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1. Introduction to the theory of Nonlinear Diffusion: Parabolic PDEs of degenerate and singular type, both in the local and nonlocal frameworks.

The aim is to introduce the student to the basic theory of Nonlinear Parabolic PDEs, which model a number of diffusion phenomena in different sciences ranging from Physics to Biology Engineering, Finance, etc. We will focus on the prototype diffusion equations: the Porous Medium Equation $\partial_t u = \Delta u^m$ $(m > 0)$, the p-Laplacian evolution equation $\partial_t = \Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ $p > 1$, and we will study different initial and boundary value problems, by prescribing initial datum at $t = 0$ and different boundary conditions (Dirichlet, Neumann, Robin, etc.) depending on the spatial domain where the problem is posed. We will address the main basic questions whose goal is to describe the solutions of the problems in the most precise way: no-representation formula is known for nonlinear PDEs, which marks a strong difference with the linear case (the prototype being the classical Heat Equation $\partial_t u = \Delta u$. This is just a sample of the extra difficulties which the nonlinear PDEs present. Depending on the interest and skills of the student, we can approach also more general equations, for instance the nonlocal p-Laplacian or the Nonlocal Filtration equation of the form $\partial_t = \mathcal{L}\varphi(u)$, where $\mathcal{L}$ is allowed to be a nonlocal diffusion operator, and φ is a possibly degenerate nonlinearity. The main example here is given when the diffusion operator is Fractional Laplacian $\mathcal{L} = -(-\Delta)^s$: in the case of the Dirichlet problem, there are at least three different realizations of the Fractional Laplacian (FL), known in the literature with different names, Restricted or simply the FL, Spectral FL, Regional FL or Censored FL. Here a list of possible topics that may be analyzed: (one or more depending on the student)

- Existence and uniqueness: the different concepts of weak solutions. Results obtained via Gradient-Flows techniques or via Nonlinear Semigroup theory.
- Boundedness, positivity and Harnack inequlities: using the celebrated De Giorgi-Nash-Moser Theory, or using the so-called Gross method or via a (new) Green function approach.
- Interior regularity: solutions are Hölder continuous, second part of the celebrated De Giorgi-Nash-Moser Theory. Higher regularity estimates: weak solutions can be classical. This requires the delicate Calderon-Zygmund and/or Schauder theories.
- Advanced topic: boundary behaviour and boundary regularity. Requires advanced techniques that involve sharp/higher regularity estimates in the interior, and sharp barriers or "almost optimal" sub/super solutions.

2. Entropy Methods for Nonlinear Diffusion Equations: a bridge between asymptotic behaviour of solutions, functional inequalities and geometry.

Understanding the large time behaviour of evolution equation is one of the fundamental issues in Nonlinear Parabolic PDEs. This can be understood by thinking about the classical Heat Equation $\partial u = \Delta u$: we know that all nonnegative integrable initial data produce solutions that behave -for large times- as the fundamental solution with the same mass (the Gaussian). This can be seen as a Central Limit Theorem (CLT) which has an incredible amount of applications. Understanding the sharp asymptotic behaviour in a precise and quantitative way would provide the analogous of a quantitative CLT, and would allow to answer question that cannot be answered by the standard version of the CLT. There is a natural connection between asymptotic rates of convergence and functional inequalities of Poincaré, Sobolev, Gagliardo-Nirenberg (or Nash), Hardy, Logarithmic Sobolev inequalities, etc. which we aim to investigate and it is interesting already at the level of linear equations, like the Heat Equation, the Fokker-Plank Equation, or the Ohrnstein-Uhlembeck equation (via the Boltzmann entropy approach for instance). Deep connections with geometry (the curvature plays a big role) are also present when posing the diffusion problem on Riemannian Manifolds and new phenomena arise. We will pay special attention to the analysis of Entropies (special Lyapunov functionals which are decreasing along the flow), in particular the relation between Entropy and Entropy-Production will be strictly related to the above mentioned functional inequalities.

The same kind of issues can be analyzed also in the nonlinear setting, for instance the connection between the sharp version of the Sobolev (or Gagliardo-Nirenberg) (SGNI in what follows) inequality and sharp asymptotic behaviour of solutions to the Fast Diffusion Equation, $\partial_t u = \Delta u^m$ $(m < 1)$, discovered in the 2000s, has opened the way to a better understanding of both aspects (asymptotic behaviour of solutions and functional inequalities) and can eventually lead to an improvement of the sharp SGNI by computing an explicit deficit (reminder term in the case when the equality is not achieved in the SGNI inequality). The underlying idea is that the equilibria of the FDE are also the optimal functions in SGNI (the functions for which equality is achieved). To understand deeply the connection between the two a priori disconnected worlds is a fascinating journey through PDEs, Analysis and Geometry (the latter surprisingly popping up even in the standard Euclidean setting). Note that when $m = \frac{N-2}{N+2}$ the FDE is a particular case of the celebrated Yamabe flow, which studies the evolution of (metrics of) Riemannian Manifolds under conformal constraints. Other intriguing geometric connections will (surprisingly) appear when analyzing the sharp asymptotic behaviour of FDE via Entropy methods through (weighted) linearization. We will also introduce the student to the basics of spectral theory.