Part 3: Inverse Boundary Value Problem for Heat Operators

(Size estimate and dynamical probe method for heat operators)

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ICMAT, Madrid, May 11, 2011

Contents

Size estimation in thermography

Joint work with M. Di Cristo, Y. Lei

Notations

$\Omega \subset \mathbb{R}^n\,(n = 2, 3)$: bounded domain with Lipschitz boundary $\partial\Omega$

$D \subset \Omega$: measurable set (inclusion), $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$: connected

$\gamma := 1 + (k-1)\chi_D$: conductivity, $k(> 1)$: constant (for simplicity)

$d_0 := \mathrm{dist}(D, \partial\Omega) > 0$

Γ^D, $\Gamma^N \subset \partial\Omega$:open sets such that

$$\Gamma^D \neq \emptyset,\, \overline{\Gamma^D} \cup \overline{\Gamma^N} = \partial\Omega,\, \Gamma^D \cap \Gamma^N = \emptyset.$$

Further, we assume that $\partial\Gamma^D$, $\partial\Gamma^N$ are Lipschitz if $n = 3$.

Mathematical framework

It is well known that, for any given $\varphi \in L^2((0,T);\overline{H}^{-1/2}(\Gamma^N))$, there exists a unique solution $u \in W(\Omega_T)$

i.e. $u \in L^2(0,T);H^1(\Omega)),\ \partial_t u \in L^2((0,T);H^{-1}(\Omega))$

to the mixed prob.:

$$
\begin{cases}
\partial_t u - \nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega_T \\
\partial_\nu u = \varphi \text{ on } \Gamma^N \times (0,T) \\
u = 0 \,\text{on}\, \Gamma^D \times (0,T) \\
u = 0 \text{ on } \Omega \times \{0\},
\end{cases}
$$

where $\Omega_T := \Omega \times (0,T)$.

Mathematical framework continued

In a similar way, we can consider a unique solution to the backward heat operator $\partial_t + \nabla \cdot \gamma \nabla$ in Ω_T and initial condition at $t = T$.

We denote the solution by u'.

That is

$$\begin{cases} \partial_t u' + \nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega_T \\ \partial_\nu u' = \varphi \text{ on } \Gamma^N \times (0, T) \\ u' = 0 \, \text{on} \, \Gamma^D \times (0, T) \\ u' = 0 \text{ on } \Omega \times \{T\}, \end{cases}$$

Mathematical framework continued

For $\alpha := T/5 > 0$, let $\eta_\alpha \in C^\infty([0,\infty))$ be such that

$$0 \le \eta_\alpha \le 1,\ \eta_\alpha(t) = 0\,(0 \le t \le \alpha/2),\ \eta_\alpha(t) = 1\,(t \ge \alpha).$$

Also, let $p \in H^1(\Omega)$ satisfy

$$\begin{cases} \nabla \cdot \gamma \nabla p = 0 \text{ in } \Omega \\ p = 0 \text{ on } \Gamma^D. \end{cases}$$

.

Then, the solution u to the previous mixed prob. with $\varphi = \eta_\alpha \partial_\nu p|_{\Gamma^N}$,

$$u(\cdot,t) \approx \eta_\alpha(t) p(\cdot)\ \ (t \gg 1)\,(\text{exponentially}).$$

$u'(\cdot,t)$ has a similar property for the Neumann data $\zeta_\alpha(t)\partial_\nu p|_{\Gamma^N}$ with $\zeta_\alpha(t) := \eta_\alpha(T-t)$.

We further assume $\partial_\nu p|_{\Gamma^N}$ is not identically zero.

Inverse problem for size estimation

The inverse problem for estimating the size $|D|$ of an unknown inclusion D is as follows.

Inverse problem

Suppose Ω is known, but D, k are unknown. Then, given a measurement $\{\varphi = \partial_\nu u|_{\Gamma^N \times (0,T)}, u|_{\Gamma^N \times (0,T)}\}$ (i.e. a Cauchy data on $\Gamma^N \times (0,T)$), estimate $|D|$.

Let

$$K := \frac{\int_{\Gamma^N_{(T_1,T_2)}} \partial_\nu u_0'(u_0 - u)}{\int_{\Gamma^N_{(T_1,T_2)}} \partial_\nu u_0' u_0} \quad (\text{indicator function}),$$

where u_0, u_0' are those u, u' for the case $D = \emptyset$ and $T_1 = (2T)/5$, $T_2 = (3T)/5$.

Theorem

Further let

$$F(\psi) := \|\psi\|_{\overline{H}^{-1/2}(\Gamma^N)} / \|\psi\|_{\overline{H}^{-1}(\Gamma^N)} \text{ (frequency function)}.$$

Then, we have the following theorem for estimating the size of D.

