In particular, $E_v^\pm = E_\pm(V, v)$ and $E_+(A, B) = E_-(B, A)$, where we write $E_\pm(V, v)$ anstelle von $E_\pm(V, \{v\})$ etc.

Moreover set

$$E(A, B) = E_+(A, B) \cup E_-(A, B).$$

As before, write $E_\pm^G(A, B)$ resp. $E^G(A, B)$ to stress the dependency of the graph G if necessary.

(e) We assume that all our graphs are *locally finite*, i.e., we have $\deg v < \infty$ for all $v \in V$.

Remarks.

(a) We allow graphs with loops (i.e., edges e with $\partial_- e = \partial_+ e$) as well as multiple edges (e.g. double edges e_1, e_2 with $\partial_+ e_1 = \partial_+ e_2$ and $\partial_- e_1 = \partial_- e_2$ or $\partial_+ e_1 = \partial_- e_2$ and $\partial_- e_1 = \partial_+ e_2$). (This is useful e.g. for quotient graphs).) A graph without loops and multiple edges is called *schlicht* ("simple").

Note that for a schicht graph G, it is sufficient to encode the edges as two-element subsets, i.e., $E \subset \{\, \{v, w\} \,|\, v, w \in V, v \neq w \,\}$, hence the data $G = (V, E)$ is enough to encode the graph.

The union of adjacent edges $E_v = E_v^+ \mathbin{\dot\cup} E_v^-$ is assumed to be *disjoint* in order to cound a loop *twice*: this is again useful for quotient graphs in order to preserve the degree. Moreover, for the degree, a loop increases the degree by 2.

Nevertheless, $E(A, B)$ is not a disjoint union (hence a loop appears once in $E(v, v)$ and twice in E_v).

(b) The graphs here always have an *orientation*, such graphs are sometimes also called *directed graphs* or *digraphs*. For certain objects (e.g. the Laplacian) the orientaion plays no role, for others it does (e.g. the discrete derivative). [1]

1.2 Definition (Substructures)**.** Let $G = (V, E, \partial)$ be a graph.

(a) A *subgraph* $G' = (V', E', \partial')$ of G is given by subsets $V' \subset V$ and $E' \subset E$ with $E' \subset E^G(V', V')$ (i.e., $\partial e \in V' \times V'$ for all $e \in E'$) and $\partial' = \partial\restriction_E$.

(b) An *induced subgraph* is a subgraph with $E^G(V', V') = E'$ (i.e., if $v_1, v_2 \in V'$, then also all edges in the original graph G have to be in G').

In particular, a subgraph obtained by just removing edges is never an induced subgraph.
Die folgende Aussage werden wir sehr oft benutzen:

1.3 Lemma (Handshake lemma)**.** *Let $G = (V, E, \partial)$ be a graph and $(a_e(v))_{e \in E_v, v \in V}$ such that $a_e(v) \in \mathbb{C}$ ($a_e(v) \geq 0$ only for a finite number of elements, then we have*

$$\sum_{v \in V} \sum_{e \in E_v^+} a_e(v) = \sum_{e \in E} a_e(\partial_+ e) \tag{1.3a}$$

$$\sum_{v \in V} \sum_{e \in E_v^-} a_e(v) = \sum_{e \in E} a_e(\partial_- e) \tag{1.3b}$$

$$\sum_{v \in V} \sum_{e \in E_v} a_e(v) = \sum_{e \in E} \sum_{v = \partial_\pm e} a_e(v). \tag{1.3c}$$

In particular, if $a_e(v) = 1$ we have

$$\sum_{v \in V} \deg^+ v = |E|, \qquad \sum_{v \in V} \deg^- v = |E| \quad \textit{und} \quad \sum_{v \in V} \deg v = 2|E|. \tag{1.3d}$$

Proof. (Induction over $n = |V|$)

□

1.4 Exercise. Why it is called *handshake lemma*? Try to give a proof as a warm up to get used to combinatorial arguments and the notation ...

[1] Attention: some authors use (for schlicht graphs) always $e = (v, w)$ and $\bar{e} = (w, v)$, if v and w are joint by an edge. In this case, the set of edges has twice as many edges as in our case. An *orientation* is then a selection of one of the elements e or $\bar{e}$ for each edge.

1.2 Matrices associated with graphs

In this part, we assume that the graph is *finite* (i.e., $|V|$ is a finite sets) — due to Lemma 1.3 and the local finiteness, we also conclude $|E| = \frac{1}{2}\sum_{v\in V} \deg v < \infty$. We set

$$n := |V| \qquad \text{and} \qquad m := |E|.$$

We are now associating matrices to a graph $G = (V, E, \partial)$. In order to do so, we have to *number* the edges and vertices, i.e., we assume

$$V = \{v_1, \dots, v_n\} \qquad \text{and} \qquad E = \{e_1, \dots, e_n\}.$$

Note that the following matrices depend on the numbering (but further quantities later on such as eigenvalues *not*!).

1.5 Definition (Matrices associated with a graph).

(a) The *adjacency matrix* $A = A^G$ of G is an $(n \times n)$-matrix defined by

$$A^G_{jk} = |E_+(v_j, v_k)| + |E_-(v_j, v_k)| = \begin{cases} 2|E(v_j, v_j)|, & j = k, \\ |E(v_j, v_k)|, & j \neq k, \end{cases}$$

(number of edges from v_j to v_k plus number of edges from v_k to v_j — loops are hence counted trice on the diagonal)

(b) The *incidence matrix* $B = B^G$ of G is the $(n \times m)$-matrix defined by

$$B^G_{ja} = \begin{cases} 1 & \text{if } \partial_+ e_a = v_j \text{ and } \partial_- e_a \neq v_j, \\ -1 & \text{if } \partial_- e_a = v_j \text{ and } \partial_+ e_a \neq v_j, \\ 0 & \text{otherwise,} \end{cases}$$

(i.e., for a loop, we set 0, see in Remark (c) below why ...)

(c) The *(combinatorial) Laplace matrix* L^G of G is the $(n \times n)$-matrix defined via $L^G := B^G(B^G)^*$ (where $(\cdot)^*$ denotes the transpose of a matrix).

Remark.

(a) The matrices A^G and L^G are symmetric, hence have n *real* eigenvalue (not necessarily all distinct). The eigenvalues of A^G are sometimes called *spectrum of the graph G* (e.g. in [CDS95]), sometimes the spectrum of a graph is the spectrum of its Laplace matrix L^G.

We will be more precise here and specify of what spectrum we talk (we also introduce weights later on, hence for each weight, there is an analogous adjacency and Laplace matrix).

(b) On a schlicht graph (no loops, no multiple edges), we have

$$A^G_{jk} = \begin{cases} 1 & \text{if } v_j \sim v_k, \\ 0 & \text{sonst,} \end{cases}$$

where $v_j \sim v_k$ means that there is an edge between v_j and v_k. In particular, the diagonal elements of A^G of a schlicht graph are always 0.

(c) Sometimes, the adjacency matrix of a *directed* graph is defined by

$$\widehat{A}^G_{jk} = |E_+(v_j, v_k)|.$$

This matrix is *not symmetric*, hence may have complex eigenvalues. Moreover, one has the in- and out-incidence matrices

$$B^{+,G}_{ja} = \begin{cases} 1 & \text{if } \partial_+ e_a = v_j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad B^{-,G}_{ja} = \begin{cases} 1 & \text{if } \partial_- e_a = v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then, one can show that $\widehat{A}^G = B^{-,G}(B^{+,G})^*$. Note that $B^G = B^{+,G} - B^{-,G}$, explaing why we set 0 for a loop in B).

Almost trivial, but an important fact is the following:

1.6 Exercise. The spectrum of L^G (or A^G) is independent of the numbering of vertices.

1.7 Lemma. *The Laplace matrix has the form $L^G = D^G - A^G$, where D^G is the* degree matrix*, a diagonal matrix with non-zero entries $D^G_{jj} = \deg v_j$.*

Proof. (Exercise?) □

Remark.

(a) We have

$$L^G_{jk} = \begin{cases} \deg^* v_k & \text{if } j = k, \\ -|E(v_j, v_k)| & \text{if } j \neq k, \end{cases}$$

where $\deg^* v := \deg v - 2|E(v,v)|$ denotes the *reduced degree*, i.e., the degree of v without loops (note that $E(v,v)$ is the set of loops, but counted one time each only).

(b) For a schlicht graph, we have

$$L^G_{jk} = \begin{cases} \deg v_k & \text{if } j = k, \\ -1 & \text{if } v_j \sim v_k, \\ 0 & \text{otherwise.} \end{cases}$$

(c) In general, the spectrum of A^G and L^G is *not related*! If the graph is *regular* (i.e., there is $r \in \mathbb{N}$ such that $\deg v = r$ for all $v \in V$), then $L^G = rI_n - A_G$ (I_n is the unit matrix), i.e., α is an eigenvalue of A^G iff $\lambda = r - \alpha$ is an eigenvalue of L^G.

1.3 Weighted graphs, associated spaces

1.8 Definition.

(a) A *weighted* (discrete) graph (G, m) is a pair given by a discrete graph $G = (V, E, \partial)$ and two functions (called *vertex and edge weight*)

$$m\colon V \longrightarrow (0,\infty), v \mapsto m(v), \quad \text{und} \quad m\colon E \longrightarrow (0,\infty), e \mapsto m(e),$$

denoted both with the same symbol m.

(b) The *relative weight* $\operatorname{rel}_m \colon V \longrightarrow (0,\infty)$ is defined as

$$\operatorname{rel}_m(v) := \frac{1}{m(v)} \sum_{e \in E_v} m(e).$$

(c) A weight m on a graph G or a weighted graph (G,m) is called *normalised*, if

$$\operatorname{rel}_m(v) = 1$$

for all $v \in V$. (In other words, $m(v) = \sum_{e \in E_v} m(e)$ for all $v \in V$).

1.9 Examples (of weights). We have two *intrinsic* weights, i.e., weights given entirely by the combinatorial graph structure:

(a) The weight $m = 1$ (i.e., $m(v) = 1$ and $m(e) = 1$ for all $v \in V$ and $e \in E$) is called *combinatorial weight*, written shortly $m = 1$. For the weighted graph with combinatorial weight, we also write $G^{\mathrm{comb}} = (G, 1)$.

For the combinatorial weight, the relative weight is the degree, i.e., $\operatorname{rel}_1(v) = \deg v$ for all $v \in V$.

(b) The weight $m(v) = \deg v$ and $m(e) = 1$ for all $v \in V$ and $e \in E$) is called *standard weight*, a bit sloppy also written as $m = \deg$. For the weighted graph with standard weight, we also write $G^{\mathrm{std}} = (G, \deg)$.

The standard weight is normalised, hence $\operatorname{rel}_{\deg}(v) = 1$ for all $v \in V$.

We now define operators on Hilbert spaces associated with a weighted graph: we work with *complex* Hilbert spaces in order to have a nice and simple spectral theory (and also to allow magnetic and periodic Laplacians later on).

1.10 Definition. Let (G,m) be a weighted graph with $G = (V, E, \partial)$. Set

$$\ell_2(V,m) := \Big\{ \varphi \colon V \longrightarrow \mathbb{C} \,\Big|\, \sum_{v \in V} |\varphi(v)|^2 m(v) < \infty \Big\}$$
$$\ell_2(E,m) := \Big\{ \eta \colon E \longrightarrow \mathbb{C} \,\Big|\, \sum_{e \in E} |\eta(e)|^2 m(e) < \infty \Big\}$$

with norms defined by

$$\|\varphi\|^2_{\ell_2(V,m)} := \sum_{v \in V} |\varphi(v)|^2 m(v) \qquad \text{and} \qquad \|\eta\|^2_{\ell_2(E,m)} := \sum_{e \in E} |\eta(e)|^2 m(e).$$

Remark. The finiteness condition only appears for *non-finite* graphs. The spaces $\ell_2(V,m)$ and $\ell_2(E,m)$ are *Hilbert spaces*, i.e., normed spaces which are complete and where the norm is induced by an inner product. In our case, the inner products are

$$\langle \varphi, \psi \rangle_{\ell_2(V,m)} = \sum_{v \in V} \varphi(v)\overline{\psi(v)} m(v) \quad \text{and} \quad \langle \eta, \alpha \rangle_{\ell_2(E,m)} = \sum_{e \in E} \eta(e)\overline{\alpha(e)} m(e).$$

For a finite graph G with combinatorial weight, we have

$$\ell_2(V,1) = \mathbb{C}^V = \{\, \varphi \colon V \longrightarrow \mathbb{C} \,|\, \varphi \text{ map} \,\} \cong \mathbb{C}^{|V|} \quad \text{and}$$
$$\ell_2(E,1) = \mathbb{C}^E = \{\, \eta \colon E \longrightarrow \mathbb{C} \,|\, \eta \text{ map} \,\} \cong \mathbb{C}^{|E|}$$

with standard inner product. The last isomorphy includes a numbering of the edges (as for the matrices associated with G).

