Topological derivatives for inverse problems

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Theoretical and Numerical Aspects of Inverse Problems and Scattering, April 2011
Tuesday 5th
- Introduction
- Topological derivative methods

Thursday 7th and Tuesday 12th
- Analytical computation of TD for sound hard obstacles
- Numerical computation of TDs (?)
Outline

1. Inverse scattering problems
2. Topological derivative methods
   - TD for shape reconstruction
   - TD for shapes and parameters
3. Other problems and conclusions
   - Improving the method
   - Other problems and boundary conditions
   - Conclusions
Medium $\mathcal{R}$ with obstacles $\Omega$: How many? how big? where? physical properties in $\Omega$?

Some applications
- Medicine (tumors, fracture)
- Geophysics (oil, gas)
- Materials (damage, cracks)
Inverse scattering problems

Topological derivative methods

Other problems and conclusions

Scattering problem
An incident wave $u_{inc}$ interacts with a medium $\mathcal{R}$ containing objects $\Omega$.

Forward (direct) problem

- The shape, size, location and physical properties of the objects are known.
- Compute the response of the system at the detectors "×".
- A well-posed problem: it has a unique solution that depends continuously on the data.

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Topological derivatives for inverse problems
**Inverse scattering problems**

**Topological derivative methods**

**Other problems and conclusions**

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**Scattering problem**

An incident wave $u_{inc}$ interacts with a medium $\mathcal{R}$ containing objects $\Omega$.

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**Forward (direct) problem**

- The shape, size, location and physical properties of the objects are known
- **Compute the response of the system at the detectors "×"**
- A well–posed problem: it has a unique solution that depends continuously on the data
Inverse scattering problems
Topological derivative methods
Other problems and conclusions

Inverse problem

- Measurements $u_{meas}$ are taken at the receptors
- Find the scatters $\Omega$ and the interior parameters s.t.
  $$u = u_{meas} \quad \text{on } \Gamma_{meas}, \quad u = \text{sol. forward problem}$$
- An ill–posed problem: it may not have a solution and if it has one, it may not depend continuously on the data

Scattering problem
An incident wave $u_{inc}$ interacts with a medium $\mathcal{R}$ containing objects $\Omega$. 

$\mathcal{R} = \mathbb{R}^n$
How do we distinguish the medium and the obstacles? They differ in their physical properties:

- elastic constants in acoustic scattering
- electric conductivity and permittivity in electromagnetic scattering
- thermal conductivity and diffusivity in thermal scattering

Type of incident radiation? Acoustic, electromagnetic, thermal waves
We assume that

- **We generate** acoustic waves
- Incident waves are time–harmonic

\[ U_{\text{inc}}(x, t) = \text{Re}[e^{-i\omega t} u_{\text{inc}}(x)], \]

- \( u_{\text{inc}} \) is a planar wave in the direction \( d \),

\[ u_{\text{inc}}(x) = e^{ikx \cdot d} \]

- The solution to the direct problem is time–harmonic

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A simple forward problem

Ω is a penetrable known obstacle. The incident field generates a scattered wave \( u_{sc} \) in \( \mathbb{R}^n \setminus \Omega \) and a transmitted wave \( u_{tr} \) in \( \Omega \). The total field

\[
u = u_{inc} + u_{sc} \quad \text{in} \quad \mathbb{R}^n \setminus \Omega \quad \text{and} \quad u = u_{tr} \quad \text{in} \quad \Omega
\]
solves

\[
\begin{cases}
\Delta u + k_e^2 u = 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega \\
\Delta u + k_i^2 u = 0 & \text{in} \quad \Omega \\
u^- = u^+ , \quad \partial_n u^- = \partial_n u^+ & \text{on} \quad \partial \Omega \\
\lim_{r \to \infty} r^{(n-1)/2} (\partial_r (u - u_{inc}) - i k_e(u - u_{inc})) = 0
\end{cases}
\]

where \( k_e, k_i > 0 \) are known.
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Original problem *(we assume that $k_i$ is known)*

