# Topological derivatives for inverse problems

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# Outline

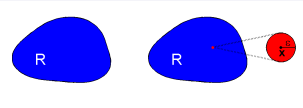
1. Definition of topological derivative

2. Topological derivative for the Neumann problem
   - Variational formulation of the forward Neumann problem
   - Shape and topological derivatives

## Definition of Topological Derivative (Sokowloski–Zochowski '99)

The TD of a shape functional $J(\mathcal{R})$ at a point $\mathbf{x} \in \mathcal{R}$ is

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R}_\varepsilon) - J(\mathcal{R})}{f(\varepsilon)}$$




where $\mathcal{R}_\varepsilon = \mathcal{R} \setminus \overline{B_\varepsilon}(\mathbf{x})$ and $f(\varepsilon) > 0$ is a monotonically decreasing function such that

1. the limit exists
2. the limit does not vanish
3. $f(\varepsilon) \to 0$ as $\varepsilon \to 0$

Usually $f$ is related to the measure of the ball

- $f(\varepsilon) = \pi\varepsilon^2$ for the Neumann and transmission probl. in 2D
- $f(\varepsilon) = -2\pi/log(\lambda\varepsilon)$ for the Dirichlet problem in 2D

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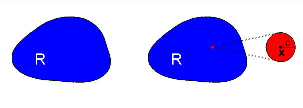


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$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R}_\varepsilon) - J(\mathcal{R})}{f(\varepsilon)}$$

- It is a scalar function of $\mathbf{x}$
- It measures sensitivity to removing balls around $\mathbf{x}$
- $D_T(\mathbf{x}, \mathcal{R}) << 0 \implies$ high probability of finding an object

Equivalently,

$$J(\mathcal{R}_\varepsilon) = J(\mathcal{R}) + f(\varepsilon) D_T(\mathbf{x}, \mathcal{R}) + g(f(\varepsilon)) \text{ as } \varepsilon \to 0$$

with

$$\lim_{\varepsilon \to 0} \frac{g(f(\varepsilon))}{f(\varepsilon)} = 0$$

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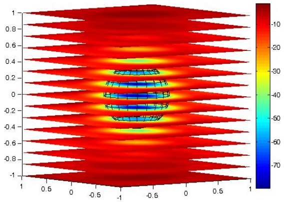
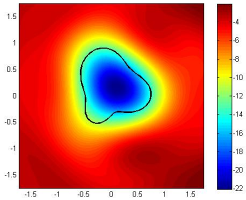
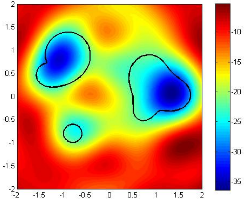
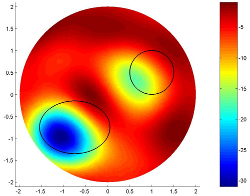
# Some examples



### Theorem (Novotny, Feijoo, Taroco & Padra '03)

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \left( \lim_{\alpha \to 0} \frac{J(\mathcal{R}_{\varepsilon+\alpha}) - J(\mathcal{R}_{\varepsilon})}{f(\varepsilon + \alpha) - f(\varepsilon)} \right)$$

# Equivalent definition based on shape sensitivity

## Definition of shape derivative

Let us consider a family of deformations along a vector field $\mathbf{V}$

$$\Phi_\alpha(\mathbf{z}) = \mathbf{z} + \alpha \mathbf{V}(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^n,\ \alpha > 0.$$

Then $J(\phi_\alpha(\mathcal{R}))$ is a scalar function of $\alpha$.

The shape derivative of $J(\mathcal{R})$ in the direction $\mathbf{V}$ is

$$D_S(\mathcal{R}) = \frac{d}{d\alpha} J(\phi_\alpha(\mathcal{R}))\Big|_{\alpha=0}$$

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Given $\mathbf{x} \in \mathcal{R}$, take $\varepsilon > 0$ and consider the vector field

$$\mathbf{V}(\mathbf{z}) = -\mathbf{n}(\mathbf{z}), \quad \mathbf{z} \in \Gamma_\varepsilon = \partial B_\varepsilon(\mathbf{x})$$

and extend it to $\mathbb{R}^n$ in such a way that it vanishes away from a narrow neighborhood of $\Gamma_\varepsilon$.