Theorem

There exist constants $C_1 > 0$ depending only on d_0, $|\Omega|$ and $C_2 > 0$, $p > 1$ depending only on d_0, $|\Omega|$, $F(\partial_\nu p|_{\Gamma^N})$ such that

$$\frac{1}{k-1} C_1 K \preceq |D| \preceq (\frac{k}{k-1})^{1/p} C_2 K^{1/p},$$

where $\preceq$ denotes $\leq$ modulo an exponentially small term for $T \gg 1$.

Remark

(i) We have a similar result for anisotropic conductors.

(ii) G. Alessandrini and E. Rosset ('98) gave a similar result for stationary heat conductors.

Ingredients of the proof

$$\frac{k-1}{k}\int_D |\nabla p_0|^2 \preceq \int_{\Gamma^N_{(T_1,T_2)}} \partial_\nu u_0'(u_0-u) \preceq (k-1)\int_D |\nabla p_0|^2, \tag{1}$$

$$\sup_D |\nabla p_0| \leq C\,(\int_{\Gamma^N} p_0 \partial_\nu p_0)^{1/2} \text{ with } p_0 = p \text{ when } D=\emptyset. \tag{2}$$

(by standard est. for ellip. eq. and Poincaré ineq.)

$$\int_D |\nabla p_0|^2 \geq C(\int_{\Gamma^N} p_0 \partial_\nu p_0)|D|^p, \tag{3}$$

(by Lipschitz propagation of smallness ($\Leftarrow$ three sphere ineq.) and A_p property ($\Leftarrow$ doubling ineq.))

· (1), (2) imply the lower estimate.

· (1), (3) imply the upper estimate.

Numerical examples

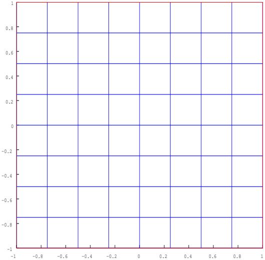


Figure 1: representation of the grid

Taking $k = 9$, we will compute the maximum and minimum of the indicator function :

$$I_T(\varphi) = \frac{\int_{\Gamma^N_{(T_1,T_2)}} \partial_\nu u_0'(u_0 - u)}{\int_{\Gamma^N_{(T_1,T_2)}} \partial_\nu u_0' u_0},$$

In all the figures below $--o--$ and $--+--$ mean the maximum value and minimum value of the indicator function, respectively.