Find $\Omega$ such that

$$u = u_{meas} \quad \text{on } \Gamma_{meas}$$

A weaker formulation

Find $\Omega$ minimizing

$$J(\Omega) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

for $u$ solving the forward problem with objects $\Omega$

- The domain $\Omega$ is the variable
- The Helmholtz transmission problem is the constraint
Constrained optimization

Original problem \((\text{we assume that } k_j \text{ is known})\)

Find \(\Omega\) such that

\[ u = u_{\text{meas}} \quad \text{on } \Gamma_{\text{meas}} \]

A weaker formulation

Find \(\Omega\) minimizing

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Some alternatives

Modified gradient methods: differ on how an initial guess is deformed from one iteration to the next in such a way that the cost functional decreases

- Classical deformations following a vector field
  - Problem: The number of scatterers has to be known from the beginning

- Level set based deformations allow changes in topology
  - Problem: Slow evolution. Initial guess?
  - Santosa 1996, Dorn 2005

- Topological derivatives
  - Provide good initial guesses
  - Fast and allow topological changes
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Definition of Topological Derivative (Sokowloski–Zochowski ’99)

The TD of a shape functional $J(\mathcal{R})$ at a point $x \in \mathcal{R}$ is

$$D_T(x, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R} \setminus B_\varepsilon(x)) - J(\mathcal{R})}{\text{Vol}(B_\varepsilon(x))}$$

- It is a scalar function of $x$
- It measures sensitivity to removing balls around $x$
- $D_T(x, \mathcal{R}) \ll 0 \implies$ high probability of finding an object

Equivalently, for $x \in \mathcal{R}$ and $h(\varepsilon) = \text{Vol}(B_\varepsilon(x))$

$$J(\mathcal{R} \setminus B_\varepsilon(x)) = J(\mathcal{R}) + h(\varepsilon)D_T(x, \mathcal{R}) + o(h(\varepsilon)) \text{ as } \varepsilon \to 0$$
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How to obtain $D_T(x, \mathcal{R})$ (Feijoo ’04)

1. Given $x \in \mathcal{R}$, take the ball $B_\varepsilon(x)$. Choose the vector field

$$V(z) = -n(z), \quad z \in \partial B_\varepsilon(x)$$

and extend $V$ to $\mathbb{R}^n$ s.t. $V = 0$ far from $\partial B_\varepsilon(x)$.

2. Consider the domain $\mathcal{R}_\tau := \{z + \tau V(z) \mid z \in \mathcal{R} \setminus B_\varepsilon(x)\}$. Then, $J(\mathcal{R}_\tau)$ is a scalar function of $\tau$.

3. Compute the shape derivative (Lagrangian formulation)

$$D_S := \frac{d}{d\tau} J(\mathcal{R}_\tau) \bigg|_{\tau=0}$$

4. Use the relation (asymptotic expansions)

$$D_T(x, \mathcal{R}) = \lim_{\varepsilon \to 0} \left( \frac{1}{\mathcal{V}(\varepsilon)} D_S \right), \quad \mathcal{V}(\varepsilon) = \text{Vol}(B_\varepsilon(x))$$
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How to obtain \( \mathbf{D}_T(\mathbf{x}, \mathcal{R}) \) (Feijoo ’04)

1. Given \( \mathbf{x} \in \mathcal{R} \), take the ball \( B_\varepsilon(\mathbf{x}) \). Choose the vector field
   \[
   \mathbf{V}(\mathbf{z}) = -\mathbf{n}(\mathbf{z}), \quad \mathbf{z} \in \partial B_\varepsilon(\mathbf{x})
   \]
   and extend \( \mathbf{V} \) to \( \mathbb{R}^n \) s.t. \( \mathbf{V} = 0 \) far from \( \partial B_\varepsilon(\mathbf{x}) \)