Remark:

$$\phi_\alpha(B_\varepsilon(\mathbf{x})) = B_{\varepsilon+\alpha}(\mathbf{x}), \quad \phi_\alpha(\mathcal{R}_\varepsilon) = \phi_\alpha(\mathcal{R} \setminus \overline{B}_\varepsilon(\mathbf{x})) = \mathcal{R}_{\varepsilon+\alpha}$$

**Theorem (Novotny, Feijoo, Taroco & Padra '03)**

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{D_S(\mathcal{R}_\varepsilon)}{f'(\varepsilon)}, \quad \text{where } \mathcal{R}_\varepsilon = \mathcal{R} \setminus \overline{B}_\varepsilon(\mathbf{x})$$

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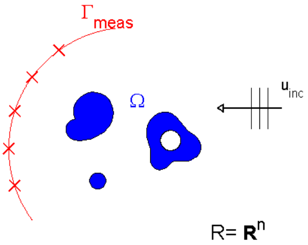


## Forward problem

$$\begin{cases} u \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Omega}) \\ \Delta u + \lambda^2 u = 0, \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\ \partial_{\mathbf{n}} u = 0, \quad \text{on } \Gamma = \partial\Omega \\ \lim_{r\to\infty} r^{1/2}\left(\partial_r(u - u_{inc}) - \imath\lambda(u - u_{inc})\right) = 0 \end{cases}$$

## Inverse problem

- Measurements $u_{meas}$ are taken at the receptors
- Find the scatters $\Omega$ s.t.

$u = u_{meas}$ on $\Gamma_{meas}$, u= sol. forward problem

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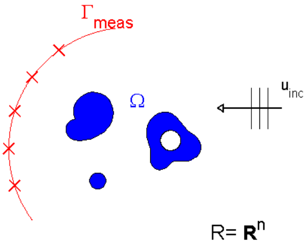


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# Variational formulation of the forward problem

## Dirichlet–to–Neumann operator

Take $R > 0$ such that $\Omega \subset B_R := B_R(\mathbf{0})$.

$$L : \begin{array}{ccc} H^{1/2}(\Gamma_R) & \longrightarrow & H^{-1/2}(\Gamma_R) \\ f & \longmapsto & \partial_{\mathbf{n}} w \end{array}$$

where $w \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{B_R})$ is the unique solution to

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## Remark

A transparent boundary condition on $\Gamma_R$:

$$\partial_{\mathbf{n}}(u - u_{inc}) = L(u - u_{inc})$$

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1. An equivalent problem in a bounded domain

$$\begin{cases} u \in H^1(B_R \setminus \overline{\Omega}) & \\ \Delta u + \lambda^2 u = 0, & \text{in } B_R \setminus \overline{\Omega} \\ \partial_{\mathbf{n}} u = 0, & \text{on } \Gamma = \partial\Omega \\ \partial_{\mathbf{n}}(u - u_{inc}) = L(u - u_{inc}), & \text{on } \Gamma_R = \partial B_R \end{cases}$$

2. An equivalent variational problem

$$\begin{cases} u \in H^1(B_R \setminus \overline{\Omega}) \\ b(\Omega; u, v) = \ell(v), \quad \forall v \in H^1(B_R \setminus \overline{\Omega}) \end{cases}$$

where

$$b(\Omega; u, v) := \int_{B_R\setminus\Omega} \nabla u \nabla \bar{v} - \lambda^2 u \bar{v} - \int_{\Gamma_R} (Lu)\bar{v}$$

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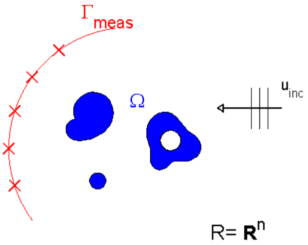


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## A weaker formulation

Find $\Omega$ minimizing

$$J(\mathbb{R}^2 \setminus \Omega) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

for $u$ solving the forward problem with objects $\Omega$.