1.4 Discrete derivatives and Laplacians

Graphs may now be again infinite.

1.11 Definition. Let (G, m) be a weighted graph with $G = (V, E, \partial)$. For $\varphi\colon V \longrightarrow \mathbb{C}$ we define the *(discrete) derivative* $d\varphi\colon E \longrightarrow \mathbb{C}$ by

$$(d\varphi)(e) = \varphi(\partial_+ e) - \varphi(\partial_- e)$$

for $e \in E$.

1.12 Proposition. *Let (G, m) be a weighted graph with $G = (V, E, \partial)$ and with bounded relative weight, i.e., with*

$$\varrho_\infty := \sup_{v \in V} \operatorname{rel}_m(v) = \sup_{v \in V} \frac{1}{m(v)} \sum_{e \in E_v} m(e) < \infty. \tag{1.12}$$

The the linear map

$$d\colon \ell_2(V, m) \longrightarrow \ell_2(E, m)$$

is bounded by $\sqrt{2\varrho_\infty}$, i.e., we have

$$\|d\varphi\|_{\ell_2(E,m)} \le \sqrt{2\varrho_\infty} \|\varphi\|_{\ell_2(V,m)}$$

for all $\varphi \in \ell_2(V, m)$ (in particular, $\|d\|_{\ell_2(V,m)\to\ell_2(E,m)} \le \sqrt{2\varrho_\infty}$ for the operator norm).

Proof. We have

$$\begin{aligned}
\|d\varphi\|^2_{\ell_2(E,m)} &= \sum_{e \in E} |\varphi(\partial_+ e) - \varphi(\partial_- e)|^2 m(e) \\
&\overset{\text{CY}}{\le} 2 \sum_{e \in E} \Big(|\varphi(\partial_+ e)|^2 + |\varphi(\partial_- e)|^2 \Big) m(e) \\
&= 2 \sum_{v \in V} \Big(\sum_{e \in E_v^+} |\varphi(\partial_+ e)|^2 m(e) + \sum_{e \in E_v^-} |\varphi(\partial_- e)|^2 m(e) \Big) \\
&= 2 \sum_{v \in V} |\varphi(v)|^2 \sum_{e \in E_v} m(e) \\
&= 2 \sum_{v \in V} |\varphi(v)|^2 \operatorname{rel}_m(v) m(v) \\
&\le 2 \sup_{v \in V} \operatorname{rel}_m(v) \sum_{v \in V} |\varphi(v)|^2 m(v) \\
&= 2\varrho_\infty \|\varphi\|^2_{\ell_2(V,m)},
\end{aligned}$$

where we used Cauchy-Young ($(a-b)^2 \le 2(a^2+b^2)$) in the first inequality and Lemma 1.3 (1.3a)–(1.3b) in the second equality. □

↓ Lecture 2, 2017-07-04 ↓

1.13 Definition. The *Laplace operator* (on functions on vertices, i.e., on 0-forms) on a weighted graph (G, m) is defined by

$$\Delta := \Delta_{(G,m)} := d^* d\colon \ell_2(V, m) \longrightarrow \ell_2(V, m).$$

1.14 Corollary (of Proposition 1.12). *Let (G, m) be a weighted graph with bounded relative weight $\varrho_\infty := \sup_{v \in V} \operatorname{rel}_m(v) < \infty$. Then the Laplace operator $\Delta = \Delta_{(G,m)}$ is bounded, self-adjoint and fulfills $0 \le \Delta \le 2\varrho_\infty$, i.e.,*

$$0 \le \langle \Delta\varphi, \varphi \rangle \le 2\varrho_\infty \langle \varphi, \varphi \rangle$$

for all $\varphi \in \ell_2(V, m)$.

Proof. The boundedness follows from the boundedness of d (implying that d^* is bounded with the same norm and hence also d^*d is bounded). For the self-adjointness, note $\Delta^* = (d^*d)^* = d^*d^{**} = d^*d = \Delta$.

We have

$$\langle \Delta\varphi, \varphi \rangle_{\ell_2(V,m)} = \langle d^*d\varphi, \varphi \rangle_{\ell_2(V,m)} = \langle d\varphi, d\varphi \rangle_{\ell_2(E,m)} = \|d\varphi\|^2_{\ell_2(E,m)}$$

for all $\varphi \in \ell_2(V, m)$. Moreover,

$$0 \le \|d\varphi\|^2_{\ell_2(E,m)} \le \|d\|^2 \|\varphi\|^2_{\ell_2(V,m)} \le 2\varrho_\infty \|\varphi\|^2_{\ell_2(V,m)}$$

by Proposition 1.12. □

Let us now see how d^* and the Laplacian act concretely:

1.15 Proposition. *The adjoint $d^* \colon \ell_2(E, m) \longrightarrow \ell_2(V, m)$ is given by*

$$(d^*\eta)(v) = \frac{1}{m(v)} \sum_{e \in E_v} \overrightarrow{\eta}_e(v) m(e), \qquad \text{where} \qquad \overrightarrow{\eta}_e(v) := \begin{cases} \eta(e), & v = \partial_+ e, \\ -\eta(e), & v = \partial_- e \end{cases}$$

denotes the oriented evaluation *of η.*

Proof. (Excercise) [2] □

Remark. One can think of d^* as as discrete *divergence*: it is the effective flux of a *vector field* η in a vertex v. Then $(d^*\eta)(v) = 0$ means that at vertex v, the in-flux equals the out-flux.

1.16 Proposition. *For the Laplacian, we have*

$$(\Delta\varphi)(v) = \frac{1}{m(v)} \sum_{e \in E_v} \big(\varphi(v) - \varphi(v_e)\big) m(e) = \operatorname{rel}_m(v)\varphi(v) - \frac{1}{m(v)} \sum_{e \in E_v} \varphi(v_e) m(e),$$

where v_e denotes the vertex on the edge e opposite to v. In particular, the Laplacian $\Delta = d^*d$ is independent of the orientation of the graph (in contrast to d and d^*).

Moreover, if G is finite, then 0 is an eigenvalue of $\Delta_{(G,m)}$ with eigenfunction being constant.

[2] We have

$$\begin{aligned}
\langle d\varphi, \eta \rangle_{\ell_2(E,m)} &= \sum_{e \in E} \big(\varphi(\partial_+ e) - \varphi(\partial_- e)\big) \overline{\eta(e)} m(e) \\
&= \sum_{v \in V} \varphi(v) \Big(\sum_{e \in E_v^+} \overline{\eta(e)} m(e) - \sum_{e \in E_v^-} \overline{\eta(e)} m(e) \Big) \\
&= \sum_{v \in V} \varphi(v) \sum_{e \in E_v^+} \overline{\overrightarrow{\eta}_v(e)} m(e) \\
&= \sum_{v \in V} \varphi(v) \Big(\frac{1}{m(v)} \sum_{e \in E_v^+} \overline{\overrightarrow{\eta}_v(e)} m(e) \Big) m(v) \\
&= \langle \varphi, d^*\eta \rangle_{\ell_2(V,m)},
\end{aligned}$$

wobei wir wieder Lemma 1.3 (1.3a)–(1.3b) genutzt haben (zweite Gleichung).

Proof. We have

$$(\Delta\varphi)(v) = \frac{1}{m(v)} \sum_{e \in E_v} (\widehat{d\varphi})_v(e) m(e).$$

Moreover, we have

$$(\widehat{d\varphi})_v(e) = \begin{cases} f(v) - f(v_e), & v = \partial_+ e, v_e = \partial_- e \\ -(f(v_e) - f(v)), & v = \partial_- e, v_e = \partial_+ e, \end{cases}$$
$$= f(v) - f(v_e),$$

using the notation v_e, and showing also the independence of the orientation.

Denote by $\mathbb{1}$ the constant function with with $\mathbb{1}(v) = 1$ for all v. If G is finite, then $\mathbb{1} \in \ell_2(V, m)$. Moreover, 0 is an eigenvalue as we have

$$(\Delta_{(G,m)}\mathbb{1})(v) = \frac{1}{m(v)} \sum_{e \in E_v} (\mathbb{1}(v) - \mathbb{1}(v_e)) = 0$$

hence $\mathbb{1}$ is an eigenvector for the eigenvalue 0. □

1.17 Definition (Weighted adjacency operator)**.** Let (G, m) be a weighted graph. The *weighted adjacency operator* $A = A_{(G,m)}$ is defined by

$$(A\varphi)(v) := \frac{1}{m(v)} \sum_{e \in E_v} m(e)\varphi(v_e)$$

for $\varphi \colon V \longrightarrow \mathbb{C}$.

Remark. The weighted adjacency operator for a normalised weight

$$(A_{(G,\deg)}\varphi)(v) = \sum_{e \in E_v} p_e(v)\varphi(v_e), \qquad \text{where} \qquad p_e(v) = \frac{m(e)}{m(v)},$$

is sometimes also called Markov or transition operator, where $p_e(v)$ is the probability that there is a transition at the vertex v along the edge e. Note that $\sum_{e \in E_v} p_e(v) = 1$.

1.18 Proposition. *Let (G, m) be a weighted graph with bounded relative weight $\varrho_\infty := \sup_{v \in V} \operatorname{rel}_m(v) < \infty$. Then $A \colon \ell_2(V, m) \longrightarrow \ell_2(V, m)$ is a bounded operator with norm bounded by ϱ_∞. Moreover,*

$$-\varrho_\infty \le A_{(G,m)} \le \varrho_\infty$$

and we have

$$\Delta_{(G,m)} = D_{(G,m)} - A_{(G,m)},$$

where $(D_{(G,m)}\varphi)(v) = \operatorname{rel}_m(v)\varphi(v)$ is the multiplication operator ("diagonal matrix") with the relative weight.

Proof. (Boundedness: exercise) [3] The inequality is a fact from functional analysis: $A = A^*$ and $\|A\| \le \varrho_\infty$ implies $-\varrho_\infty \le A \le \varrho_\infty$. [4] The equation for $\Delta_{(G,m)}$ follows from Proposition 1.16. □

In certain cases we can relate $\Delta_{(G,m)}$ with $A_{(G,m)}$ by an affine transformation (generalising the normalised weights where $\mathrm{rel}_m = r = 1$):

1.19 Definition. Let $r > 0$. We say that a weighted graph (G, m) is *r-regular* if the relative weight is constant $\mathrm{rel}_m = r$, i.e., if $rm(v) = \sum_{e \in E_v} m(e)$ for all $v \in V$.

Example.

(a) Any normalised weight is 1-regular, in particular, the standard weight.

(b) Moreover, the combinatorial weight is r-regular iff the graph is r-regular (i.e, if $\deg v = r$ for all $v \in V$).

An immediate consequence of Proposition 1.16 is:

1.20 Corollary. *Let (G, m) be an r-regular weighted graph, then*

$$\Delta_{(G,m)} = r - A_{(G,m)}.$$

In particular, λ is an eigenvalue of $\Delta_{(G,m)}$ iff $\alpha = r - \lambda$ is an eigenvalue of $A_{(G,m)}$, and the multiplicity is preserved.

[3] We have (similarly as in Proposition 1.12)

$$\begin{aligned}
\|A\varphi\|^2_{\ell_2(V,m)} &= \sum_{v \in V} \frac{1}{m(v)^2} \Big| \sum_{e \in E_v} \varphi(v_e) m(e) \Big|^2 m(v) \\
&\overset{\text{CS}}{\le} \sum_{v \in V} \frac{1}{m(v)} \sum_{e \in E_v} |\varphi(v_e)|^2 m(e) \sum_{e \in E_v} m(e) \\
&= \sum_{e \in E} \sum_{v = \partial_\pm e} |\varphi(v_e)|^2 m(e)\, \mathrm{rel}_m(v) && \text{(Lemma 1.3, Def. } \mathrm{rel}_m) \\
&\le \varrho_\infty \sum_{e \in E} \sum_{v = \partial_\pm e} |\varphi(v_e)|^2 m(e) && (\mathrm{rel}_m \text{ beschränkt}) \\
&= \varrho_\infty \sum_{e \in E} \sum_{v = \partial_\pm e} |\varphi(v)|^2 m(e) && (v = \partial_\pm e \text{ durchläuft } v \text{ and } v_e) \\
&= \varrho_\infty \sum_{v \in V} |\varphi(v)|^2 \sum_{e \in E_v} m(e) && \text{(Lemma 1.3)} \\
&= \varrho_\infty \sum_{v \in V} |\varphi(v)|^2 m(v)\, \mathrm{rel}_m(v) && (\text{Def. } \mathrm{rel}_m) \\
&\le \varrho_\infty^2 \sum_{v \in V} |\varphi(v)|^2 m(v) && (\mathrm{rel}_m \text{ beschränkt}) \\
&\le \varrho_\infty^2 \|\varphi\|^2_{\ell_2(V,m)}.
\end{aligned}$$

[4] Namely we have

$$\begin{aligned}
\varrho_\infty \ge \|A_{(G,m)}\| &= \sup\big\{ \|A_{(G,m)}\varphi\|_{\ell_2(V,m)} \,\big|\, \varphi \in \ell_2(V,m), \|\varphi\|_{\ell_2(V,m)} = 1 \big\} \\
&= \sup\big\{ |\langle A_{(G,m)}\varphi, \psi\rangle_{\ell_2(V,m)}| \,\big|\, \varphi, \psi \in \ell_2(V,m), \|\varphi\|_{\ell_2(V,m)} = 1, \|\psi\|_{\ell_2(V,m)} = 1 \big\} \\
&\ge \sup\big\{ |\langle A_{(G,m)}\varphi, \varphi\rangle_{\ell_2(V,m)}| \,\big|\, \varphi \in \ell_2(V,m), \|\varphi\|_{\ell_2(V,m)} = 1 \big\}.
\end{aligned}$$

and $|\langle A_{(G,m)}\varphi, \varphi\rangle| \le \varrho\langle\varphi, \varphi\rangle$ follows for general $\varphi \in \ell_2(V,m)$.