2. Consider the domain \( \mathcal{R}_\tau := \{ \mathbf{z} + \tau \mathbf{V}(\mathbf{z}) \mid \mathbf{z} \in \mathcal{R} \setminus B_\varepsilon(\mathbf{x}) \} \).
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Transmission problem: $u^- = u^+, \partial_n u^- = \partial_n u^+$

Case I: No a priori information on the obstacles, $\mathcal{R} = \mathbb{R}^n$, $\Omega = \emptyset$

Theorem. For any $x \in \mathbb{R}^n$ the topological derivative of

$$J(\mathbb{R}^n) = \frac{1}{2} \int_{\Gamma_{\text{meas}}} |u - u_{\text{meas}}|^2$$

is

$$D_T(x, \mathbb{R}^n) = \text{Re} \left[ (k_i^2 - k_e^2) u(x) w(x) \right]$$

where $u$ and $w$ solve forward and adjoint problems with $\Omega = \emptyset$
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Forward problem with $\Omega = \emptyset$:

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\begin{align*}
\Delta u + k^2_{e} u &= 0 \quad \text{in } \mathbb{R}^n \\
\lim_{r \to \infty} r^{(n-1)/2}(\partial_r (u - u_{\text{inc}}) - ik_{e} (u - u_{\text{inc}})) &= 0
\end{align*}
\]

Therefore, $u = u_{\text{inc}}(x) = e^{ik_{e}x \cdot d}$

Adjoint problem with $\Omega = \emptyset$:

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\begin{align*}
\Delta w + k^2_{e} w &= (u_{\text{meas}} - u)\delta_{\Gamma_{\text{meas}}} \quad \text{in } \mathbb{R}^n \\
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Therefore, $w = \int_{\Gamma_{\text{meas}}} G_{k_{e}}(x - y)(u_{\text{meas}} - u)(y)dy$

- The true obstacles enter in the TD through the measured data at the adjoint field
- "Free" computation: $\text{TD} = \text{Re} \left[ (k^2_i - k^2_{e}) u(x) w(x) \right]$
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\Delta u + k_e^2 u = 0 & \text{in } \mathbb{R}^n \\
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- "Free" computation: \( \text{TD} = \text{Re} \left[ (k_i^2 - k_e^2) u(x) w(x) \right] \)
Some examples
"×"= observation points, 24 incident directions in $[0, 2\pi)$, $k_e = 2$ and $k_i = 1/2$. Level of noise=1%
Similar results when

- Observation points are further
- + observation points
- + incident directions
- + noise
Results depend on the wave length (1 w.l. = $2\pi/k$):

1\textsuperscript{st} row: $k_e = 2$ and $k_i = 1/2$

2\textsuperscript{nd} row: $k_e = 4$ and $k_i = 1$
Case II: $\Omega_{\text{ap}}$ first guess, $\mathcal{R} = \mathbb{R}^n \setminus \Omega_{\text{ap}}$, $\Omega = \Omega_{\text{ap}}$

**Theorem.** For any $x \in \mathbb{R}^n \setminus \Omega_{\text{ap}}$ the topological derivative of

$$J(\mathbb{R}^n \setminus \Omega_{\text{ap}}) = \frac{1}{2} \int_{\Gamma_{\text{meas}}} |u - u_{\text{meas}}|^2$$

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**Forward problem with** $\Omega = \Omega_{ap}$:

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\begin{aligned}
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\Delta u + k_i^2 u &= 0 \quad \text{in } \Omega_{ap} \\
\begin{array}{l}
u^- = u^+, \
\partial_n u^- = \partial_n u^+ \quad \text{on } \partial \Omega_{ap}
\end{array} \\
\lim_{r \to \infty} r^{(n-1)/2} \left( \partial_r (u - u_{inc}) - ik_e (u - u_{inc}) \right) &= 0
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**Adjoint problem with** $\Omega = \Omega_{ap}$:

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\begin{aligned}
\Delta w + k_e^2 w &= (u_{meas} - u) \delta_{\Gamma_{meas}} \quad \text{in } \mathbb{R}^n \setminus \Omega_{ap} \\
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The boundary conditions influence $u$ and $w$!
Inverse scattering problems
Topological derivative methods
Other problems and conclusions