## Our monotone iterative method

1. Compute the TD when $\Omega = \emptyset$
2. Take $\Omega_1 = \{\mathbf{x}, D_T(\mathbf{x}, \mathbb{R}^2) < -C_1\}, \; C_1 > 0$
3. For j=1:jmax
   - Compute the TD in $\mathbb{R}^n \setminus \Omega_j$
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# Computation of $D_T(\mathbf{x}, \mathbb{R}^2)$

- Recall that the TD can be computed as

$$D_T(\mathbf{x}, \mathbb{R}^2) = \lim_{\varepsilon \to 0} \frac{D_S(\mathbb{R}^2 \setminus B_\varepsilon)}{f'(\varepsilon)}, \quad B_\varepsilon := B_\varepsilon(\mathbf{x})$$

where (as we will see), $f(\varepsilon) = \pi \varepsilon^2$

- By definition

$$D_S(\mathbb{R}^2 \setminus B_\varepsilon) = \frac{d}{d\alpha} J(\phi_\alpha(\mathbb{R}^2 \setminus B_\varepsilon))\Big|_{\alpha=0}$$

where $\phi_\alpha(\mathbf{z}) = \mathbf{z} + \alpha \mathbf{V}(\mathbf{z})$ and $\mathbf{V}(\mathbf{z}) = -\mathbf{n}(\mathbf{z})$ when $\mathbf{z} \in \Gamma_\varepsilon$ and vanishes away from a narrow neighborhood of $\Gamma_\varepsilon$. Therefore,

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# Computation of $D_S(\mathbb{R}^2 \setminus B_\varepsilon)$

## Theorem

$$D_S(\mathbb{R}^2 \setminus B_\varepsilon) = \text{Re} \int_{\Gamma_\varepsilon} \lambda^2 u_\varepsilon w_\varepsilon - \nabla u_\varepsilon \nabla w_\varepsilon$$

where $u_\varepsilon$ solves the forward problem in $B_\varepsilon$:

$$\begin{cases} u_\varepsilon \in H^1(B_R \setminus \overline{B_\varepsilon}) \\ b(B_\varepsilon, u_\varepsilon, v) = \ell(v) \quad \forall v \in H^1(B_R \setminus \overline{B_\varepsilon}) \end{cases}$$

and $w_\varepsilon$ solves the adjoint problem

$$\begin{cases} w_\varepsilon \in H^1(B_R \setminus \overline{B_\varepsilon}) \\ b(B_\varepsilon, w_\varepsilon, v) = \ell_{meas}(v) \quad \forall v \in H^1(B_R \setminus \overline{B_\varepsilon}) \end{cases}$$

with $\ell_{meas}(v) = \int_{\Gamma_{meas}} \overline{(u_{meas} - u_\varepsilon)} v$.

Remark: The adjoint problem

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is equivalent to

$$\begin{cases} \Delta w_\varepsilon + \lambda^2 w_\varepsilon = \overline{(u_\varepsilon - u_{meas})} \delta_{\Gamma_{meas}}, \quad \text{in } \mathbb{R}^2 \setminus \overline{B_\varepsilon} \\ \partial_{\mathbf{n}} w_\varepsilon = 0, \quad \text{on } \Gamma_\varepsilon \\ \lim_{r\to\infty} r^{1/2}(\partial_r w_\varepsilon - \imath \lambda w_\varepsilon) = 0 \end{cases}$$

## Proof of the theorem

- Step 1:

$$D_S(\mathbb{R}^2 \setminus B_\varepsilon) = \mathrm{Re}\left( \frac{d}{d\alpha} b(B_{\varepsilon+\alpha}; u_\varepsilon, p_\varepsilon)\Big|_{\alpha=0} \right)$$

where $p_\varepsilon$ solves ($w_\varepsilon = \bar{p}_\varepsilon$)

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$$\frac{d}{d\alpha} b(B_{\varepsilon+\alpha}; u_\varepsilon, p_\varepsilon)\Big|_{\alpha=0} = \int_{\Gamma_\varepsilon} \lambda^2 u_\varepsilon \bar{p}_\varepsilon - \nabla u_\varepsilon \nabla \bar{p}_\varepsilon$$

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**Proof of Step 2**:

$$b(B_{\varepsilon+\alpha}; u_\varepsilon, p_\varepsilon) = \int_{B_R \setminus B_{\varepsilon+\alpha}} (\nabla u_\varepsilon \nabla \bar{v}_\varepsilon - \lambda^2 u_\varepsilon \bar{v}_\varepsilon) d\mathbf{z}_{\varepsilon+\alpha} - \underbrace{\int_{\Gamma_R} (L u_\varepsilon) \bar{v}_\varepsilon}_{\text{indep. of } \alpha}$$