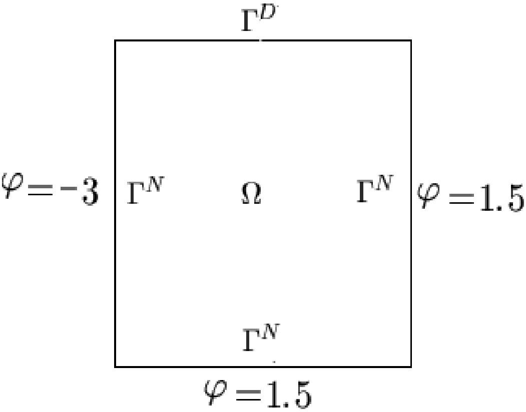


Figure 2: boundary data

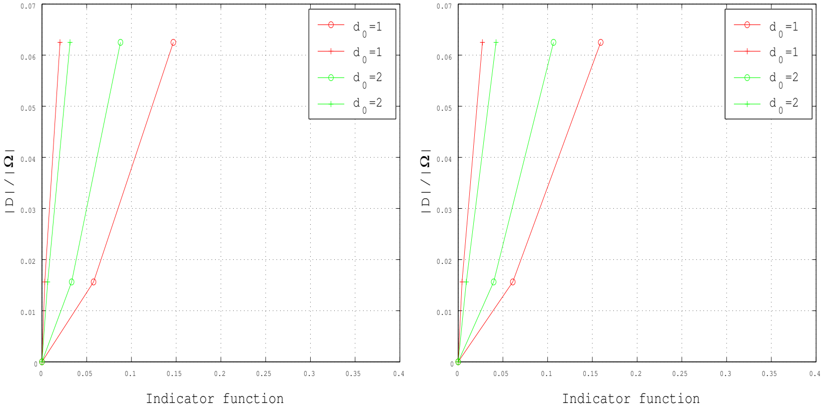


Figure 3: Influence of d_0 for square inclusion: $T = 2$, right: $T = 3$

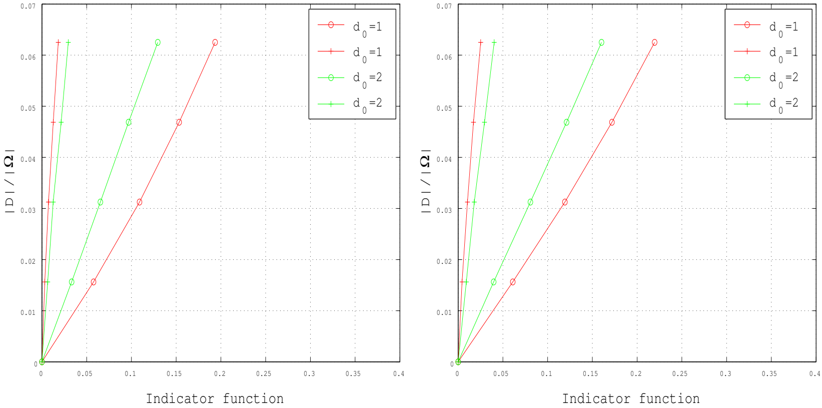


Figure 4: Influence of d_0 for inclusion of general shape: $T = 2$, right: $T = 3$

• **The behavior of the maximum value is more sensitive.**

• **If the inclusion is closer to the boundary, then the maximum value of indicator function can be achieved by smaller size inclusions.**

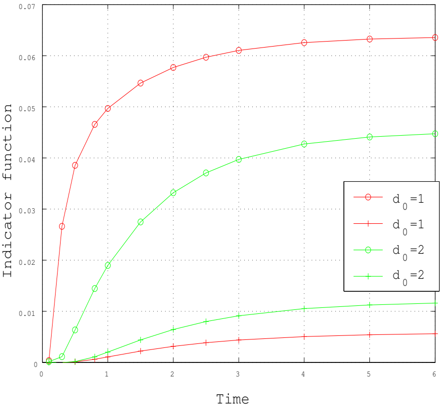


Figure 5: T=[0:6], square inclusion.

The bounds of the indicator function become more stable as T increases.

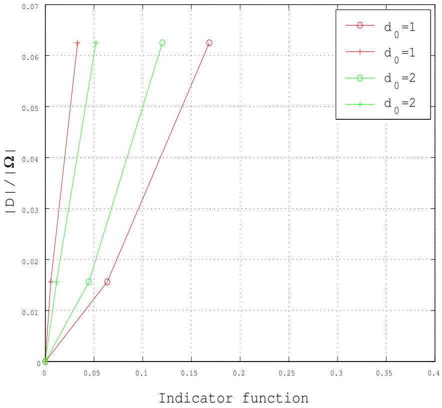


Figure 6: Influence of d_0 for square inclusion, in the time $T = 6$.

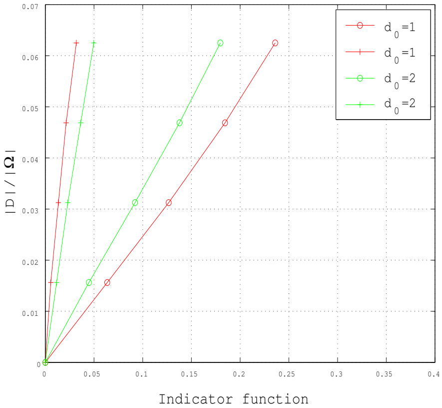


Figure 7: Influence of d_0 for inclusion of general shape, in the time $T = 6$.

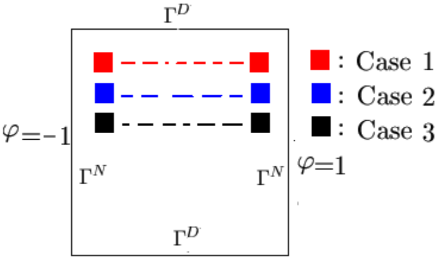


Figure 8: different location of the inclusion

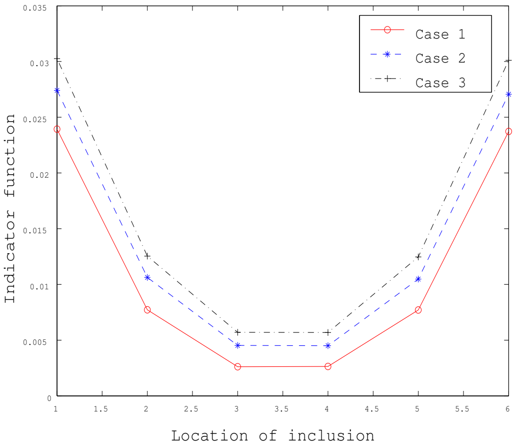


Figure 9: value of the indicator function related to the location of the inclusion

If the inclusion is closer to the cooling boundary, then the value of the indicator function becomes smaller.

