Matteo Bonforte

1. Introduction to the theory of Nonlinear Diffusion: Parabolic PDEs of degenerate and singular type, both in the local and nonlocal frameworks.

The aim is to introduce the student to the basic theory of Nonlinear Parabolic PDEs, which model a number of diffusion phenomena in different sciences ranging from Physics to Biology Engineering, Finance, etc. We will focus on the prototype diffusion equations: the Porous Medium Equation $\partial_t u = \Delta u^m$ ($m > 0$), the p-Laplacian evolution equation $\partial_t = \Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ $p > 1$, and we will study different initial and boundary value problems, by prescribing initial datum at $t = 0$ and different boundary conditions (Dirichlet, Neumann, Robin, etc.) depending on the spatial domain where the problem is posed. We will address the main basic questions whose goal is to describe the solutions of the problems in the most precise way: no-representation formula is known for nonlinear PDEs, which marks a strong difference with the linear case (the prototype being the classical Heat Equation $\partial_t u = \Delta u$. This is just a sample of the extra difficulties which the nonlinear PDEs present. Depending on the interest and skills of the student, we can approach also more general equations, for instance the nonlocal p-Laplacian or the Nonlocal Filtration equation of the form $\partial_t = \mathcal{L}\varphi(u)$, where $\mathcal{L}$ is allowed to be a nonlocal diffusion operator, and φ is a possibly degenerate nonlinearity. The main example here is given when the diffusion operator is Fractional Laplacian $\mathcal{L} = -(-\Delta)^s$: in the case of the Dirichlet problem, there are at least three different realizations of the Fractional Laplacian (FL), known in the literature with different names, Restricted or simply the FL, Spectral FL, Regional FL or Censored FL. Here a list of possible topics that may be analyzed: (one or more depending on the student)

- Existence and uniqueness: the different concepts of weak solutions. Results obtained via Gradient-Flows techniques or via Nonlinear Semigroup theory.
- Boundedness, positivity and Harnack inequlities: using the celebrated De Giorgi-Nash-Moser Theory, or using the so-called Gross method or via a (new) Green function approach.
- Interior regularity: solutions are Hölder continuous, second part of the celebrated De Giorgi-Nash-Moser Theory. Higher regularity estimates: weak solutions can be classical. This requires the delicate Calderon-Zygmund and/or Schauder theories.
- Advanced topic: boundary behaviour and boundary regularity. Requires advanced techniques that involve sharp/higher regularity estimates in the interior, and sharp barriers or "almost optimal" sub/super solutions.