1.21 Remark. For concrete calculations, it is often useful to write the operators as a matrix. For this, we need an orthonormal basis (ONB) $\{\, \delta_v \,|\, v \in V \,\}$ of $\ell_2(V, m)$, namely

$$\delta_v(w) := \begin{cases} 1/m(v)^{1/2}, & v = w, \\ 0, & v \neq w \end{cases}$$

[then $\langle \delta_v, \delta_w \rangle_{\ell_2(V,m)} = m(v)/(m(v)^{1/2} m(v)^{1/2}) = 1$ for $v = w$ and 0 otherwise].

The matrices associated with $A_{(G,m)}$ and $\Delta_{(G,m)}$ with respect to this orthonormal basis are dann

$$\begin{aligned} (A_{(G,m)})_{vw} := \langle A_{(G,m)} \delta_v, \delta_w \rangle &= \sum_{v' \in V} A_{(G,m)} \delta_v(v') \delta_w(v') m(v') = A_{(G,m)} \delta_v(w) \sqrt{m(w)} \\ &= \frac{1}{\sqrt{m(w)}} \sum_{e \in E_w} m(e) \delta_v(w_e) \\ &= \begin{cases} \dfrac{2m(E(v,v))}{m(v)}, & v = w, \\ \dfrac{m(E(v,w))}{\sqrt{m(v)m(w)}}, & v \neq w, \end{cases} \end{aligned}$$

where we set $m(E') := \sum_{e \in E'} m(e)$ (loops counted once!). Similarly,

$$\begin{aligned} (\Delta_{(G,m)})_{vw} := \langle \Delta_{(G,m)} \delta_v, \delta_w \rangle &= \sum_{v' \in V} \Delta_{(G,m)} \delta_v(v') \delta_w(v') m(v') = \Delta_{(G,m)} \delta_v(w) \sqrt{m(w)} \\ &= \frac{1}{\sqrt{m(w)}} \sum_{e \in E_w} m(e) \Big(\delta_v(w) - \delta_v(w_e) \Big) \\ &= \begin{cases} \dfrac{m(E_v) - 2m(E(v,v))}{m(v)} = \dfrac{m(E_v^*)}{m(v)}, & v = w, \\ -\dfrac{m(E(v,w))}{\sqrt{m(v)m(w)}}, & v \neq w, \end{cases} \end{aligned}$$

where E_v^* are the edges adjacent with v without loops.

Example. Let G be a finite graph.

(a) For the combinatorial weight $m = 1$, $\{\, \delta_v \,|\, v \in V \,\}$ is the standard ONB. We then have

$$(A_{G^{\mathrm{comb}}})_{vw} = \begin{cases} 2|E(v,v)|, & v = w, \\ |E(v,w)|, & v \neq w, \end{cases} \quad \text{and} \quad (\Delta_{G^{\mathrm{comb}}})_{vw} = \begin{cases} \deg^*(v), & v = w, \\ -|E(v,w)|, & v \neq w, \end{cases}$$

In particular (after a numbering of the vertices), $(A_{G^{\mathrm{comb}}})_{v_j,v_k})_{j,k}$ resp. $(L_{G^{\mathrm{comb}}})_{v_j,v_k})_{j,k}$ is the adjacency resp. Laplace matrix A^G resp. L^G defined in Definition 1.5.

(b) For a schlicht graph with standard weight $m = \deg$, we have

$$(A_{(G,\deg)})_{vw} = \begin{cases} 0, & v = w, \\ \dfrac{1}{\sqrt{(\deg v)(\deg w)}}, & v \sim w \\ 0, & \text{otherwise} \end{cases} \qquad \text{and}$$

$$(\Delta_{(G,\deg)})_{vw} = \begin{cases} 1, & v = w, \\ -\dfrac{1}{\sqrt{(\deg v)(\deg w)}}, & v \sim w \\ 0, & \text{otherwise.} \end{cases}$$

1.22 Examples (Some examples of matrices for the standard weight)**.**

(a) Let $G = L_n$ be the path graph with n vertices and $n-1$ edges, then we have

$$A_{L_n^{\text{std}}} = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & \dots & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/2 & & 0 & 0 & 0 \\ 0 & 1/2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1/2 & 0 \\ 0 & 0 & 0 & & 1/2 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & \dots & 0 & 1/\sqrt{2} & 0 \end{pmatrix}$$

und

$$\Delta_{L_n^{\text{std}}} = \begin{pmatrix} 1 & -1/\sqrt{2} & 0 & \dots & 0 & 0 & 0 \\ -1/\sqrt{2} & 1 & -1/2 & & 0 & 0 & 0 \\ 0 & -1/2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1/2 & 0 \\ 0 & 0 & 0 & & -1/2 & 1 & -1/\sqrt{2} \\ 0 & 0 & 0 & \dots & 0 & -1/\sqrt{2} & 1 \end{pmatrix}.$$

2 First results on spectral graph theory

2.1 The spectrum

Let $T\colon \mathscr{H} \longrightarrow \mathscr{H}$ be a bounded operator in a Hilbert space $\mathscr{H}$ (e.g. $\mathscr{H} = \ell_2(V, m)$ (i.e., there exists $C \leq 0$ with $\|T\varphi\|_{\mathscr{H}} \leq C\|\varphi\|_{\mathscr{H}}$ for all $\varphi \in \mathscr{H}$).

2.1 Definition. The *spectrum of* T is defined by

$$\begin{aligned} \sigma(T) &:= \left\{ \lambda \in \mathbb{C} \,\middle|\, T - \lambda \mathbb{1} \text{ is not bijective} \right\} \\ &= \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda \text{ eigenvalue} \right\} \cup \left\{ \lambda \in \mathbb{C} \,\middle|\, T - \lambda \mathbb{1} \text{ injektive, but not surjective} \right\}. \end{aligned}$$

2.2 Remark. For finite-dimensional Hilbert spaces we have

$$\sigma(T) = \{ \lambda \in \mathbb{C} \mid T - \lambda \mathbb{1} \text{ is not injektiv} \},$$

since “bijektive” and “injektive” of a linear map $\mathscr{H} \to \mathscr{H}$ lon a finite dimensional space $\mathscr{H} \cong \mathbb{C}^n$ is the same. Therefore, the spectrum consists only of eigenvalues ($\lambda \in \mathbb{C}$ with $\ker(T - \lambda \mathbb{1}) \neq \{0\}$). The second part exists hence only in infinite dimensional spaces.

Now an important characterisation of eigenvalues of finite multiplicity: (for simplicity only stated for finite dimensional operators): [without proof, this version is from Linear Algebra ...]

2.3 Theorem (Min-max principle). *Let T be a linear operator in a finite dimensional Hilbert space $\mathscr{H}$. Denote by $\lambda_k(T)$ the k-th eigenvalue, ordered increasingly and repeated according to the multiplicity. Then we have*

$$\lambda_k(T) = \inf_{D_k \subset \mathscr{H}} \sup_{\varphi \in D_k \setminus \{0\}} \frac{\langle T\varphi, \varphi \rangle_{\mathscr{H}}}{\|\varphi\|^2_{\mathscr{H}}},$$

where D_k runs through all k-dimensional subspaces of $\mathscr{H}$.

2.4 Example. Let $\mathscr{H} = \mathbb{C}^3$ and T (afer diagonalisation) be of the form

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

i.e., $\sigma(T) = \{1, 2, 5\}$. We have

$$\langle Tx, x \rangle_{\mathscr{H}} = |x_1|^2 + 2|x_2|^2 + 5|x_3|^2$$

and

$$\sup_{\varphi \in D_k \setminus \{0\}} \frac{\langle T\varphi, \varphi \rangle_{\mathscr{H}}}{\|\varphi\|^2_{\mathscr{H}}} = \begin{cases} 1, & D_k \subset \mathbb{C} \times \{(0,0)\} \\ 2, & D_k \subset \mathbb{C}^2 \times \{0\}, \exists x \in D_k \colon x_2 \neq 0 \\ 5, & D_k \subset \mathbb{C}^3, \exists x \in D_k \colon x_3 \neq 0. \end{cases}$$

If D_1 runs through all 1-dimensional spaces, the smallest value is 1, ai.e., $\lambda_1(T) = 1$; if D_2 runs through all 2-dimensional spaces, the smallest value is 2, hence $\lambda_2(T) = 2$ and finally for $D_3 = \mathbb{C}^3$ we have $\lambda_3(T) = 5$.

Remark. From $S \leq T$ (in the sense that $\langle S\varphi, \varphi \rangle \leq \langle T\varphi, \varphi \rangle$ for all $\varphi \in \mathscr{H}$) we conclude

$$\lambda_k(S) \leq \lambda_k(T);$$

this follows immediately from the corresponding inequality for the *Rayleigh quotient*

$$\frac{\langle S\varphi, \varphi \rangle}{\|\varphi\|^2} \leq \frac{\langle T\varphi, \varphi \rangle}{\|\varphi\|^2}$$

and Theorem 2.3. In particular, $a \leq T \leq b$ implies that all eigenvalues of T are in $[a, b]$.

Remark. We have more generally (also for $\mathscr{H}$ infinite dimensional, T bounded): If T is self-adjoint (i.e. $T = T^*$), then $\sigma(T) \subset \mathbb{R}$; if $T \geq 0$, then $\sigma(T) \subset [0, \infty)$, if $a \leq T \leq b$, then $\sigma(T) \subset [a, b]$. [Functional analysis ...]

2.5 Remark. Let T_j be (self-adjoint) operators in a Hilbert space $\mathscr{H}_j$ ($j = 1, \dots, r$) and let $\mathscr{H} = \mathscr{H}_1 \oplus \dots \oplus \mathscr{H}_r$. Let $T := T_1 \oplus \dots \oplus T_r$, then

$$\sigma(T) = \sigma(T_1) \cup \dots \cup \sigma(T_r),$$

and the multiplicity of an eigenvalue λ is the sum of the multiplicities of λ of all T_j.

Let us now come back to graphs:
From Corollary 1.14 ($0 \leq \Delta \leq 2\varrho_\infty$) and Proposition 1.18 ($-\varrho_\infty \leq A \leq \varrho_\infty$) we conclude:

2.6 Corollary. *Assume that* (G, m) *is a weighted graph with bounded relative weight. Then we have*

$$\sigma(\Delta_{(G,m)}) \subset [0, 2\varrho_\infty] \quad \text{and} \quad \sigma(A_{(G,m)}) \subset [-\varrho_\infty, \varrho_\infty]$$

Some concrete examples of spectra:

2.7 Examples (of spectra of graphs)**.** Note that for the combinatorial weight, the maixmal relative weight is $\varrho_\infty = \sup_v \deg v$.