Dynamical probe method

Important preliminary estimates

Joint work with J.Fan, K.Kim and S.Nagayasu

Domain and operators

$\Omega \subset \mathbb{R}^n$: b'dd domain (heat conductor), $\partial\Omega : C^2$.

$\gamma(x) = (\gamma_{jk}(x))$: defined a.e. in Ω, symm, pos. def. (conductivity)

$$\lambda|\xi|^2 \leq \sum \gamma_{jk}(x)\xi_j\xi_k \leq \Lambda|\xi|^2.$$

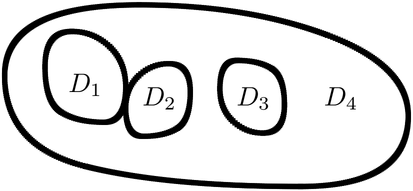
We further assume the followings for Ω and γ.

Domain and operators (continued)

Let $\Omega = \left(\bigcup_{m=1}^{L} \overline{D_m} \right) \setminus \partial\Omega$.

$\gamma^{(m)} \in C^{\mu}(\overline{D_m})$ $(0 < \mu < 1)$, $\gamma(x) = \gamma^{(m)}(x)$ $(x \in D_m)$.

Each separated D_m is of $C^{1,\alpha}$ smooth with $0 < \alpha \leq 1$ and non-separated one is the limit of the separated one.



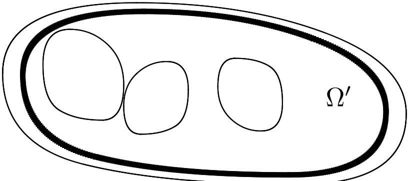
Gradient estimate

Theorem 1 (Fan, Kim, Nagayasu and N**)**

Let $\Omega' \Subset \Omega$, $0 < t_0 < T$. *Any sol* u *to (P):* $\partial_t u - \nabla \cdot (\gamma \nabla u) = 0$ *in* $\Omega \times (0, T)$ *has the following interior regularity est:*

$$\text{sup}_{t_0<t<T} \|u(\cdot, t)\|_{C^{1,\alpha'}(\overline{\Omega' \cap D_m})} \le C \|u\|_{L^2(\Omega \times (0,T))},$$

where $0 < \alpha' \le \text{min}(\mu, \frac{\alpha}{2(\alpha+1)})$ *and* C *is indep of the dist between inclusions.*



Gradient estimate of fundamental solution

By applying our main theorem and a scaling argument, we obtain pointwise grad. est. for $0 < t - s < T$,

$$|\nabla_x E(x,t;y,s)| \leq \frac{C_T}{(t-s)^{\frac{n+1}{2}}} \exp\left(-\frac{c|x-y|^2}{t-s}\right)$$

of fund sol $E(x,t;y,s)$ for $\partial_t - \nabla \cdot (\gamma \nabla \cdot)$.

Remarks

(i) We can obtain a similar estimate for non-homog parabolic eq :

$$\partial_t u - \nabla \cdot (\gamma \nabla u) = g + \nabla \cdot f.$$

(ii) Recently, Haigang Li-Yanyan Li extended the result to time dependent parabolic systems. (preprint) But the inclusions are independent of time.

(iii) The time dependent inclusions case is an open problem.

(iv) The elliptic case was proved by Li-Vogelius for scalar equations and Li-Nirenberg ([LN]) for systems, which answered to the Babuška's conjecture. Babuška et al (1999) numerically observed that the gradient est of sol is indep of the distances between inclusions.

Idea of Proof

Idea of proof:

- Some interior estimates (Lemma).

· (ref. Ladyzenskaja-Rivkind-Uralceva)

- Apply [LN] to (P).

Proof

Lemma 2

Let $\widetilde{\Omega} \Subset \Omega$, $0 < t_0 < T$. *Any sol* u *to*

$$\partial_t u - \nabla \cdot (A \nabla u) = 0 \text{ in } \Omega \times (0, T) =: Q$$

has the following estimates:

$$\begin{aligned} \sup_{t_0 < t < T} \|u(\cdot, t)\|_{L^2(\widetilde{\Omega})} &\le C \|u\|_{L^2(Q)} \text{ (standard)}, \\ \|u\|_{L^\infty(\widetilde{\Omega} \times (t_0, T))} &\le C \|u\|_{L^2(Q)} \text{ (Di Giorgi's arg.)}, \\ \|u_t\|_{L^2(\widetilde{\Omega} \times (t_0, T))} &\le C \|u\|_{L^2(Q)} \text{ ([LRU])}. \end{aligned}$$

Remark 3

(i) *This lemma holds for* $A \in L^\infty$.
(ii) LRU=Ladyzenskaja-Rivkind-Uralceva.