TD for shape reconstruction
TD for shapes and parameters

Same examples as before with $\Omega = \emptyset$

Initial guess $\Omega_1$ superimposed on the TD when $\Omega = \Omega_1$
An iterative method

Algorithm

1. Compute the TD when $\Omega = \emptyset$
2. Take $\Omega_1 = \{x, D_T(x, \mathbb{R}^n) < -C_1\}, \; C_1 > 0$
3. For $j=1:j_{max}$
   - Compute the TD in $\mathbb{R}^n \setminus \Omega_j$
   - Select $\Omega_{j+1} \supset \Omega_j$

$$\Omega_{j+1} = \Omega_j \cup \{x, D_T(x, \mathbb{R}^n \setminus \Omega_j) < -C_{j+1}\}$$
How to choose $C_j$?

- **First step:** $\Omega_1 = \{ x, D_T(x, \mathbb{R}^2) < -C_1 \}$
  
  $$C_1 = \frac{3}{5} \min D_T$$

  - Accept $C_1$ if $J_1 < J_0$
  - Otherwise $C'_1 < C_1$

- **Iterations:** $\Omega_{j+1} = \Omega_j \cup \{ x, D_T(x, \mathbb{R}^2 \setminus \Omega_j) < -C_{j+1} \}$
  
  $$C_{j+1} = \frac{9}{10} \min D_T$$

Stopping criteria?

$$J \approx 0 \quad \text{or} \quad \Omega_j \approx \Omega_{j+1} \quad \text{or} \quad |u_\delta - u_{meas}| < 1.2\delta$$
How to choose $C_j$?

**First step:** $\Omega_1 = \{x, D_T(x, \mathbb{R}^2) < -C_1\}$

$$C_1 = \frac{3}{5} |\min D_T|$$

- Accept $C_1$ if $J_1 < J_0$
- Otherwise $C'_1 < C_1$

**Iterations:** $\Omega_{j+1} = \Omega_j \cup \{x, D_T(x, \mathbb{R}^2 \setminus \Omega_j) < -C_{j+1}\}$

$$C_{j+1} = \frac{9}{10} |\min D_T|$$

Stopping criteria?

$$J \approx 0 \lor \Omega_j \approx \Omega_{j+1} \lor |u_\delta - u_{meas}| < 1.2\delta$$
How to choose $C_j$?

- **First step:** $\Omega_1 = \{ x, D_T(x, \mathbb{R}^2) < -C_1 \}$

\[
C_1 = \frac{3}{5} \left| \min D_T \right|
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Stopping criteria?

\[
J \approx 0 \; \text{or} \; \Omega_j \approx \Omega_{j+1} \; \text{or} \; |u_\delta - u_{meas}| < 1.2\delta
\]
Implementation in 2D: $TD = \text{Re} \left[ (k_i^2 - k_0^2) u(x) w(x) \right]$

- Computation of $TD$ when $\Omega = \emptyset$:
  $$u = u_{\text{inc}} \quad \text{and} \quad w = \int_{\Gamma_{\text{meas}}} G(x - y)(u_{\text{meas}} - u)(y)dy$$

- Computation of $TD$ when $\Omega = \Omega_j$:
  - $u$ and $w$ solve HTP with $\Omega = \Omega_j = \bigcup_{i=1}^d \Omega_i$
  - To apply BEM, we assume that $\Omega_i$ is star–shaped:

  $$x^i(t) = (c^i_x, c^i_y) + r^i(t)(\cos(t), \sin(t))$$

  We approximate

  $$r^i(t) \approx a_0^i + \sum_{k=1}^K (a_k^i \cos(kt) + b_k^i \sin(kt))$$

  - Solve a least squares problem to obtain $a^i$s, $b^i$s such that

    $$\Omega_j = \Omega_{j-1} \cup \{x, TD_{j-1} << 0\}$$
Inverse scattering problems
Topological derivative methods
Other problems and conclusions