(a) For the path graph L_n with n vertices the (combinatorial) adjacency spectrum is

$$\sigma(A^{L_n}) = \Big\{2\cos\Big(\frac{\pi}{n+1}\Big), 2\cos\Big(\frac{2\pi}{n+1}\Big), \dots, 2\cos\Big(\frac{n\pi}{n+1}\Big)\Big\} \subset [-2, 2]$$

(all eigenvalues simple), i.e.,

$$\sigma(A^{L_1}) = \{0\}, \quad \sigma(A^{L_2}) = \{1, -1\}, \quad \sigma(A^{L_3}) = \{\sqrt{2}, 0, -\sqrt{2}\},$$
$$\sigma(A^{L_4}) = \{(1+\sqrt{5})/2, (-1+\sqrt{5})/2, (1-\sqrt{5})/2, -(1+\sqrt{5})/2\}.$$

Moreover, the (combinatorial) Laplace spectrum is

$$\sigma(L^{L_n}) = \Big\{0 = 2 - 2\cos\Big(\frac{0\pi}{n}\Big), 2 - 2\cos\Big(\frac{\pi}{n}\Big), \dots, 2 - 2\cos\Big(\frac{(n-1)\pi}{n}\Big)\Big\} \subset [0, 4]$$

(all eigenvalues simple) i.e.,

$$\sigma(L^{L_1}) = \{0\}, \quad \sigma(L^{L_2}) = \{0, 2\}, \quad \sigma(L^{L_3}) = \{0, 1, 3\},$$
$$\sigma(L^{L_4}) = \{0, 2-\sqrt{2}, 2, 2+\sqrt{2}\}.$$

(b) For the cyclic graph C_n with n vertices, the (combinatorial) adjacency spectrum is

$$\sigma(A^{C_n}) = \Big\{2 = 2\cos\Big(\frac{2\pi 0}{n}\Big), 2\cos\Big(\frac{2\pi}{n}\Big), \dots, 2\cos\Big(\frac{2\pi(n-1)}{n}\Big)\Big\} \subset [-2, 2],$$

i.e.,

$$\sigma(A^{C_1}) = \{2\}, \quad \sigma(A^{C_2}) = \{2, -2\}, \quad \sigma(A^{C_3}) = \{2, -1_2\},$$
$$\sigma(A^{C_4}) = \{(2, 0_2, -2\},$$

Moreover, the (combinatorial) Laplace spectrum is

$$\sigma(L^{C_n}) = \Big\{0 = 2 - 2\cos\Big(\frac{2\pi 0}{n}\Big), 2 - 2\cos\Big(\frac{2\pi}{n}\Big), \dots, 2 - 2\cos\Big(\frac{2\pi(n-1)}{n}\Big)\Big\} \subset [0, 4],$$

i.e.,

$$\sigma(L^{C_1}) = \{0\}, \quad \sigma(L^{C_2}) = \{0, 4\}, \quad \sigma(L^{C_3}) = \{0, 3_2\},$$
$$\sigma(L^{C_4}) = \{0, 2_2, 4\}.$$

(c) For the complete graph K_n with n vertices, the (combinatorial) adjacency spectrum is

$$\sigma(A^{K_n}) = \{(n-1)_1, (-1)_{n-1}\},$$

i.e.,

$$\alpha_1(K_n) = n - 1, \quad \alpha_2(K_n) = \dots = \alpha_n(K_n) = -1.$$

The (combinatorial) Laplace spectrum is

$$\sigma(L^{K_n}) = \{0_1, n_{n-1}\},$$

i.e.,

$$\lambda_1(K_n) = 0, \quad \lambda_2(K_n) = \dots = \lambda_n(K_n) = n.$$

↓ **Lecture 3, 2017-07-05** ↓

2.2 Connectedness and the spectrum

2.8 Definition. Let $G = (V, E, \partial)$ be a graph.

(a) A *path* P of length n in G is a sequence $v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n$ of vertices v_j $(j = 0, \dots, n)$ and edges e_j $(j = 1, \dots, n)$ such that

$$(\partial_+ e_j = v_j \text{ and } \partial_- e_j = v_{j-1}) \quad \text{or} \quad (\partial_- e_j = v_j \text{ and } \partial_+ e_j = v_{j-1}).$$

We call v_0 and v_n the *initial* resp. *terminal* vertex of the path P, and we say that v_0 and v_n are *connected by a path* if there is such a path P.

(b) We say that G is *connected* if there is a path from any vertex of G to any other.

2.9 Remark.

(a) The relation

$$v \leftrightsquigarrow w \qquad \Leftrightarrow \qquad \text{“}v \text{ and } w \text{ are connected by a finite path”}$$

is an equivalence relation.

Let $V_0 := [v_0] := \{\, v \in V \mid v \leftrightsquigarrow v_0 \,\}$ be the equivalence class of $v_0 \in V$ and $E_0 := E(V_0, V_0)$. Then $G_0 = (V_0, E_0, \partial\restriction_{E_0})$ is a subgraph, called *connected component of* v_0. Any graph G has a unique decomposition into disjoint connected components $G = \dot\bigcup_j G_j$.

(b) Assume now that there are finitely many components G_j (say r), then a (weighted) Laplacian on G fulfils

$$\Delta_{(G,m)} = \Delta_{(G_1,m)} \oplus \dots \oplus \Delta_{(G_r,m)}$$

on $\ell_2(V, m) = \ell_2(V(G_1), m) \oplus \dots \oplus \ell_2(V(G_r), m)$, and hence

$$\sigma(\Delta_{G,m}) = \sigma(\Delta_{(G_1,m)}) \cup \dots \cup \sigma(\Delta_{(G_r,m)}),$$

(see Remark 2.5), and the multiplicity of each eigenvalue is the multiplicity of the eigenvalue on each component. A similar remark holds for $A_{(G,m)}$.

Remark. In a connected graph we have

$$|V| < \infty \quad \Leftrightarrow \quad |E| < \infty :$$

“$\Rightarrow$”: see beginning of Section 1.2.
“$\Leftarrow$”: In a connected graph, $\deg v \geq 1$ (otherwise v would be isolated). Hence we have

$$|V| = \sum_{v \in V} 1 \leq \sum_{v \in V} \deg v = 2|E| < \infty$$

by (1.3d).

2.10 Theorem. *Let (G, m) be a finite weighted graph. Then we have*

$$G \text{ is connected} \quad \Leftrightarrow \quad 0 \text{ is a simple eigenvalue of } \Delta_{(G,m)}.$$

Moreover, the multiplicity of the eigenvalue 0 of $\Delta_{(G,m)}$ gives the number of connected components of G.

Proof. "⇒": 0 is an eigenvalue by Proposition 1.16. Let φ be another eigenfunction with eigenvalue 0, then

$$0 = \langle \Delta_{(G,m)}\varphi, \varphi\rangle = \|d\varphi\|^2_{\ell_2(E,m)}$$

hence $d\varphi = 0$, i.e., $\varphi(\partial_+ e) = \varphi(\partial_- e)$ for all $e \in E$. Let $v_0 \in V$. Let $V_0 := \{ v \in V \mid \varphi(v) = \varphi(v_0) \}$. Then V_0 contains all vertices v which can be connected by a path with v_0. Since G is connected, $V_0 = V$, and hence φ is constant, i.e, $\varphi \in \mathbb{C}\mathbb{1}$. We have therefore shown that $\ker \Delta_{(G,m)}$ is one-dimensional, as claimed.

"⇐": We show the contraposition: If G is not connected, let $G = G_1 \cupdot \dots \cupdot G_r$ be the decomposition into connected components with $r \geq 2$. Each component has 0 as (simple) eigenvalue 0 (by the first part), so $\Delta_{(G,m)}$ has 0 as eigenvalue of multiplicity $r \geq 2$, i.e., 0 is not a simple eigenvalue.

The last part also shows the last assertion. □

Remark.

(a) The eigenvectors of the eigenvalue 0 have in a general (not connected) finite graph the form $u = \sum_{s=1}^r \alpha_s \mathbb{1}_{V(G_s)}$, where $G = G_1 \cupdot \dots \cupdot G_r$ is the decomposition into connected components and $\mathbb{1}_{V'}$ is the indicator function of $V' \subset V$.

(b) The above assertion is only true (in general) for the Laplacian. For the (combinatorial) adjacency operator (i.e., the adjacency matrix) of a non-regular graph, one cannot see from the spectrum whether the graph is connected or not, see next example.

The following example shows that the adjacency matrix (the combinatorial adjacency operator) does not encode the connectedness of the graph (it is also an example of so-called *(adjacency) isospectral* or *co-spectral* graphs (w.r.t. the adjacency matrix), i.e., two graphs having the same (combinatorial) adjacency spectrum).

2.11 Example. Let $K_{m,n}$ be the complete bipartite (see Definition 2.13 later) graph (i.e., each of the m vertices is connected with each of the n vertices. In particular, $K_{1,n}$ is a star graph with central vertex of degree n and n adjacent vertices of degree 1.

(a) Let $G_1 = K_{1,4}$ and $G_2 = K_1 \cupdot C_4$. Then we have

$$\sigma(A^{G_1}) = \{2_1, 0_3, (-2)_1\}$$

(multiplicities are written as subscripts) and $\sigma(A^{K_1}) = \{0\}$, and $\sigma(A^{C_4}) = \{2, 0_2, -2\}$ hence

$$\sigma(A^{G_2}) = \sigma(A^{K_1 \cupdot C_4}) = \sigma(A^{K_1}) \cup \sigma(A^{C_4}) = \{2_1, 0_3, (-2)_1\}.$$

(b) Let G_1 be the graph with seven vertices, one central of degree 3 connected with three vertices of degree 2 connected each with a vertex of degree 1. Then

$$\sigma(A^{G_1}) = \{2_1, 1_2, 0_1, (-1)_2, (-2)_1\}.$$

Let $G_2 = K_1 \cupdot C_6$. Then we have $\sigma(A^{K_1}) = \{0\}$ and

$$\begin{aligned}\sigma(A^{C_6}) &= \Big\{2, 2\cos\Big(\frac{\pi}{3}\Big) = 1, 2\cos\Big(\frac{2\pi}{3}\Big) = -1, 2\cos(\pi) = -2, 2\cos\Big(\frac{4\pi}{3}\Big) = -1, 2\cos\Big(\frac{5\pi}{3}\Big) = 1\Big\} \\ &= \{2_1, 1_2, (-1)_2, -2_1\}\end{aligned}$$

and

$$\sigma(A^{K_1 \cupdot C_6}) = \sigma(A^{K_1}) \cup \sigma(A^{C_6}) = \{2_1, 1_2, 0_1, (-1)_2, -2_1\}$$

A simple consequence of Corollary 1.20 and Theorem 2.10 is:

2.12 Corollary. *Assume that (G, m) is an r-regular weighted graph. Then r is is an eigenvalue of the weighted adacency operator $A_{(G,m)}$. Moreover, we have*

$$G \text{ is connected} \quad \Leftrightarrow \quad r \text{ is a simple eigenvalue of } A_{(G,m)}.$$

Finally, the multiplicity of r is the number of connected components of G.

Proof. Note that $0 \in \sigma(\Delta_{(G,m)})$ iff $r \in \sigma(A_{(G,m)})$ by Corollary 1.20. The statement then follows from Theorem 2.10. □

2.3 Bipartiteness and the spectrum

2.13 Definition. A discrete graph $G = (V, E, \partial)$ is *bipartite* or 2-*colourable*, if there is a partition $V = A \cup B$ with $E = E(A, B)$. In other words: each vertex in A is *only* connected with a vertex in B and vice versa.

Examples.

(a) All path graphs L_n are bipartite.

(b) The cycle graphs C_n are bipartite iff n is odd (the number of edges is $n - 1$)

(c) A graph with loops is never bipartite.

Exercise. A graph is bipartite iff it contains no closed paths of *odd* length.

Now a nice relation between the spectrum of the (weighted) adjacency operator and its geometry:

2.14 Theorem. *Let (G, m) be a weighted graph where $G = (V, E, \partial)$ is finite.*

(a) Then the following assertions are equivalent:

(i) The graph G is bipartite.

(ii) The set $\sigma(A_{(G,m)})$ is symmetric with repect to 0, i.e., If $\mu \in \sigma(A_{(G.m)})$ is an eigenvalue, then $-\mu \in \sigma(A_{(G,m)})$ is also an eigenvalue with same multiplicity.

(iii) We have $\lambda_k(A_{(G,m)}) = -\lambda_{n-k+1}(A_{(G,m)})$, for $k = 1, \dots, n$ with $n = |V|$.

In particular, a bipartite graph with an odd number of vertices has always 0 as an eigenvalue.

(b) ***"Inverse problem":*** *Let G be connected and let α be the largest eigenvalue of $A_{(G,m)}$. If $-\alpha$ is also an eigenvalue of $A_{(G,m)}$, then G is bipartite.*

Proof. (a) (i) $\Longrightarrow$ (ii): Let $V = A \cup B$ be a partition of the bipartite graph G with $E = E(A, B)$. Let $T = \mathbb{1}_A - \mathbb{1}_B$ be the multiplication operator given by $(T\varphi)(v) = \varphi(v)$ if $v \in A$ and $(T\varphi)(v) = -\varphi(v)$ if $v \in B$. Then we have

$$TA_{(G,m)} = -A_{(G,m)}T, \tag{2.14a}$$

since

$$(TA_{(G,m)}u)(v) = \begin{cases} \dfrac{1}{m(v)} \sum_{e \in E_v} m(e)u(v_e), & v \in A, \\ -\dfrac{1}{m(v)} \sum_{e \in E_v} m(e)u(v_e), & v \in B. \end{cases}$$

On the other side, we have

$$(A_{(G,m)}Tu)(v) = \frac{1}{m(v)} \sum_{e \in E_v} m(e)(Tu)(v_e) = \begin{cases} -\dfrac{1}{m(v)} \sum_{e \in E_v} m(e)u(v_e), & v \in A, \\ \dfrac{1}{m(v)} \sum_{e \in E_v} m(e)u(v_e), & v \in B, \end{cases}$$

as G contains no loops and $v \in A$ iff $v_e \in B$ and $v \in B$ iff $v_e \in A$. Hence we have shown

Let now μ be an eigenvalue with eigenvector u, then

$$\mu Tu = T\mu u = TA_{(G,m)}u = -A_{(G,m)}Tu,$$

and because $u \neq 0$, we also have $Tu \neq 0$, hence Tu is an eigenvector of $A_{(G,m)}$ with eigenvalue $-\mu$. For the multiplicity, note that T is bijective, hence we have (first and third equality)

$$\dim\ker(A_{(G,m)} - \mu) = \dim\ker(T(A_{(G,m)} - \mu)) = \dim\ker(-A_{(G,m)}T - \mu T) = \dim\ker(-A_{(G,m)} - \mu).$$

The hard part . . . We show first (b). Let A be the matrix assocated with $A_{(G,m)}$ with respect to the ONB $\{\, \delta_v \,|\, v \in V \,\}$ of Remark 1.21. Moreover let μ be the largest eigenvalue of A (then $\mu \geq 0$ as μ and $-\mu$ are both eigenvalues and μ is the largest one).