Proof

Let $\widetilde{\Omega}_3 \Subset \widetilde{\Omega}_2 \Subset \widetilde{\Omega}_1 \Subset \widetilde{\Omega}_0 := \Omega$, $0 < \delta_1 < \delta_2 < T$. Then

$$(*)\ \sup_{\delta_2 < t < T} \|u(\cdot, t)\|_{L^2(\widetilde{\Omega}_2)} \le C\|u\|_{L^2(Q)},$$

$$(**)\ \|u_t\|_{L^2(\widetilde{\Omega}_1 \times (\delta_1, T))} \le C\|u\|_{L^2(Q)}.$$

Since $\partial_t u_t - \nabla \cdot (A \nabla u_t) = 0$, we have

$$(***)\ \|u_t\|_{L^\infty(\widetilde{\Omega}_2 \times (\delta_2, T))} \le C\|u_t\|_{L^2(\widetilde{\Omega}_1 \times (\delta_1, T))} \le C'\|u\|_{L^2(Q)}\ .$$

Now we fix $t \in (\delta_2, T)$: $\nabla \cdot (A \nabla u) = u_t \in L^\infty(\widetilde{\Omega}_2)$.

Proof

Then by [LN], we have

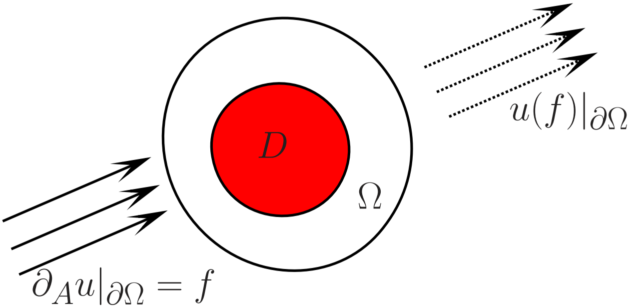
$$\|u(\cdot,t)\|_{C^{1,\alpha'}(\overline{D}_m\cap\widetilde{\Omega}_3)}$$
$$\leq C\Big(\|u(\cdot,t)\|_{L^2(\widetilde{\Omega}_2)}+\|u_t(\cdot,t)\|_{L^\infty(\widetilde{\Omega}_2)}\Big).$$

Taking $\sup_{\delta_2<t<T}$, we have by (*), (**), (***),

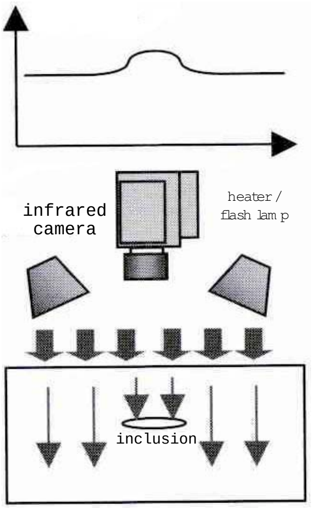
$$\sup_{\delta_2<t<T}\|u(\cdot,t)\|_{C^{1,\alpha'}(\overline{D}_m\cap\widetilde{\Omega}_3)}$$
$$\leq C\Big(\sup_{\delta_2<t<T}\|u(\cdot,t)\|_{L^2(\widetilde{\Omega}_2)}+\|u_t\|_{L^\infty(\widetilde{\Omega}_2\times(\delta_2,T))}\Big)$$
$$\leq C\|u\|_{L^2(Q)}.$$

Thermography and Dynamical Probe Method

Active thermography



Principle of active thermography



Dynamical probe method for anisotropic heat conductors

Joint work with K.Kim

Mixed problem (set up)

$\Omega \subset \mathbb{R}^n \ (1 \le n \le 3)$: bounded domain,

$$\partial\Omega \ : \ C^2 \, (n = 2, 3), \quad \partial\Omega = \overline{\Gamma^D} \cup \overline{\Gamma^N},$$

where Γ^D, Γ^N are open subsets of $\partial\Omega$ such that $\Gamma^D \cap \Gamma^N = \emptyset$ and $\partial\Gamma^D$, $\partial\Gamma^N$ are C^2 if they are nonempty.

$D \subset \Omega$: open set (**separated inclusion(s)**), $\overline{D} \subset \Omega$,
$\partial D : C^{1,\alpha} \ (0 < \alpha \le 1)$, $\Omega \setminus \overline{D}$: connected.

Heat conductivity:

$$\gamma(x) = A(x) + (\tilde{A}(x) - A(x))\chi_D \ : \text{positive definite for each } x \in \overline{\Omega},$$

where $A, \ \tilde{A} \in C^1(\overline{\Omega})$ are positive definite and $\tilde{A} - A$ is always positive definite or negative in a neigh. of ∂D, χ_D is the char func of D.