Doing a change of variables to $\mathbb{R}^2 \setminus B_\varepsilon$ and integrating by parts,

$$\begin{aligned}
&\frac{d}{d\alpha} b(B_{\varepsilon+\alpha}; u_\varepsilon, p_\varepsilon)\Big|_{\alpha=0} \\
&= \int_{\mathbb{R}^2 \setminus B_\varepsilon} (\Delta u_\varepsilon + \lambda^2 u_\varepsilon) \nabla \bar{p}_\varepsilon \cdot \mathbf{V} + \int_{\mathbb{R}^2 \setminus B_\varepsilon} (\Delta \bar{p}_\varepsilon + \lambda^2 \bar{p}_\varepsilon) \nabla u \cdot \mathbf{V} \\
&+ \int_{\Gamma_\varepsilon \cup \Gamma_R} (\nabla u_\varepsilon \nabla \bar{p}_\varepsilon) \mathbf{V} \cdot \mathbf{n} - \int_{\Gamma_\varepsilon \cup \Gamma_R} \lambda^2 u_\varepsilon \bar{p}_\varepsilon \mathbf{V} \cdot \mathbf{n} \\
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$$+ \int_{\Gamma_\varepsilon \cup \Gamma_R} (\nabla u_\varepsilon \nabla \bar{p}_\varepsilon) \mathbf{V} \cdot \mathbf{n} - \int_{\Gamma_\varepsilon \cup \Gamma_R} \lambda^2 u_\varepsilon \bar{p}_\varepsilon \mathbf{V} \cdot \mathbf{n}$$

$$= \int_{\Gamma_\varepsilon} \lambda^2 u_\varepsilon \bar{p}_\varepsilon - \nabla u_\varepsilon \nabla \bar{p}_\varepsilon$$

Proof of Step 2:

$$b(B_{\varepsilon+\alpha}; u_\varepsilon, p_\varepsilon) = \int_{B_R \setminus B_{\varepsilon+\alpha}} (\nabla u_\varepsilon \nabla \bar{v}_\varepsilon - \lambda^2 u_\varepsilon \bar{v}_\varepsilon) d\mathbf{z}_{\varepsilon+\alpha} - \underbrace{\int_{\Gamma_R} (L u_\varepsilon) \bar{v}_\varepsilon}_{\text{indep. of } \alpha}$$

Doing a change of variables to $\mathbb{R}^2 \setminus B_\varepsilon$ and integrating by parts,

$$\frac{d}{d\alpha} b(B_{\varepsilon+\alpha}; u_\varepsilon, p_\varepsilon)\Big|_{\alpha=0}$$

$$= \int_{\mathbb{R}^2 \setminus B_\varepsilon} (\Delta u_\varepsilon + \lambda^2 u_\varepsilon) \nabla \bar{p}_\varepsilon \cdot \mathbf{V} + \int_{\mathbb{R}^2 \setminus B_\varepsilon} (\Delta \bar{p}_\varepsilon + \lambda^2 \bar{p}_\varepsilon) \nabla u \cdot \mathbf{V}$$

$$+ \int_{\Gamma_\varepsilon \cup \Gamma_R} (\nabla u_\varepsilon \nabla \bar{p}_\varepsilon) \mathbf{V} \cdot \mathbf{n} - \int_{\Gamma_\varepsilon \cup \Gamma_R} \lambda^2 u_\varepsilon \bar{p}_\varepsilon \mathbf{V} \cdot \mathbf{n}$$

$$= \int_{\Gamma_\varepsilon} \lambda^2 u_\varepsilon \bar{p}_\varepsilon - \nabla u_\varepsilon \nabla \bar{p}_\varepsilon$$