The graph G is connected; hence we apply the Perron-Frobenius theorem (Theorem 2.21, see later) to $-A$ and obtain, that $-\mu$ is a *simple* eigenvalue of $-A$, and there is a corresponding eigenfunction $f \in \mathbb{C}^V$ such that $f(v) > 0$ for all $v \in V$.

The strategy is now the following: we will analyse the eigenfunction with opposite sign (i.e., μ for A) and show that it never vanishes and its sign gives the desired colouring of G.

Assume now that $u \in \mathbb{C}^V$ is an eigenvector of A with eigenvalue $-\mu \leq 0$ and $u \cdot u = 1$ (standard inner product). As A has only real entries and is symmetric (self-adjoint), we can choose u to have only real entries.

Let $|u| = (|u(v)|)_{v \in V}$, then $|u| \cdot |u| = u \cdot u = 1$ and

$$\mu = |-\mu| = |Au \cdot u| = \Big| \sum_{v,w \in V} A_{vw}u(w)u(v) \Big| \leq \sum_{v,w \in V} A_{vw}|u(w)||u(v)| = A|u| \cdot |u| \leq \mu \qquad (2.14b)$$

(because $A_{vw} \geq 0$ for all v, w (first inequality) and because for a symmetric matrix, we have $Au \cdot u \leq \mu u \cdot u$, if μ is the largest eigenvalue, follows e.g. from Theorem 2.3 for $-A$, second inequality). In particular, we have equality in (2.14b), hence $|u| \in \mathbb{C}^V$ fulfils the equation

$$\mu|u| \cdot |u| = A|u| \cdot |u|$$

for the largest eigenvalue. We conclude from the last equation (and the fact, that μ is the largest eigenvalue) by a variational argument that $A|u| = \mu|u|$ and hence $|u| = cf$ for some $c > 0$. In particular, $u(v) \neq 0$ for all $v \in V$.

We also conclude from (2.14b) that $A_{vw}u(v)u(w)$ has always the same sign (because $\leq$ is also $=$ in the middle!). It must be non-positive as $-\mu \leq 0$.

If $A_{vw} > 0$, i.e., $E(v,w) \neq \emptyset$, then $u(v)u(w) < 0$ for all such $v, w \in V$. Set now $A := \{\, v \in V \,|\, u(v) > 0 \,\}$ and $B := \{\, v \in V \,|\, u(v) < 0 \,\}$, then $A \cap B = \emptyset$ and $A \cup B = V$. Moreover, $E(A, A) = \emptyset$ and $E(B, B) = \emptyset$ (otherwise there are $v, w \in V$ mit $u(v)u(w) > 0$). We have hence shown that G is bipartite.

(ii) $\Longrightarrow$ (i): If the graph is not connected,apply the argument of (b) to each connected component. The symmetry of the spectrum assurs that if μ is an eigenvalue of the weighted adjacency operator on (G_r, m), then so is $-\mu$ (recall that the symmetry respects the multiplicity).

(ii)$\Leftrightarrow$(iii) follows from the numbering of the n eigenvalues. ☐

Exercise. Can one omit the assumption of connectedness in Theorem 2.14 (b)? Find a counterexample of a graph, where with α also $-\alpha$ is always an eigenvalue, but the graph is not bipartite.

↓ Lecture 4, 2017-07-06 ↓

From the spectrum of a *Laplace* operator (of a non-regular weighted graph (G, m), we cannot necessarily see the bipartiteness (second example):

2.15 Examples. We choose the combinatorial weight here.

(a) Let $G = C_n$, then $\sigma(A^{C_n}) = \{\, 2\cos(2k\pi/n) \mid k = 0, \dots, n-1 \,\}$. The largest eigenvalue is 2. Now, $-2 \in \sigma(A^{C_n})$ iff n is even, (then $2\cos(2k\pi/n) = -2$ for $k = n/2$), hence C_n bipartite.

(b) **Isospectral pair, one bipartite, the other not:** We cannot see the bipartiteness from the (combinatorial) Laplace spectrum: Let G_1 and G_2 be two graphs with 6 vertices and 7 edges; G_1 is C_4 and C_3 with one vertex identified, G_2 is C_4 with $K_{1,3}$ as diagonal glued in We have (after some numbering of the vertices)

$$L^{G_1} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix} \quad \text{und} \quad L^{G_2} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

and

$$\sigma(L^{G_1}) = \sigma(L^{G_2}) = \{0, 3 - \sqrt{5}, 2, 3, 3, 3 + \sqrt{5}\}.$$

The graph G_1 is not bipartite, but G_2 is.

Nevertheless, for r-regular weighted graphs we have (recall that $\lambda = r - \alpha$) (and in particular for the standard Laplacian ($r = 1$):

2.16 Corollary. *Let (G, m) be a finite r-regular weighted graph. Then we have*

(a) The following are equialent:

(i) The graph G is bipartite

(ii) $\sigma(\Delta_{(G,m)})$ is symmetric w.r.t. r (i.e., if λ is an eigenvalue then also $2r - \lambda$ is an eigenvalue with the same multiplicity).

(iii) We have $\lambda_k(\Delta_{(G,m)}) = 2r - \lambda_{n-k+1}(\Delta_{(G,m)})$, for $k = 1, \dots, n$ with $n = |V|$.

In particular, if a bipartite graph has an odd number of vertices, r is always a Laplace eigenvalue.

(b) If G is connected and if $2r$ is an eigenvalue of $\Delta_{(G,m)}$, then G is bipartite.

(c) The largest eigenvalue of $\Delta_{(G,m)}$ of a bipartite connected graph is $\lambda_n(\Delta_{(G,m)}) = 2r$ and simple.

Proof. Follows immeadiate from Theorem 2.14 and Corollary 1.20. For the last assertion: $\lambda_1(\Delta_{(G,m)}) = 0$ is smallest eigenvalue, corresponds to $\alpha = r$. □

2.17 Remark. Note that the "hard part" of Theorem 2.14, can be easier seen for r-regular weighted graphs for the Laplacians as follows (here, 0 is always an eigenvalue, so the assertion simplifies a bit): *Assume that* (G, m) *is* r*-regular finite and connected. If* $2r \in \sigma(\Delta_{(G,m)})$*, then* G *is bipartite.* First note that $2r$ is the maximal possible eigenvalue by Corollary 2.6. By Theorem 2.3 (actually a dual version, i.e., applied with $-\Delta_{(G,m)}$) we have

$$2r = \sup_{u \in \ell_2(V,m)} \frac{\langle \Delta_{(G,m)} u, u \rangle}{\|u\|^2}.$$

Actually, the supremum can be replaced by a maximum, i.e., there is a (normalised) function $u \in \ell_2(V, m)$ such that $2r = \langle \Delta_{(G,m)} u, u \rangle = \|du\|^2_{\ell_2(E,m)}$.

We have now reduced the question to the follwing one: *When the norm of* d *is acheived?* Looking at the proof of Proposition 1.12 for a finite graph, we see that we "lost" equality at two instances, the second one when taking the supremum (so choose $v_0 \in V$ such that $\operatorname{rel}_m(v) = \varrho_\infty$) and when applying Cauchy-Young in $|\varphi(\partial_+ e) - \varphi(\partial_- e)|^2 \le 2(|\varphi(\partial_+ e)|^2 - |\varphi(\partial_+ e)|^2$. Equality in the latter holds iff $\varphi(\partial_+ e) = -\varphi(\partial_- e)$ for all $e \in E$. If such a function $\varphi \ne 0$ exists, then G is bipartite (starting from φ with $\varphi(v_0) = 1$, we obtain $\varphi(v) = -1$ for all v with $v_e = v_0$ (being at distance 1) etc.); choose $A = \{\, v \in A \,|\, \varphi(v) > 0 \,\}$ and $B = \{\, v \in V \,|\, \varphi(v) < 0 \,\}$. As G is connected, we have $A \cup B = V(G)$.

Here are some more results what one can see from the spectrum (see [Chu97, Lemma 1.7]):

2.18 Theorem. *Let* $G = (V, E, \partial)$ *be a finite graph with* $n = |V|$ *vertices. Assume that* G *is simple*[5]*. Denote by* $\Delta = \Delta_{(G,\deg)}$ *the standard Laplacian.*

(a) We have $\sum_{k=1}^n \lambda_k(\Delta) \le n$*. Equality holds iff* G *has no isolated vertices.*

(b) If $n \ge 2$*, then* $\lambda_2(\Delta) \le \dfrac{n}{n-1}$*. Equality holds iff* $G = K_n$ *(the complete graph with* n *vertices).*

(c) If G *has no isolated vertices, then* $\lambda_n(\Delta) \ge \dfrac{n}{n-1}$*.*

(d) If G *is not the complete graph* K_n *then* $\lambda_2(\Delta) \le 1$*.*

Idea of proof. (a) follows from taking the trace.
(b)–(c) follow from (a) and $\lambda_1(\Delta) = 0$.
(d) By Theorem 2.3, the second eigenvalue is

$$\lambda_2(\Delta) = \inf_{u \perp \mathbb{1}} \frac{\langle \Delta_{(G,m)} u, u \rangle}{\|u\|^2} = \inf_{u \perp \mathbb{1}} \frac{\|du\|^2_{\ell_2(E,m)}}{\|u\|^2} \tag{2.18}$$

(the first eigenvalue is 0 with eigenfunction $\mathbb{1}$), hence

$$\lambda_2(\Delta) \le \frac{\|du\|^2_{\ell_2(E,m)}}{\|u\|^2}$$

i.e., we just have to find a nice "test function" orthgonal to the constant one and calculate the so-called Rayleigh quotient.

[5] Check, which results extend to the general case ...)

As G is not complete, there are two vertices $a, b \in V$ without an edge. Let

$$u(v) = \begin{cases} \deg b, & v = a, \\ -\deg a, & v = b, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle u, \mathbb{1}\rangle = (\deg b)(\deg a) - (\deg b)(\deg a) = 0$, $\|du\|^2 = (\deg a)^2 + (\deg b)^2$ (as there is no edge between a and b) and $\|u\|^2 = (\deg b)^2 \deg a + (\deg a)^2 \deg b$, hence

$$\frac{\|du\|^2_{\ell_2(E,m)}}{\|u\|^2} = \frac{(\deg a)^2 + (\deg b)^2}{(\deg b)^2 \deg a + (\deg a)^2 \deg b} =\leq \frac{(\deg a)^2 + (\deg b)^2}{(\deg b)^2 + (\deg a)^2} \leq 1 \qquad \square$$

Actually, we have shown in (d) that

$$\lambda_1(\Delta) \leq \max_{a,b\in V} \frac{(\deg a)^2 + (\deg b)^2}{(\deg b)^2 \deg a + (\deg a)^2 \deg b}.$$

2.4 Perron-Frobenius and the first eigenvector

Consider now the eigenvector associated to the smallest eigenvalue 0 of the matrix associated with a Laplacian on a weighted graph:
(Terminology of Colin de Verdière ...)

2.19 Definition. Let V be a finite set.

(a) We call

$$SO(V) := \{\, L = (L_{vw})_{v,w\in V} \in \mathbb{R}^{V\times V} \mid L_{vw} = L_{wv}, L_{vw} \leq 0 \;\forall v, w \in V, v \neq w \,\}$$

the *set of discrete Schrödinger operators* on V.

(b) For $L \in SO(V)$, let E be the set of (unordered) pairs $\{v, w\}$ ($v \neq w$) such that $L_{vw} < 0$. Then (V, E) is a (undirected) schlicht graph G_L, called the *graph associated* with L.

(c) We say that L is *irreducible* if G_L is connected.

Remark.

Note that $SO(V)$ is not a vector space, since with $L \in SO(V) \setminus \{0\}$ we don't have $-L \notin SO(V)$ (in general).

