The conjecture of Birch and Swinnerton-Dyer

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Elliptic curves A (rational) *elliptic curve* is a curve E defined by an equation of the form

$$E\colon y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}, \quad 4a^3 + 27b^2 \neq 0. \tag{1}$$

It was already known to the ancient greeks that there is a geometric procedure that allows one to add points on E. If one includes a 'point at infinity', the set of solutions (x, y) to the equation in (1) therefore forms an abelian group $E(\mathbb{Q})$. Much later, in 1922, Mordell proved that $E(\mathbb{Q})$ is moreover finitely generated, and hence that there is an isomorphism

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus \Delta \tag{2}$$

with an integer $r \geq 0$ and a finite abelian group Δ. Here the group Δ is well-understood; it can be effectively computed and Mazur has proved that the list of its possible isomorphism types consists of only 15 small groups. In comparison, the integer r, called the *rank* of the elliptic curve E, is rather illusive. It is an open question whether there exist elliptic curves of arbitrarily large rank, and we know of no algorithm that computes r for a given elliptic curve and probably terminates.

The conjecture Remarkably, however, we expect that r is linked to a certain holomorphic function attached to E, namely its *Hasse–Weil L-series* $L(E, s)$. More precisely, the famous conjecture of Birch and Swinnerton-Dyer, one of the central open questions in number theory and a Millennium prize problem, predicts that

$$r \stackrel{?}{=} \operatorname{ord}_{s=1} L(E, s), \tag{3}$$

that is, the rank of E is expected to be equal to the order of vanishing of $L(E, s)$ at $s = 1$. In addition, the conjecture includes a precise formula for the first non-vanishing term in the Taylor series expansion of $L(E, s)$ at $s = 1$:

$$\tfrac{1}{r!} \cdot L^{(r)}(E, 1) \stackrel{?}{=} \frac{|Ш| \cdot \Omega \cdot \mathrm{Reg} \cdot \mathrm{Tam}}{|\Delta|^2}. \tag{4}$$

Here the right hand side is composed of important arithmetic invariants of E: The *Tate–Shafarevich group* Ш, a mysterious group whose finiteness is a difficult open conjecture in its own right, the Néron period Ω, the regulator Reg, the product of Tamagawa numbers Tam, and Δ the torsion subgroup of $E(\mathbb{Q})$ as in (2).

Project description Depending on previous knowledge and interests, this project can involve working through the basics of elliptic curves necessary to understand the statements of (3) and (4), or can even aim to look at some of the techniques used to attack the conjecture in special cases.

References The standard textbook on the theory of elliptic curves is [Sil09]. For a more detailed introduction to the conjecture of Birch–Swinnerton-Dyer, and further reading, we refer to the survey articles [BST21; Gro11; SD67; Tat95; Wil06; Zha14].

References

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