# Computation of $D_T(\mathbf{x}, \mathbb{R}^2)$

## Theorem

$$D_T(\mathbf{x}, \mathbb{R}^2) = \mathrm{Re}(\lambda^2 u(\mathbf{x}) w(\mathbf{x}) - 2\nabla u(\mathbf{x}) \nabla w(\mathbf{x}))$$

where

$$\begin{aligned} u(\mathbf{x}) &= u_{inc}(\mathbf{x}) = e^{ik_e \mathbf{x}\cdot\mathbf{d}} \\ w(\mathbf{x}) &= \int_{\Gamma_{meas}} G_{k_e}(\mathbf{x}-\mathbf{y})(\overline{u_{meas} - u})(\mathbf{y}) dl_{\mathbf{y}} \end{aligned}$$

# The last proof!!

$$D_T(\mathbf{x}, \mathbb{R}^2) = \lim_{\varepsilon \to 0} \frac{1}{f'(\varepsilon)} \mathrm{Re} \int_{\Gamma_\varepsilon} \lambda^2 u_\varepsilon \overline{p}_\varepsilon - \nabla u_\varepsilon \nabla \bar{p}_\varepsilon$$

This limit is computed using asymptotic expansions of $u_\varepsilon$ and $w_\varepsilon$ as $\varepsilon \to 0$: $u_\varepsilon(\mathbf{z}) = u(\mathbf{z}) + v_\varepsilon(\mathbf{z})$ and $w_\varepsilon(\mathbf{z}) = w(\mathbf{z}) + p_\varepsilon(\mathbf{z})$. Then,

$$\begin{cases} \Delta v_\varepsilon + \lambda^2 v_\varepsilon = 0, \quad \text{in } \mathbb{R}^2 \setminus \overline{B_\varepsilon} \\ \partial_{\mathbf{n}} v_\varepsilon = -\partial_{\mathbf{n}} u(\mathbf{z}) = -\nabla u(\mathbf{x}) \cdot \mathbf{n} + O(\varepsilon), \quad \text{on } \Gamma_\varepsilon \\ \lim_{r \to \infty} r^{1/2}(\partial_r v_\varepsilon - \imath \lambda v_\varepsilon) = 0 \end{cases}$$

and doing the change of variables $\mathbf{y} = (\mathbf{z} - \mathbf{x})/\varepsilon$,

$$\begin{cases} \Delta v_\varepsilon + \varepsilon^2 \lambda^2 v_\varepsilon = 0, \quad \text{in } \mathbb{R}^2 \setminus \overline{B_1(\mathbf{0})} \\ \partial_{\mathbf{n}} v_\varepsilon = -\varepsilon \nabla u(\mathbf{x}) \cdot \mathbf{n} + O(\varepsilon^2), \quad \text{on } \Gamma_\varepsilon \\ \lim_{r \to \infty} r^{1/2}(\partial_r v_\varepsilon - \imath \lambda \varepsilon v_\varepsilon) = 0 \end{cases}$$

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Expanding now $v_\varepsilon(\mathbf{y}) = a(\mathbf{y}) + \varepsilon b(\mathbf{y}) + O(\varepsilon^2)$, we find that $a = 0$ and $b(\mathbf{y}) = \nabla u(\mathbf{x}) \cdot \mathbf{y}/\|\mathbf{y}\|^2 = \nabla u(\mathbf{x}) \cdot g(\mathbf{y})$. Therefore,

$$u_\varepsilon(\mathbf{z}) = u(\mathbf{x}) + O(\varepsilon)$$

$$\frac{\partial u_\varepsilon}{\partial z_j}(\mathbf{z}_j) = \frac{\partial u}{\partial z_j}(\mathbf{z}_j)(\mathbf{x}) + \nabla u(\mathbf{x})\frac{\partial g}{\partial y_j}(\mathbf{y}) + O(\varepsilon)$$

Similar expressions for $w_\varepsilon$ and $\nabla w_\varepsilon$. Then

$$\int_{\Gamma_\varepsilon} u_\varepsilon(\mathbf{z}) w_\varepsilon(\mathbf{z})\, d\ell_{\mathbf{z}} \approx 2\pi\varepsilon\, u(\mathbf{x}) w(\mathbf{x})$$

$$\int_{\Gamma_\varepsilon} \nabla u_\varepsilon(\mathbf{z}) \nabla w_\varepsilon(\mathbf{z})\, d\ell_{\mathbf{z}} \approx 4\pi\varepsilon \nabla u(\mathbf{x}) \nabla w(\mathbf{x})$$

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