2.20 Examples.

(a) Let (G, m) be a finite weighted graph and $V = V(G)$ with $L_{(G,m)}$ being the matrix of the Laplace operator $\Delta_{(G.m)}$ (see Remark 1.21). Then $L_{(G,m)} \in SO(V)$, since the matrix is symmetric, has only real entries and all non-diagonal entries are non-positive. Similarly, the matrix A assocaited with the weighted adjacency operator fulfils $-A_{(G,m)} \in SO(V)$.

(b) For

$$L_1 = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -3 \\ 0 & -3 & 1 \end{pmatrix} \quad \text{und} \quad L_1 = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -3 \\ 1 & -3 & 1 \end{pmatrix}$$

we have $L_1 \in SO^3$, but $L_2 \notin SO^3$.

Remark. We can reconstruct the combinatorial graph from $L \in SO(V)$ (if the graph was schlicht), but can we also reconstruct the weights? More precisely, given $L = L_{(G,m)} \in SO(V)$ can we calculate the weight functions $m\colon V \longrightarrow (0,\infty)$ and $m\colon E \longrightarrow (0,\infty)$ just from the matrix L?

The following result is (probably) known from Linear Algebra:

2.21 Theorem (Perron-Frobenius). *Let $L \in SO(V)$ be irreducible and $\lambda_1(L)$ be the smallest eigenvalue of L. Then $\lambda_1(L)$ has multiplicity 1 and there is a corresponding eigenvector with all entries strictly positive.*

2.22 Example. Let us just check Theorem 2.21 in the case of a matrix L associated with a Laplace operator $L_{(G,m)}$ of a weighted schlicht graph:

Clearly, $L \in SO(V(G))$ (as $L_{vw} < 0$ iff $v \sim w$ in G). Moreover,

$$L_{vw} = -\frac{m(vw)}{\sqrt{m(v)m(w)}} \tag{2.22a}$$

where we write $vw \in E$ for an edge and

$$L_{vv} = \frac{\sum_{w\sim v} m(vw)}{m(v)}. \tag{2.22b}$$

If G is connected, then 0 is a simple eigenvalue (see Theorem 2.10) and an eigenfunction for 0 is given by

$$\varphi_1(v) = \sqrt{m(v)}.$$

This can be just checked:

$$\begin{aligned}(L\varphi)(v) = \sum_{w\in V} L_{vw}\varphi(w) &= L_{vv}\varphi(v) - \sum_{w\sim v}\frac{m(vw)}{\sqrt{m(v)m(w)}}\sqrt{m(w)}\\ &= \frac{\sum_{w\sim v} m(vw)}{\sqrt{m(v)}} - \sum_{w\sim v}\frac{m(vw)}{\sqrt{m(v)}} = 0.\end{aligned}$$

In particular, all entries of φ are non-negative. Hence we can recover the vertex weight from the entries of the first eigenvector.

2.23 Corollary. *Let $L \in SO(V)$ be irreducible with smallest eigenvalue 0, then there is a (schlicht) weighted graph (G,m) such that L is the matrix associated with $\Delta_{(G,m)}$ (in the ONB $\{\,\delta_v \mid v \in V\,\}$. The weights are unique (up to a common factor for the vertex and edge weight, i.e., we cannot distinguish between $m\colon V \longrightarrow (0,\infty$, $m\colon E \longrightarrow (0,\infty$ and $\tau m\colon V \longrightarrow (0,\infty$, $\tau m\colon E \longrightarrow (0,\infty$.*

Proof. Let G_L be the graph associated with L. Apply Theorem 2.21 for the smallest eigenvalue 0 and obtain the corresponding eigenvector φ with strictly positive entries. Set

$$m(v) := \varphi_1(v)^2 \quad \text{und} \quad m(vw) := -L_{vw}\varphi_1(v)\varphi_1(w). \tag{2.23}$$

Then we have $m(vw) > 0$ (for $v \neq w$) iff $L_{vw} < 0$. Moreover

$$(L_{(G,m)})_{vw} = -\frac{m(vw)}{\sqrt{m(v)m(w)}} = \frac{L_{vw}\varphi_1(v)\varphi_1(w)}{\varphi_1(v)\varphi_1(w)} = L_{vw}$$

if $v \sim w$ and

$$\begin{aligned}(L_{(G,m)})_{vv} = \frac{\sum_{w\sim v} m(vw)}{m(v)} &= \frac{\sum_{w\sim v}(-L_{vw}\varphi_1(v)\varphi_1(w))}{\varphi_1(v)^2} \\ &= -\frac{1}{\varphi(v)}\sum_{w\sim v} L_{vw}\varphi_1(w) \\ &= \frac{1}{\varphi(v)} L_{vv}\varphi(v) = L_{vv}\end{aligned}$$

using (2.22a) and (2.22b) (in the second last equation, we used the fact that the lowest eigenvalue is actually 0) for all other v, w. In particular, we have

$$L = L_{(G,m)}.$$

The uniqueness up to a common factor follows from (2.23). □

3 Isoperimetric constants and Cheeger's inequality

Aim: find a *geometric* lower bound on the second (first non-zero) eigenvalue of the Laplacian on a graph. Assume that (G, m) is a finite weighted graph. In this section we write

$$\lambda_k(G, m) := \lambda_k(\Delta_{(G,m)})$$

for the k-th eigenvalue of the weighted Laplacian on (G, m) (sorted in increasing order, respecting multiplicity).

A small motivation from differential geometry . . .

3.1 Remark. Let M be a d-dimensional Riemannian manifold (e.g., if $d = 2$ a curved surface in $\mathbb{R}^3$) and $\Delta_M \geq 0$ its Laplacian (describing e.g. waves or diffusion). Denote the eigenvalues by $\lambda_k(M)$. Then $\lambda_1(M) = 0$ (constant eigenfunction) and the second eigenvalue is positive iff M is connected. Cheeger proved (in his PhD thesis 1970) that

$$\lambda_2(M) \geq \frac{h(M)^2}{4}. \tag{3.1a}$$

Here, the so-called *isoperimetric* or *Cheeger's constant*) is defined by

$$h(M) := \inf\{\, h(M,S) \,|\, S \subset M \text{ offen}, \partial S \text{ smooth} \,\}, \tag{3.1b}$$

where

$$h(M,S) := \frac{\operatorname{vol}_{d-1} \partial S}{\min\{\operatorname{vol}_d S, \operatorname{vol}_d(M \setminus S)\}}.$$

For example, $h(M) = 0$ iff M is disconnected. The isoperimetric constant is now a quantitative generalisation of disconnectedness ($h(M)$ "small" means "badly connected"). For example, the isoperimetric constant is small if M decomposes in two pieces of similar volume that has a small girth. The second eigenvalue also gives a measure how fast diffusion between the two parts is (small means slow diffusion, large second eigenvalue means fast diffusion).

The isoperimetric constant and Cheeger's inequality is interesting because it relates a *geometric* property (isoperimetric constant) with an *analytic* (or algebraic) one, the second eigenvalue.

A related optimisation problem in dimension 2 is the following *isoperimetric problem*: Find a subset S with given area (say $\operatorname{vol}_2 S = 1$) such that the circumference $\operatorname{vol}_1 \partial S$ is small. The solution here is the circle (but a formal proof is not that obvious . . .)

We will now do a similar concept on a graph: divide V into two pieces S and S^c of similar size. As boundary we will choose the edges between S and S^c (i.e., $\partial_E S := E(S, S^c)$).

For a weighted graph (G, m) with $G = (V, E, \partial)$ and subsets $V' \subset V$, $E' \subset E$ we write

$$m(V') := \sum_{v \in V'} m(v) \qquad \text{und} \qquad m(E') := \sum_{e \in E'} m(e);$$

i.e., we consider m also as a point measure.

3.2 Definition. Let (G, m) be a weighted graph. The *isoperimetric* resp. *Cheeger constant* of (G, m) is defined by

$$h(G, m) := \inf\{\, h(G, m, S) \mid S \subset V, S \neq \emptyset, S \neq V \,\}, \tag{3.2}$$

where

$$h(G, m, S) := \frac{m(\partial_E S)}{\min\{m(S), m(S^c)\}}, \qquad \partial_E S := E(S, S^c).$$

3.3 Remark.

(a) Here, $\partial_E S := E(S, S^c)$ is the so-called *edge boundary* of S, i.e., a subset of the edges. One can also use

$$\partial_V S := \{\, v \in S^c \mid \text{there is an edge with one endpoint } v \text{ and the other in } S \,\},$$

the so-called *vertex boundary* of S.

(b) $S \subset 2^V$ (power sets: set of subsets of V), $|2^V| = 2^{|V|}$ is finite if $|V|$ is finite, but very large! Hence the infimum (is is actually a minimum) in the isoperimetric constant could be calculated by brute force, but the number $2^{|V|}$ of subsets is growing very fast! …if one assumes that there are about 10^{85} atoms in space, then this number is exceeded already for $|V| = 283$ (i.e., $2^{283} > 10^{85}$). Impossible to just try — where should one store the results, even if one only needs one atom per bit? Calculating eigenvalues (e.g. of a (1000×1000)-matrix) is much quicker.

↓ Lecture 5, 2017-07-07 ↓

3.4 Remark. (a) Note that we have

$$h(G, m, S) = \frac{\|d\mathbb{1}_S\|^2_{\ell_2(E,m)}}{\|\mathbb{1}_S\|^2_{\ell_2(V,m)}},$$

and hence

$$h(G, m) = \inf\Big\{\, \|d\mathbb{1}_S\|^2_{\ell_2(E,m)} \|\mathbb{1}_S\|^2_{\ell_2(V,m)} \,\Big|\, S \subset V, 0 < m(S) < m(V)/2 \Big\}$$

(b) **Recall: the min-max principle for the second eigenvalue of a Laplacian:** For the second eigenvalue of the weighted Laplacia $\Delta_{(G,m)}$ we have

$$\lambda_2(\Delta_{G,m}) = \inf\Big\{\, \frac{\langle f, \Delta_{G,m} f\rangle_{\ell_2(V,m)}}{\|f\|^2_{\ell_2(V,m)}} \,\Big|\, f \perp \mathbb{1} \Big\} = \inf\Big\{\, \frac{\|df\|^2_{\ell_2(E,m)}}{\|f\|^2_{\ell_2(V,m)}} \,\Big|\, f \perp \mathbb{1} \Big\} \tag{3.4}$$

(recall: $\mathbb{1}$ is the constant function with value 1 on V).

(c) **Partitioning of graphs:** There are higher analogues: Let π be a *k-subpartition of V*, i.e., $\pi = \{S_1, \dots, S_k\}$ such that $S_i \neq \emptyset$, S_j finite and $S_i \cap S_j = \emptyset$ if $i \neq j$ (not necessarily $S_1 \cup \dots \cup S_k = V$). Set

$$h_k(G, m, \pi) := \frac{m(E(S_i, S_j))}{\min\{m(S_i), m(S_j)\}}$$

and

$$h_k(G, m) := \inf\Big\{ h_k(G, m, \pi) \,\Big|\, \pi \text{ } k\text{-subpartion of } V \Big\}$$

Then $h_2(G, m) = h(G, m)$. The case $k = 1$ does not make much sense in our situation here (finite graph), as $h_1(G, m) = 0$ (just choose $S = V$, then $E(V, \emptyset) = \emptyset$). Nevertheless this case becomes more interesting if one treats so-called *Dirichlet* problems (i.e., functions vanish on a subset ∂V of V, called *boundary*), hence the constant eigenfunction is excluded. Also, $h_1(G, m)$ is of interest for infinite graphs and magnetic Laplacians ...

Cheeger's inequality now states a *lower* bound on $\lambda_2(G, m)$. We first start with an easier task: an upper bound (using again the "test function" argument as in the proof of Theorem 2.18 (d)).