$H^p(\partial\Omega), H^{p,q}(\Omega \times (0,T))$: usual Sobolev spaces

$$(p, q \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \text{ or } p = \frac{1}{2})$$

ex. For $p, q \in \mathbb{Z}_+$, $g \in H^{p,q}(\Omega \times (0,T))$ iff

$$||g||_{H^{p,q}(\Omega\times(0,T))} := \left(\sum_{\substack{|\alpha|+2k\leq p \\ k\leq q}} \int_{\Omega\times(0,T)} \left|\partial_x^\alpha \partial_t^k g\right|^2 dtdx \right)^{1/2} < \infty$$

$$L^2((0,T); H^p(\partial\Omega)) := \{f \;;\; \int_0^T ||f(\cdot,t)||^2_{H^p(\partial\Omega)} dt < \infty\}$$

Mixed problem (forward problem)

Given $f \in L^2((0,T); \overline{H}^{\frac{1}{2}}(\Gamma^D))$, $g \in L^2((0,T); \dot{H}^{-\frac{1}{2}}(\overline{\Gamma^N}))$,

(?) $\exists!$ weak solution
$u = u(f,g) \in W(\Omega_T) := \{u \in H^{1,0}(\Omega_T),\ \partial_t u \in L^2((0,T); H^1(\Omega)^*)\}$:

$$
\begin{cases}
\mathcal{P}_D u(x,t) := \partial_t u(x,t) - \mathrm{div}_x(\gamma(x)\nabla_x u(x,t)) = 0 \text{ in } \Omega_T \\
u(x,t) = f(x,t) \text{ on } \Gamma_T^D,\ \partial_A u(x,t) := \nu \cdot A\nabla u(x,t) = g(x,t) \text{ on } \Gamma_T^N \\
u(x,0) = 0 \text{ for } x \in \Omega,
\end{cases}
$$

where ν is the outer unit normal of $\partial\Omega$,
$\overline{H}^{\frac{1}{2}}(\Gamma^D)$, $\dot{H}^{-\frac{1}{2}}(\overline{\Gamma^N})$ are Hörmander's notations of Sobolev sp,

$\Omega_T = \Omega_{(0,T)} := \Omega \times (0,T)$, $\partial\Omega_T = \partial\Omega_{(0,T)} := \partial\Omega \times (0,T)$.
(cylindrical sets)

This is a well-posed problem.

Measured data

Neumann-to-Dirichlet map Λ_D:

For fixed $f \in L^2((0,T); \overline{H}^{\frac{1}{2}}(\Gamma^D))$, define

$$\begin{aligned} \Lambda_D : L^2((0,T); \dot{H}^{-\frac{1}{2}}(\overline{\Gamma^N})) &\to L^2((0,T); \overline{H}^{\frac{1}{2}}(\Gamma^N)) \\ g &\mapsto u(f,g)|_{\Gamma_T^N}. \end{aligned}$$

Inverse boundary value problem

Reconstruct the unknown inclusion D from Λ_D.

Known results I

∗ H. Bellout (1992): Local uniqueness and stability.

∗ A. Elayyan and V. Isakov (1997): Global uniqueness using the localized *Neumann-to-Dirichlet map*.

∗ M. Di Cristo and S. Vessella (2010): Optimal stability estimate (i.e. log type stability estimate) even for time dependent inclusions.

∗ Y. Daido, H. Kang and G. Nakamura (2007) (Inverse Problems) : Introduced the dynamical probing method for 1-D case.

∗ Y. Daido, Y. Lei, J. Liu and G. Nakamura (2009) (Applied Mathematics and Computation) Numerical implementations of 1-D dynamical probe method for non-stationary heat equation.

Known results II

∗ Y. Lei, K. Kim and G. Nakamura (2009) (Journal of Computational Mathematics) Theoretical and numerical studies for 2-D dynamical probe method.

∗ M. Ikehata and M. Kawashita (2009) (Inverse Problems) Extracted some geometric information of an unknown cavity using CGO solution and asymptotic analysis.

∗ V. Isakov, K. Kim and G. Nakamura (2010) (Ann. Scola Superior di Pisa) Gave the theoretical basis of dynamical probe method.

