2. Entropy Methods for Nonlinear Diffusion Equations: a bridge between asymptotic behaviour of solutions, functional inequalities and geometry.

Understanding the large time behaviour of evolution equation is one of the fundamental issues in Nonlinear Parabolic PDEs. This can be understood by thinking about the classical Heat Equation $\partial u = \Delta u$: we know that all nonnegative integrable initial data produce solutions that behave -for large times- as the fundamental solution with the same mass (the Gaussian). This can be seen as a Central Limit Theorem (CLT) which has an incredible amount of applications. Understanding the sharp asymptotic behaviour in a precise and quantitative way would provide the analogous of a quantitative CLT, and would allow to answer question that cannot be answered by the standard version of the CLT. There is a natural connection between asymptotic rates of convergence and functional inequalities of Poincaré, Sobolev, Gagliardo-Nirenberg (or Nash), Hardy, Logarithmic Sobolev inequalities, etc. which we aim to investigate and it is interesting already at the level of linear equations, like the Heat Equation, the Fokker-Plank Equation, or the Ohrnstein-Uhlembeck equation (via the Boltzmann entropy approach for instance). Deep connections with geometry (the curvature plays a big role) are also present when posing the diffusion problem on Riemannian Manifolds and new phenomena arise. We will pay special attention to the analysis of Entropies (special Lyapunov functionals which are decreasing along the flow), in particular the relation between Entropy and Entropy-Production will be strictly related to the above mentioned functional inequalities.

The same kind of issues can be analyzed also in the nonlinear setting, for instance the connection between the sharp version of the Sobolev (or Gagliardo-Nirenberg) (SGNI in what follows) inequality and sharp asymptotic behaviour of solutions to the Fast Diffusion Equation, $\partial_t u = \Delta u^m$ $(m < 1)$, discovered in the 2000s, has opened the way to a better understanding of both aspects (asymptotic behaviour of solutions and functional inequalities) and can eventually lead to an improvement of the sharp SGNI by computing an explicit deficit (reminder term in the case when the equality is not achieved in the SGNI inequality). The underlying idea is that the equilibria of the FDE are also the optimal functions in SGNI (the functions for which equality is achieved). To understand deeply the connection between the two a priori disconnected worlds is a fascinating journey through PDEs, Analysis and Geometry (the latter surprisingly popping up even in the standard Euclidean setting). Note that when $m = \frac{N-2}{N+2}$ the FDE is a particular case of the celebrated Yamabe flow, which studies the evolution of (metrics of) Riemannian Manifolds under conformal constraints. Other intriguing geometric connections will (surprisingly) appear when analyzing the sharp asymptotic behaviour of FDE via Entropy methods through (weighted) linearization. We will also introduce the student to the basics of spectral theory.