An introduction to topological fluid mechanics

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Abstract: The goal of this course is to introduce certain geometric aspects of inviscid and incompressible fluid flows, which are described by the solutions to the Euler equations. Among other things, we will review Arnold's theorem on the topological structure of stationary fluids in compact manifolds, and Moffatt's theorem on the topological interpretation of helicity in terms of knot invariants. The recent realization theorems of vortex lines and vortex tubes of arbitrarily complicated topology for stationary solutions to the Euler equations will also be introduced. Our objective is not to provide detailed proofs of all the stated results but to introduce the main ideas and methods behind certain selected topics of the subject known as "Topological Fluid Mechanics".

Lecture 1: The time-dependent Euler equations.

1.1 Introduction: analytic and geometric formulations.
1.2 Helmholtz's transport of vorticity.
1.3 The classical conservation laws.
1.4 A KAM-type conservation law.

Lecture 2: The stationary Euler equations.

2.1 Introduction: definitions.
2.2 A few exact solutions.
2.3 Moffatt's MHD scenario.
2.3 Arnold's structure theorem and corollaries.

Lecture 3: Periodic orbits and helicity.

3.1 Existence of periodic orbits for analytic solutions.
3.2 Helicity and Moffatt's theorem.
3.3 Uniqueness of helicity and a problem on integrability of vector fields.

Lecture 4: Beltrami flows.

4.1 Introduction: definitions and Arnold's conjectures.

4.2 Beltrami fields with nonconstant proportionality factor.
4.2 Flexibility of Beltrami fields.

Lecture 5: Beltrami flows II.

5.1 A realization theorem for knotted vortex lines.
5.1 A realization theorem for knotted vortex tubes.
5.2 KAM for Beltrami flows.

References:

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