Dynamical probe method (fundamental solutions)

For $(y,s),(y',s')\in\mathbb{R}^n\times\mathbb{R}$, $(x,t)\in\Omega_T$,

$\Gamma(x,t;y,s)$: fundamental solution of $\mathcal{P}_\emptyset := \partial_t - \nabla\cdot(A(x)\nabla)$

$\Gamma^*(x,t;y',s')$: fundamental solution of $\mathcal{P}^*_\emptyset := -\partial_t - \nabla\cdot(A(x)\nabla)$

$G(x,t;y,s)$, $G^*(x,t;y',s')$:

$$\begin{cases} \mathcal{P}_\emptyset G(x,t;y,s) = \delta(x-y)\delta(t-s) \text{ in } \Omega_T, \\ G(\cdot,\cdot;y,s) = 0 \text{ on } \Gamma_T^D, \\ G(x,t;y,s) = 0 \text{ for } x\in\Omega,\ t\le s \end{cases}$$

$$\begin{cases} \mathcal{P}^*_\emptyset G^*(x,t;y',s') = \delta(x-y)\delta(t-s') \text{ in } \Omega_T, \\ G^*(\cdot,\cdot;y',s') = 0 \text{ on } \Gamma_T^D, \\ G^*(x,t;y',s') = 0 \text{ for } x\in\Omega,\ t\ge s' \end{cases}$$

$G(x,t;y,s)-\Gamma(x,t;y,s)$, $G^*(x,t;y',s')-\Gamma^*(x,t;y',s')$: C^1_t, C^2_x in Ω_T.

Dynamical probe method (Runge's approximation)

$\exists$ $\{v^{0j}_{(y,s)}\}$, $\{\psi^{0j}_{(y',s')}\} \in H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ for $\forall\varepsilon > 0$ s.t.

$$\begin{cases} \mathcal{P}_\emptyset v^{0j}_{(y,s)} = 0 & \text{in } \Omega_{(-\varepsilon,T+\varepsilon)}, \\ v^{0j}_{(y,s)} = 0 & \text{on } \Gamma^D \times (-\varepsilon, T+\varepsilon), \\ v^{0j}_{(y,s)}(x,t) = 0 & \text{if } -\varepsilon < t \le 0, \\ v^{0j}_{(y,s)} \to G(\cdot,\cdot;y,s) & \text{in } H^{2,1}(U \times (-\varepsilon', T+\varepsilon')) \text{ as } j \to \infty, \end{cases}$$

$$\begin{cases} \mathcal{P}^*_\emptyset \psi^{0j}_{(y',s')} = 0 & \text{in } \Omega_{(-\varepsilon,T+\varepsilon)}, \\ \psi^{0j}_{(y',s')} = 0 & \text{on } \Gamma^D \times (-\varepsilon, T+\varepsilon), \\ \psi^{0j}_{(y',s')}(x,t) = 0 & \text{if } T \le t < T+\varepsilon, \\ \psi^{0j}_{(y',s')} \to G^*(\cdot,\cdot;y',s') & \text{in } H^{2,1}(U \times (-\varepsilon', T+\varepsilon')) \text{ as } j \to \infty \end{cases}$$

for $0 < \forall\varepsilon' < \varepsilon$, $\forall U \subset \Omega$: open s.t.

$\overline{U} \subset \Omega$, $\Omega \setminus \overline{U}$: connected, ∂U : Lipschitz, $\overline{U} \not\ni y$, y',
and $-\varepsilon < s$, $s' < T+\varepsilon$.

Dynamical probe method (Runge approx funcs)

Let v, ψ satisfy

$$\begin{cases} \mathcal{P}_\emptyset v = 0 \text{ in } \Omega_T, \\ v = f \text{ on } \Gamma_T^D, \\ \partial_A v = 0 \text{ on } \Gamma_T^N, \\ v(x,0) = 0 \text{ for } x \in \Omega, \end{cases} \qquad \begin{cases} \mathcal{P}_\emptyset^* \psi = 0 \text{ in } \Omega_T, \\ \psi = 0 \text{ on } \Gamma_T^D, \\ \partial_A \psi = g \text{ on } \Gamma_T^N, \\ \psi(x,T) = 0 \text{ for } x \in \Omega. \end{cases}$$

For $j = 1, 2, \cdots$, we define

$$\begin{cases} v_{(y,s)}^j := v + v_{(y,s)}^{0j} \to V_{(y,s)} := v + G(\cdot,\cdot;y,s) \\ \psi_{(y',s')}^j := \psi + \psi_{(y',s')}^{0j} \to \Psi_{(y',s')} := \psi + G^*(\cdot,\cdot;y',s'). \end{cases}$$

in $H^{2,1}(U_T)$ as $j \to \infty$.

$\{v_{(y,s)}^j\}, \{\psi_{(y',s')}^j\}$: *Runge's approximation functions*

Pre-indicator function

Definition 4

$(y, s), (y', s') \in \Omega_T$

$\{v^j_{(y,s)}\}, \{\psi^j_{(y',s')}\} \subset W(\Omega_T)$: *Runge's approximation functions*

Pre-indicator function :

$$I(y', s'; y, s)$$
$$= \lim_{j\to\infty} \int_{\Gamma_T^N} \left[\partial_A v^j_{(y,s)}|_{\Gamma_T^N}\ \psi^j_{(y',s')}|_{\Gamma_T^N} - \Lambda_D(\partial_A v^j_{(y,s)})|_{\Gamma_T^N}\ \partial_A \psi^j_{(y's')}|_{\Gamma_T^N}\right]$$

whenever the limit exists.