3.5 Theorem. *Let (G, m) be a finite weighted graph. Then we have $2h(G, m) \geq \lambda_2(G, m)$.*

Proof. **A test function argument ...** Let $h(G, m) = h(G, m, S)$ (i.e. $S \subset V$ realises the minimum; S is also called *optimal cut*). Set

$$f(v) := \begin{cases} a := \Big(\dfrac{m(S^c)}{m(S)m(V)}\Big)^{1/2}, & v \in S, \\ -b := -\Big(\dfrac{m(S)}{m(S^c)m(V)}\Big)^{1/2}, & v \in S^c. \end{cases}$$

Then we have

$$\langle f, \mathbb{1} \rangle_{\ell_2(V,m)} = \sum_{v \in S} a m(v) - \sum_{v \in S^c} b m(v) = a m(S) - b m(S^c) = 0$$

and

$$\|f\|^2_{\ell_2(V,m)} = \sum_{v \in S} a^2 m(S) + b^2 m(S^c) = \frac{m(S^c) + m(S)}{m(V)} = 1$$

because of the choice of a and b (one could start with general values for a and b and then solve the resulting system). Moreover, we have

$$\begin{aligned} \|df\|^2_{\ell_2(E,m)} &= \sum_{e \in E} \Big|(df)(e)\Big|^2 m(e) \\ &= \sum_{e \in E} \Big| f(\partial_+ e) - f(\partial_- e)\Big|^2 m(e) \\ &= \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v} |f(v) - f(v_e)|^2 m(e) \\ &= \frac{1}{2} \Big(\sum_{v \in S} \sum_{e \in E_v, v_e \in S^c} |f(v) - f(v_e)|^2 m(e) + \sum_{v \in S^c} \sum_{e \in E_v, v_e \in S} |f(v) - f(v_e)|^2 m(e) \Big) \\ &= \frac{1}{2} \Big(m\Big(E(S, S^c)\Big) + m\Big(E(S^c, S)\Big) \Big) (a + b)^2 \\ &= m(E(S, S^c))(a + b)^2 \end{aligned}$$

(Lemma 1.3 for the third equation) and $|f(v) - f(w)|^2 = (a+b)^2$ for the fifth equation) — the cases $v, v_e \in S$ and $v, v_e \in S^c$ disappear, as here the difference is 0. In addition, we have

$$\begin{aligned}(a+b)^2 &= \frac{1}{m(V)}\Big(\Big(\frac{m(S^c)}{m(S)}\Big)^{1/2} + \Big(\frac{m(S)}{m(S^c)}\Big)^{1/2}\Big)^2 \\ &= \frac{1}{m(V)m(S)m(S^c)}\Big(m(S^c) + m(S)\Big)^2 \\ &= \frac{m(V)}{m(S)m(S^c)} = \frac{1}{m(S)} + \frac{1}{m(S^c)} \\ &\le \frac{2}{\min\{m(S), m(S^c)\}}.\end{aligned}$$

Together we conclude from Theorem 2.3 for $k = 2$ (see also (3.4)) that

$$\lambda_2(G,m) \le \frac{\|df\|^2_{\ell_2(E,m)}}{\|f\|^2_{\ell_2(V,m)}} = m(E(S,S^c))(a+b)^2 \le \frac{2m(E(S,S^c))}{\min\{m(S), m(S^c)\}} = h(G,m,S) = h(G,m),$$

as f is a "test function", i.e., a function admissible when calculating the infimum in (3.4) ist. □

We now need one preparation for Cheeger's inequality $\lambda_2(G,m) \ge h(G,m)^2/(2\varrho_\infty)$:

3.6 Proposition (Coarea formula). *Let $\Phi\colon V \longrightarrow \mathbb{R}$ be a function, then*

$$\sum_{e\in E} |d\Phi(e)| m(e) = \int_0^\infty m(\partial_E\{\, v \in V \mid \Phi(v) > t\,\})\,\mathrm{d}t.$$

Remark. On a manifold of dimension d or an open subset M of $\mathbb{R}^d$ we have for a (smooth or Lipschitz) function $\Phi\colon M \longrightarrow [0,\infty)$ the following continuous analogue (the usual *coarea formula*):

$$\int_M |\nabla\Phi|\,\mathrm{d}M = \int_0^\infty \mathrm{vol}_{d-1}\Big(\partial\{\, x \in M \mid \Phi(x) > t\,\}\Big)\,\mathrm{d}t.$$

In some sense, this is a version of the *Cavallieri* principle. Hence, we can think of $m(\partial_E S) = m(E(S,S^c)$ as a sort of *surface measure*, similarly, as $\mathrm{vol}_{d-1}(\partial S)$ is the surface measure of $S \subset M$ (with smooth or Lipschitz boundary).

Proof. For an edge $e \in E$ let $I_e := [\Phi(v), \Phi(v_e))$ be a half-open interval, where v and v_e are incident with e such that $\Phi(v) \le \Phi(v_e)$. Then we have

$$\|d\Phi\|_{\ell_1(E,m)} := \sum_{e\in E} |\mathrm{d}\Phi(e)| m(e) = \sum_{e\in E} |\Phi(\partial_+ e) - \Phi(\partial_- e)| m(e) = \sum_{e\in E} |I_e| m(e), \tag{3.6a}$$

where $|I_e| = |\Phi(\partial_+ e) - \Phi(\partial_- e)|$ denotes the Lebesgue measure. Moreover, we have

$$\begin{aligned} e \in \partial_E\{\, v \in V \mid \Phi(v) > t\,\} \quad &\Leftrightarrow \quad \exists v, w \in V\colon e \text{ ist Kante zwischen } v \text{ and } w,\ \Phi(v) \le t < \Phi(w) \\ &\Leftrightarrow \quad t \in I_e. \end{aligned}$$

Therefore, we have

$$m\Big(\partial_E\{\, v \in V \mid \Phi(v) > t\,\}\Big) = m\Big(\{\, e \in E \mid t \in I_e\,\}\Big) = \sum_{e\in E} \mathbb{1}_{I_e}(t) m(e),$$

where

$$\mathbb{1}_{I_e}(t) := \begin{cases} 1, & t \in I_e, \\ 0, & t \notin I_e. \end{cases}$$

Finally, we have

$$\int_{-\infty}^{\infty} m\Big(\partial_E\{\, x \in M \mid \Phi(x) > t \,\}\Big)\,\mathrm{d}t = \int_{-\infty}^{\infty} \sum_{e\in E} \mathbb{1}_{I_e}(t) m(e)\,\mathrm{d}t$$
$$= \sum_{e\in E} \int_{-\infty}^{\infty} \mathbb{1}_{I_e}(t) m(e)\,\mathrm{d}t = \sum_{e\in E} |I_e| m(e) \overset{(3.6a)}{=} \sum_{e\in E} |d\Phi(e)| m(e),$$

(the integral $\int_{-\infty}^{\infty}$ is actually finite). □

We can now prove the main result of this section:

3.7 Theorem (Cheeger's inequality). *Let (G, m) be a finite weighted graph, then*

$$\lambda_2(G, m) \geq \frac{h(G, m)^2}{2\varrho_\infty},$$

where $\lambda_2(G, m)$ is the second eigenvalue of the weighted Laplacian on (G, m), and where ϱ_∞ is the maximum of the relative weight.

Remark. Reclal that for the standard weight we have $\varrho_\infty = 1$, and for the combinatorial weight we have $\varrho_\infty = \max_{v\in V} \deg v$, the maximal degree.

The idea in the proof is to use the sign decomposition of the second eigenfunction f as a good candidate for an optimal cut (i.e., $S = \{\, v \in V \mid f(v) > 0 \,\}$):

Proof. Let f be an eigenfunction of $\Delta = \Delta_{(G,m)}$ with eigenvalue $\lambda_2 = \lambda_2(G, m)$. As there is a matrix associated with Δ having only real entries and as $\Delta = \Delta^*$, we can assume that f is real-valued. Let

$$V_+ := \{\, v \in V \mid f(v) > 0 \,\} \quad \text{und} \quad V_- := \{\, v \in V \mid f(v) \leq 0 \,\}.$$

Without loss of generality, assume that

$$m(V_+) \leq m(V_-) \tag{3.7a}$$

(otherwise consider $-f$). Let

$$g(v) := \begin{cases} f(v), & v \in V_+, \\ 0, & v \in V_-, \end{cases}$$

i.e., $g := \max\{f, 0\}$. In particular, we have

$$f(v) \leq g(v) \qquad \text{for all } v \in V. \tag{3.7a'}$$

Moreover, we have (using the shorthand notation $\|\cdot\|_{V_+} = \|\cdot\|_{\ell_2(V_+,m)}$ etc.)

$$\langle f, \Delta_{(G,m)} f\rangle_{V_+} = \sum_{v\in V_+} f(v) \sum_{e\in E_v} m(e)(f(v) - f(v_e)) \geq \sum_{v\in V_+} g(v) \sum_{e\in E_v} m(e)(g(v) - g(v_e))$$
$$= \sum_{v\in V} g(v) \sum_{e\in E_v} m(e)(g(v) - g(v_e))$$
$$= \langle g, \Delta_{(G,m)} g\rangle_V, \tag{3.7a''}$$

where we used that $f(v) = g(v)$ for $v \in V_+$ and $-f(v_e) \geq -g(v_e)$ for $v_e \in V$, according to (3.7a') (first inequality). For the second equality, note that $g(v) = 0$ for $v \in V_-$ holds.

In particular, we have

$$\lambda_2 = \lambda_2 \frac{\|f\|^2_{V_+}}{\|f\|^2_{V_+}} \overset{\Delta f = \lambda_2 f}{=} \frac{\langle f, \Delta_{(G,m)} f\rangle_{V_+}}{\|f\|^2_{V_+}} \overset{(3.7\text{a"})}{\geq} \frac{\langle g, \Delta_{(G,m)} g\rangle_V}{\|g\|^2_V} \tag{3.7a'''}$$

(using again $\|f\|^2_{V_+} = \|g\|^2_V$ for the last equality).
We now show the following estimate (here, $\|\cdot\|_E = \|\cdot\|_{\ell_2(E,m)}$):

$$\Big(\sum_{e\in E} m(e)\Big|(d(g^2))(e)\Big|\Big)^2 \leq 2\varrho_\infty \|g\|^2_V \|dg\|^2_E. \tag{3.7b}$$

Proof of (3.7b)**.**

$$\begin{aligned}
\Big(\sum_{e\in E} m(e)\Big|(d(g^2))(e)\Big|\Big)^2 &= \Big(\sum_{e\in E} m(e)\Big|g(\partial_+e)^2 - g(\partial_-e)^2\Big|\Big)^2 \\
&= \Big(\sum_{e\in E} m(e)\Big|g(\partial_+e) - g(\partial_-e)\Big|\Big|g(\partial_+e) + g(\partial_-e)\Big|\Big)^2 \\
&\overset{\text{CS}}{\leq} \sum_{e\in E} m(e)\Big|g(\partial_+e) - g(\partial_-e)\Big|^2 \cdot \sum_{e\in E} m(e)\Big|g(\partial_+e) + g(\partial_-e)\Big|^2 \\
&\overset{\text{CY}}{\leq} \sum_{e\in E} m(e)\Big|(dg)(e)\Big|^2 \cdot 2\sum_{e\in E} m(e)\Big(\Big|g(\partial_+e)\Big|^2 + \Big|g(\partial_-e)\Big|^2\Big),
\end{aligned} \tag{3.7b'}$$

where "$\overset{\text{CS}}{\leq}$" means that we applied the Cauchy-Schwarz inequality

$$\Big|\sum_{e\in E} m(e) a_e b_e\Big|^2 \overset{\text{CS}}{\leq} \sum_{einE} m(e)|a_e|^2 \cdot \sum_{einE} m(e)|b_e|^2,$$

and the Cauchy-Young inequality $(a+b)^2 \leq 2a^2 + 2b^2$ at "$\overset{\text{CY}}{\leq}$" in the second sum. Moreover, we have

$$\sum_{e\in E} m(e)\Big|g(\partial_+e)\Big|^2 = \sum_{v\in V}\sum_{e\in E_v^+} m(e)\Big|g(\underbrace{\partial_+e}_{=v})\Big|^2 = \sum_{v\in V} m(E_v^+)\Big|g(v)\Big|^2$$

according to Lemma 1.3. Analogously, we have

$$\sum_{e\in E} m(e)\Big|g(\partial_-e)\Big|^2 = \sum_{v\in V}\sum_{e\in E_v^-} m(e)\Big|g(\underbrace{\partial_-e}_{=v})\Big|^2 = \sum_{v\in V} m(E_v^-)\Big|g(v)\Big|^2,$$

so that

$$\begin{aligned}
2\sum_{e\in E} m(e)\Big(\Big|g(\partial_+e)\Big|^2 + \Big|g(\partial_-e)\Big|^2\Big) &= 2\sum_{v\in V}\Big(m(E_v^+) + m(E_v^-)\Big)\Big|g(v)\Big|^2 \\
&= 2\sum_{v\in V} \deg_m v\Big|g(v)\Big|^2 \leq 2\underbrace{\sup_{v\in V}\frac{\deg_m v}{m(v)}}_{=\varrho_\infty}\|g\|^2_V
\end{aligned}$$

follows. In particular, we can continue estimate (3.7b') by

$$\Big(\sum_{e\in E} m(e)\Big|(d(g^2))(e)\Big|\Big)^2 \leq \sum_{e\in E} m(e)\Big|(dg)(e)\Big|^2 \cdot 2\varrho_\infty\|g\|^2_V = 2\varrho_\infty\|dg\|^2_E\|g\|^2_V,$$

i.e., we have shown (3.7b). □

Combining (3.7a''') and (3.7b), we obtain

$$\lambda_2 \overset{(3.7a''')}{\geq} \frac{\langle g, \Delta_G g\rangle_V}{\|g\|_V^2} = \frac{\|dg\|_E^2}{\|g\|_V^2} \overset{(3.7b)}{\geq} \frac{\Big(\sum_{e\in E} m(e)\Big|(d(g^2))(e)\Big|\Big)^2}{2\varrho_\infty \|g\|_V^4} \tag{3.7c}$$