TD for shape reconstruction
TD for shapes and parameters

Topological derivatives for inverse problems
Inverse scattering problems
Topological derivative methods
Other problems and conclusions

TD for shape reconstruction
TD for shapes and parameters

Topological derivatives for inverse problems

María–Luisa Rapún
Outline

1. Inverse scattering problems

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Direct and inverse problems

Direct problem

\[
\begin{align*}
\Delta u + k_e^2 u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\
\Delta u + k_i^2 u &= 0 \quad \text{in } \Omega \\
 u^- = u^+ , \quad \partial_n u^- = \partial_n u^+ &\quad \text{on } \partial \Omega \\
 \lim_{r \to \infty} r^{(n-1)/2} (\partial_r (u - u_{inc}) - i k_e (u - u_{inc})) &= 0
\end{align*}
\]

Inverse problem

Find \( \Omega \) and \( k_i \)
Idea

- In the first computation of the TD, i.e. when $\Omega = \emptyset$, we do not need to know $k_i$:

$$D_T(x, \mathbb{R}^2) = \text{Re} \left[ (k_i^2 - k_e^2)u(x)w(x) \right]$$

where $u = u_{\text{inc}}$ and $w = \int_{\Gamma_{\text{meas}}} G_k(x - y)(u_{\text{meas}} - u)(y) dl_y$

- We compute the TD taking $k_i^0 \approx k_e$ to get an initial guess $\Omega_1$
- In the next step, we update $k_i$ by a gradient method
- Update $\Omega$, update $k_i$
Idea

- In the first computation of the TD, i.e. when $\Omega = \emptyset$, we do not need to know $k_i$:

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In the first computation of the TD, i.e. when $\Omega = \emptyset$, we do not need to know $k_i$:

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where $u = u_{inc}$ and $w = \int_{\Gamma_{meas}} G_k(x - y) (u_{meas} - u)(y) dl_y$

We compute the TD taking $k_i^0 \approx k_e$ to get an initial guess $\Omega_1$

In the next step, we update $k_i$ by a gradient method

Update $\Omega$, update $k_i$
Piecewise homogeneous material
Heterogeneous materials

Original

Reconstruction

María–Luisa Rapún

Topological derivatives for inverse problems
Outline

1. Inverse scattering problems

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   - Conclusions
A non–monotone iterative method

The previous scheme was monotone: it generates a sequence of increasing approximations. If at some step a spurious region is included, it cannot be removed.

Non–monotone schemes need a generalized definition of topological derivative:

- \( x \in \mathbb{R}^n \setminus \Omega_{ap} \),

\[
D_T(x, \mathbb{R}^n \setminus \Omega_{ap}) = \lim_{\varepsilon \to 0} \frac{J(\mathbb{R}^n \setminus \Omega_{ap}) - J(\mathbb{R}^n \setminus (\Omega_{ap} \cup B_\varepsilon(x)))}{\text{Vol}(B_\varepsilon(x))}
\]

- \( x \in \Omega_{ap} \),

\[
D_T(x, \mathbb{R}^n \setminus \Omega_{ap}) = \lim_{\varepsilon \to 0} \frac{J(\mathbb{R}^n \setminus \Omega_{ap}) - J(\mathbb{R}^n \setminus (\Omega_{ap} \setminus B_\varepsilon(x)))}{\text{Vol}(B_\varepsilon(x))}
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$$D_T(x, \mathbb{R}^n \setminus \Omega_{ap}) = \lim_{\varepsilon \to 0} \frac{J(\mathbb{R}^n \setminus \Omega_{ap}) - J((\Omega_{ap} \cup B_{\varepsilon}(x)))}{Vol(B_{\varepsilon}(x))}$$

