Projects for Beca Intro

by Leo Margolis

Project 1: Units in group rings

The group ring RG of a group G over a ring R is the free R-module with basis G. As a module it has an additive structure, but it also inherits a natural multiplication by extending distributively and R-linearly the multiplication of G. Group rings are fundamental objects in the representation theory of groups, where they are used to prove many deep theorems on the structure of finite groups, but they are also interesting objects in themselves which live at the intersection of several areas of mathematics.

In this project the goal is to understand problems on the connection of G and the unit group of a group ring RG and possibly study some new examples. Recall that an element x in a ring is a unit if there is an element x' such that $xx' = x'x = 1$. There are various problems in the area, the exact formulation depending on the choice of properties of R and G. One typical questions is which properties of G are determined by the ring structure of RG, for example it is clear that RG is commutative if and only if G is abelian. But in some situations much more can be said - understanding when G can even be determined completely is a key problem. Another kind of question is how close any unit of RG is to being an element of G, e.g. the orders of the units and the elements of G might be the same.

The methods to study these questions rely on group theory and the representation theory of groups.

Project 2: Coset partitions of groups

A theorem from the 1950's, which confirmed a conjecture of Erdös, states: in a partition of the integers by arithemtic progressions $a_1 + m_1\mathbb{Z}$,...,$a_n + m_n\mathbb{Z}$, necessarily $m_i = m_j$ for some $i \neq j$. Here by partition we mean that the union of all the progressions $(a_i + m_i\mathbb{Z})$ equals $\mathbb{Z}$ and the intersection of $(a_i + m_i\mathbb{Z})$ and $(a_j + m_j\mathbb{Z}) = \emptyset$ is empty for any $i \neq j$.

Herzog and Schönheim conjectured in 1974 that a similar statement should hold when $\mathbb{Z}$ is replaced by any group, so whenever a group G is partitioned by cosets $g_1U_1, ..., g_nU_n$, for some $g_1, ..., g_n \in G$ and subgroups $U_1, ..., U_n$ of G, then $[G : U_i] = [G : U_j]$ for some $1 \leq i, j \leq n$. Though easy to formulate this conjecture remains open and is only known for some restricted classes of groups. In this sense there are various examples where one could study the conjecture and obtain new results, starting with elementary group theoretic methods.

Maximal number of students: 2