Finally, we show that

$$\sum_{e\in E} m(e)\Big|(d(g^2))(e)\Big| \geq h(G,m)\|g\|_V^2. \tag{3.7d}$$

(Then the assertion $\lambda_2 \geq (h(G,m)\|g\|_V^2)^2/(2\varrho_\infty\|g\|_V^4) = h(G)^2/2\varrho_\infty$ follows)

Proof of (3.7d)**.** Let $S_t := \{\, v \in V \mid g(v)^2 > t \,\}$. Since

$$m(S_t) \leq m(V_+) \overset{(3.7a)}{\leq} m(V_-) \leq m(S_t^{c}),$$

we have $\min\{m(S_t), m(S_t^c)\} = m(S_t)$ and since $h(G,m)$ is the mimimum of all such sets S, we have $h(G,m) \leq m(\partial_E S_t)/m(S_t)$. Moreover,

$$\begin{aligned}
\sum_{e\in E} m(e)\Big|(d(g^2))(e)\Big| &= \int_0^\infty m(\partial_E S_t)\,\mathrm{d}t && \text{(Proposition 3.6)}\\
&\geq h(G,m)\int_0^\infty m(S_t)\,\mathrm{d}t && \text{(Def. von } h(G,m))\\
&= h(G,m)\int_0^\infty \sum_{v\in V} \mathbb{1}_{S_t}(v)m(v)\,\mathrm{d}t\\
&= h(G,m)\sum_{v\in V}\int_0^\infty \mathbb{1}_{S_t}(v)m(v)\,\mathrm{d}t\\
&= h(G,m)\sum_{v\in V}\int_0^\infty \mathbb{1}_{[0,g(v)^2)}(t)\,\mathrm{d}t\; m(v)\\
&= h(G,m)\sum_{v\in V} g(v)^2 m(v)\\
&= h(G,m)\|g\|_V^2
\end{aligned}$$

hence we have shown (3.7d). □

Together, we have proven Theorem 3.7. □

3.8 Definition. A family $(G_n, m_n)_n$ of finite weighted graphs is called *expander family*, if

$$\inf_n h(G_n, m_n) > 0 \quad \text{und} \quad \sup_n \varrho_\infty(G_n, m_n) < \infty$$

holds (and if $(|V(G_n)|)_{n\in\mathbb{N}}$ is strictly increasing).

Remark. The last condition only shall exclude trivial cases such as a constant family).
For combinatorial weights, (G_n) is an expander family, iff $\inf_n h(G_n) > 0$ and if the degree is uniformly bounded independently of n (i.e., $sup_n \max_{v\in V(G_n)} \deg v < \infty$). Most times, one requires that G_n is a regular graph (then $\varrho_\infty(G_n) = r$)
The construction of classes of expander families is quite difficult and requires some knowledge of algebra.

By Cheeger's inequality Theorem 3.7 (and the simpler inequality Theorem 3.5), we obtain:

3.9 Corollary. *A family $(G_n, m_n)_n$ is an expander family iff*

$$\inf_n \lambda_2(G_n, m_n) > 0 \quad \text{and} \quad \sup_n \varrho_\infty(G_n, m_n) < \infty$$

holds (and if $(|V(G_n)|)_{n\in\mathbb{N}}$ is strictly increasing).

Proof. By Theorems 3.7 and 3.5 we have that $\inf_n h(G_n, m_n) > 0$ iff $\inf_n \lambda_2(G_n, m_n)$. □

Cheeger's inequality enters in the "hard" conclusion that a lower bound on $\lambda_2(G_n, m_n)$ gives a lower bound on $h(G_n, m_n)$.

3.10 Examples.

(a) Let $G = L_n$ be the path graph n vertices, then $\lambda_2(L_n, 1) = 2 - 2\cos(\pi/n)$ (combinatorial weight), see Example 2.7 (a). Here, $\varrho_\infty(L_n, 1) = 2$.

We can calculate Cheeger's constant here also directly: let $S = \{0, 1, \dots, k\}$ with $k \le n/2$, then $h(L_n, 1, S) = 1/k$, the latter is minimal if k is maximal, i.e., $k = n/2$ for n even or $k = (n-1)/2$ for n odd. Hence (without formal proof ...)

$$h(L_n, 1) = \frac{1}{\lceil (n-1)/2 \rceil} = \begin{cases} 2/n, & n \text{ even,} \\ 2/(n-1), & n \text{ odd} \end{cases}$$

In particular, $\inf_n h(L_n, 1) = 0$, i.e.,$(L_n, 1)_n$ is not an expander family.

How good is Cheeger's inequality here?

$$2h(L_n, 1) \ge 2 - 2\cos\Big(\frac{\pi}{n+1}\Big) = \lambda_2(L_n, 1) \ge \frac{h(L_n, 1)^2}{2\varrho_\infty} = \frac{1}{4\lceil n/2 \rceil^2}.$$

Concretely, we have

$$\begin{aligned} 2h(L_1, 1) = 2 \ge 2 = \lambda_2(L_1, 1) &\ge \frac{h(L_1, 1)^2}{4} = \frac{1}{4}, \\ 2h(L_2, 1) = 2 \ge 1 = \lambda_2(L_2, 1) &\ge \frac{h(L_2, 1)^2}{4} = \frac{1}{4}, \\ 2h(L_3, 1) = 1 \ge 2 - \sqrt{2} = \lambda_2(L_3, 1) &\ge \frac{h(L_3, 1)^2}{4} = \frac{(1/2)^2}{4} = \frac{1}{16}, \\ 2h(L_n, 1) = \frac{2}{\lceil (n-1)/2 \rceil} \ge \underbrace{2 - 2\cos\Big(\frac{\pi}{n}\Big)}_{\ge \frac{\pi^2}{n^2}} = \lambda_2(L_n, 1) &\ge \frac{h(L_n, 1)^2}{4} = \frac{1}{4\lceil (n-1)/2 \rceil^2}. \end{aligned}$$

since $\cos x \ge 1 - x^2/2$, also $x^2 \ge 2 - 2\cos x$. Hence the estimate is not optimal but of the right order. For n large, we have on the left hand side a factor $\pi^2 \approx 9.87$ and on the right hand side 1.

(b) For the standard weigth we have

$$\sigma(\Delta_{(L_n, \deg)}) = \Big\{0 = 1 - \cos\Big(\frac{0\pi}{n}\Big), 1 - \cos\Big(\frac{\pi}{n}\Big), \dots, 1 - \cos\Big(\frac{n\pi}{n}\Big) = 2\Big\}$$

and $\varrho_\infty(L_n, \deg) = 1$.

Here, Cheeger's constant is acheived with optimal cut S (e.g. the first $\lceil (n-1)/2 \rceil$ vertices) given by

$$h(L_n, \deg) = h(L_n, \deg, S) = \frac{1}{1 + 2(\lceil n/2 \rceil - 1)} = \frac{1}{2\lceil n/2 \rceil - 1} = \begin{cases} \dfrac{1}{n}, & n \text{ ungerade}, \\ \dfrac{1}{n-1}, & n \text{ gerade}. \end{cases}$$

Hence again $\inf_n h(L_n, \deg) = 0$, i.e. $(L_n, \deg)_n$ and is no expander family.

(c) Let $G = C_n$ be the cyclic graph with n vertices. These graphs are 2-regular, hence $h(C_n, 1) = 2h(C_n, \deg)$ and $\lambda_2(C_n, 1) = 2\lambda_2(C_n, \deg)$. We have $\lambda_2(C_n, 1) = 2 - 2\cos(2\pi/n)$ (combinatorial weight), see Example 2.7 (b). Dann ist $\varrho_\infty(C_n, 1) = 2$.

One can see that

$$h(C_n, 1) = \begin{cases} \dfrac{4}{n}, & n \text{ even}, \\ \dfrac{4}{n-1}, & n \text{ odd} \end{cases}$$

$(n \geq 2)$. Hence $(C_n, 1)_n$ is again no expander family.

Cheegers inequality Theorem 3.7 and Theorem 3.5 say that

$$2h(C_n, 1) \geq 2 - 2\cos\Big(\frac{2\pi}{n}\Big) = \lambda_2(C_n, 1) \geq \frac{h(C_n, 1)^2}{2\varrho_\infty} = \begin{cases} \dfrac{4}{n^2}, & n \text{ gerade}, \\ \dfrac{4}{(n-1)^2}, & n \text{ ungerade} \end{cases}$$

Concretely, we have

$$\begin{aligned}
2h(C_2, 1) &= 4 \geq 4 = \lambda_2(C_2, 1) \geq \frac{h(C_2, 1)^2}{4} = 1, \\
2h(C_3, 1) &= 4 \geq 3 = \lambda_2(C_3, 1) \geq \frac{h(C_3, 1)^2}{4} = \frac{2^2}{4} = 1, \\
2h(C_4, 1) &= 2 \geq 2 = \lambda_2(C_4, 1) \geq \frac{h(C_4, 1)^2}{4} = \frac{1}{4}, \\
2h(C_n, 1) &= \frac{8}{n} \geq \underbrace{2 - 2\cos\Big(\frac{2\pi}{n}\Big)}_{\geq \frac{4\pi^2}{n^2}} = \lambda_2(C_n, 1) \geq \frac{h(C_n, 1)^2}{4} = \frac{4}{n^2} \quad (n \text{ even}) \\
2h(C_n, 1) &= \frac{8}{n-1} \geq \underbrace{2 - 2\cos\Big(\frac{2\pi}{n}\Big)}_{\geq \frac{4\pi^2}{n^2}} = \lambda_2(C_n, 1) \geq \frac{h(C_n, 1)^2}{4} = \frac{4}{(n-1)^2} \quad (n \text{ odd})
\end{aligned}$$

as above.

(d) The same is true for the standard weight, as $h(C_n, \deg) = 2/n$ (n even) and $h(C_n, \deg) = 2/(n-1)$ (n odd), i.e. $(C_n, \deg)_n$ is no expander family. As all (finite) 2-regular graphs are isomorphic to C_n for some $n \in \mathbb{N}$, there cannot be an expander family for 2-regular graphs.

(e) Let $G = K_n$ be the complete graph with n vertices; this graph is $(n-1)$-regular. In particular, $h(K_n, 1) = (n-1)h(K_n, \deg)$ and $\lambda_2(K_n, 1) = (n-1)\lambda_2(K_n, \deg)$. Here, $\lambda_2(K_n, 1) = n$

(see Example 2.7 (c)). Since $\varrho_\infty(K_n, 1) = n - 1$, the relative weight is not bounded and hence $(C_n, 1)_n$ is not an expander family (although the eigenvalue family is now bounded from below by 1; it is even tending to ∞.

Once can also see (using the high symmetry of K_n) that

$$h(K_n, 1) = \begin{cases} \dfrac{n}{2}, & n \text{ even}, \\ \dfrac{n+1}{2}, & n \text{ odd}. \end{cases}$$

Cheeger's inequality Theorem 3.7 and Theorem 3.5 say that

$$2h(K_n, 1) \geq n = \lambda_2(K_n, 1) \geq \frac{h(K_n, 1)^2}{2\varrho_\infty} = \frac{h(K_n, 1)^2}{2(n-1)}.$$

Concretely,

$$\begin{aligned} 2h(K_2, 1) = 2 \geq 2 = \lambda_2(K_2, 1) &\geq \frac{h(K_2, 1)^2}{2} = \frac{1}{2}, \\ 2h(K_3, 1) = 4 \geq 3 = \lambda_2(K_3, 1) &\geq \frac{h(K_3, 1)^2}{4} = \frac{2^2}{4} = 1, \\ 2h(K_4, 1) = 4 \geq 4 = \lambda_2(K_4, 1) &\geq \frac{h(K_4, 1)^2}{6} = \frac{2}{3}, \\ 2h(K_n, 1) = n = \lambda_2(K_n, 1) &\geq \frac{h(K_n, 1)^2}{2(n-1)} = \frac{n^2}{8(n-1)} \quad (n \text{ even}) \\ 2h(K_n, 1) = n + 1 \geq n = \lambda_2(K_n, 1) &\geq \frac{h(K_n, 1)^2}{2(n-1)} = \frac{(n+1)^2}{8(n-1)} \quad (n \text{ odd}). \end{aligned}$$

(f) For $(K_n, \deg)$ we have

$$h(K_n, \deg) = \frac{1}{n-1} \cdot h(K_n, 1) = \begin{cases} \dfrac{n}{2(n-1)}, & n \text{ even}, \\ \dfrac{n+1}{2(n-1)}, & n \text{ odd}, \end{cases}$$

i.e., $\inf_n h(K_n, \deg) = 1/2 > 0$. Moreover, $\varrho_\infty(K_n, \deg) = 1$, hence $(K_n, \deg)_n$ is an expander family (but a not very interesting one, as with increasing n, also the degree increases as $n - 1$).

Other examples can be found e.g. in [Mah16, Sec. 8.4.2].