2. $x \in \Omega_{ap}$,

$$D_T(x, \mathbb{R}^n \setminus \Omega_{ap}) = \lim_{\varepsilon \to 0} \frac{J(\mathbb{R}^n \setminus \Omega_{ap}) - J((\Omega_{ap} \setminus B_{\varepsilon}(x)))}{Vol(B_{\varepsilon}(x))}$$
Algorithm

1. Compute the TD when $\Omega = \emptyset$ (as before)
2. Take $\Omega_1 = \{x, D_T(x, \mathbb{R}^n) < -C_1\}$, $C_1 > 0$ (as before)
3. For $j=1:j_{\text{max}}$
   - Compute the TD in $\mathbb{R}^n$ when $\Omega = \Omega_j$
   - Select $\Omega_{j+1}$

   \[ \text{if } x \in \mathbb{R}^n \setminus \Omega_j \text{ and } TD < -C_j \implies x \in \Omega_{j+1} \]

   \[ \text{if } x \in \Omega_j \text{ and } TD > C'_j \implies x \notin \Omega_{j+1} \]
Reconstruction of an annular region. Points are added or removed at each step. The hole is recovered after 6 steps.
1. Inverse scattering problems

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Other problems and boundary conditions

- Dirichlet (sound–soft objects): \( u|_{\Gamma} = 0 \)

- Neumann (sound–hard objects): \( \partial_n u|_{\Gamma} = 0 \)

\[
D_T(x) = \text{Re}(2\nabla u(x) \nabla w(x) - \lambda^2 \epsilon u(x) w(x))
\]
Other problems and boundary conditions

- **Dirichlet (sound–soft objects):** $u|_{\Gamma} = 0$

- **Neumann (sound–hard objects):** $\partial_n u|_{\Gamma} = 0$

$$D_T(\mathbf{x}) = \text{Re}(2 \nabla u(\mathbf{x}) \nabla w(\mathbf{x}) - \lambda_e^2 u(\mathbf{x}) w(\mathbf{x}))$$
Other problems and boundary conditions

- General transmission problems (penetrable objects):
  \[ u^+ = u^-, \quad \alpha_+ \partial_n u^+ = \alpha_- \partial_n u^- \]

- Non–constant parameters (heterogeneous materials):
  \[ k = k(x), \quad \alpha = \alpha(x) \]
Non–steady problems

- We combine topological derivatives in space with Laplace transforms in time.
- The observation of the system over an interval of time allows for better reconstructions than in the time–harmonic case.
Time–harmonic simulation

Fig. 2. Topological derivative with data from 24 incident sources (\(\bullet\)) at 25 sampling points (\(\times\)) for different frequencies: (a) \(\omega = 2\), (b) \(\omega = 4\), (c) \(\omega = 7\), (d) \(\omega = 10\).
Fig. 5. Topological derivative of the cost functional (20) with weight function $j_{t_k}$: (a) 2 source points (••), 3 observation points (×), $N = 60$ for times in $[0.2, 0.45]$; (b) 3 source points (••), 4 observation points (×), $N = 30$ for times in $[0.2, 0.45]$; (c) 4 source points (••), 5 observation points (×), $N = 18$ for times in $[0.2, 0.45]$; (d) 5 source points (••), 6 observation points (×), $N = 12$ for times in $[0.2, 0.45]$. 

Time–dependent simulation
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The topological derivative is a powerful tool to solve inverse problems dealing with shape reconstruction in different areas: acoustics, photothermal problems,...

The TD gives a good approximation of the number, size and location of the objects buried in a medium.

Iterative procedures improve their shape, and catch small objects, if missed in the first trial.

The algorithm for shape reconstruction can be combined with a gradient method to recover both shapes and parameters.
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Conclusions

- The topological derivative is a powerful tool to solve inverse problems dealing with **shape reconstruction in different areas**: acoustics, photothermal problems, ...
- The TD gives a **good approximation** of the number, size and location of the objects buried in a medium.
- **Iterative procedures improve** their shape, and catch small objects, if missed in the first trial.
- The algorithm for shape reconstruction can be combined with a gradient method to recover both **shapes and parameters**.