Reflected solution

Lemma 5

$y \notin \overline{D}$, $0 < s < T$, $\{v^j_{(y,s)}\} \subset W(\Omega_T)$: *Runge's approximation functions,*

$u^j_{(y,s)} := u(f, \partial_A v^j_{(y,s)}|_{\Gamma^N_T})$, $w^j_{(y,s)} := u^j_{(y,s)} - v^j_{(y,s)}$

Then, $w^j_{(y,s)}$ *has a limit* $w_{(y,s)} \in W(\Omega_T)$ *satisfying*

$$\begin{cases} \mathcal{P}_D w_{(y,s)} = \mathrm{div}_x((\tilde{A} - A)\chi_D \nabla_x V_{(y,s)}) \text{ in } \ \Omega_T, \\ w_{(y,s)} = 0 \text{ on } \Gamma^D_T, \ \partial_A w_{(y,s)} = 0 \text{ on } \ \Gamma^N_T \\ w_{(y,s)}(x,0) = 0 \text{ for } \ x \in \Omega. \end{cases}$$

$w_{(y,s)}$: reflected solution

Representation formula

Theorem 6

For $y, y' \notin \overline{D}$, $0 < s,\ s' < T$ *such that* $(y, s) \neq (y', s')$, *the pre-indicator function* $I(y', s'; y, s)$ *has the representation formula in terms of the reflected solution* $w_{(y,s)}$ *:*

$$I(y', s'; y, s) = -w_{(y,s)}(y', s') - \int_{\partial\Omega_T} w_{(y,s)} \partial_A \Gamma^*(\cdot, \cdot; y', s') d\sigma dt$$

Main result (indicator function)

Definition 7

$C := \{c(\lambda)\,;\, 0 \leq \lambda \leq 1\}$ *: non-selfintersecting* C^1 *curve in* $\overline{\Omega}$*,*
$c(0), c(1) \in \partial\Omega$ *(We call this* C *a needle.)*

Then, for each $c(\lambda) \in \Omega$ *and each fixed* $s \in (0, T)$*,*

indicator function (mathematical testing machine)

$$J(c(\lambda), s) := \lim_{\epsilon \downarrow 0} \limsup_{\delta \downarrow 0} |I(c(\lambda - \delta), s + \epsilon^2; c(\lambda - \delta), s)|$$

whenever the limit exists.

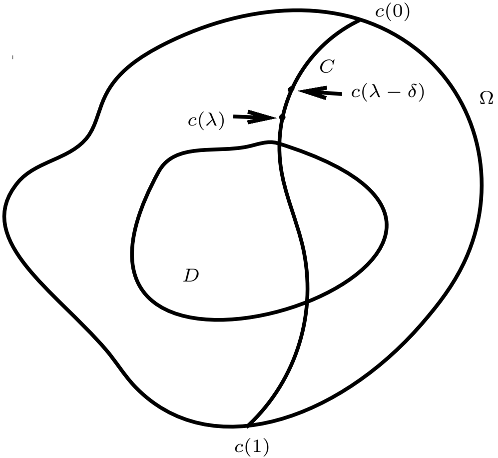


Figure 10: Domains Ω, D, and a curve C

Seperated inclusions case result (theorem)

Theorem 8

Let D consist of separated inclusions, and C, $c(\lambda)$ be as in the definition above.

Fix $s \in (0, T)$.

(i) $C \subset \Omega \setminus \overline{D}$ *except* $c(0)$ *and* $c(1)$

$\Longrightarrow J(c(\lambda), s) < \infty$ for all λ, $0 \leq \lambda \leq 1$

(ii) $C \cap \overline{D} \neq \emptyset$

λ_s $(0 < \lambda_s < 1)$ *s.t.* $c(\lambda_s) \in \partial D$, $c(\lambda) \in \Omega \setminus \overline{D}$ $(0 < \lambda < \lambda_s)$

$\Longrightarrow$

$$\lambda_s = \sup\{\, 0 < \lambda < 1\,;\, J(c(\lambda'), s) < \infty \quad \text{for any} \quad 0 < \lambda' < \lambda\,\}.$$

Remark :

(i) A numerical realization of this reconstruction scheme has been done for isotropic conductivities.

(ii) If $\Gamma^D \neq \emptyset$ and $f(\cdot, t) = 0 = g(\cdot, t)$ $(t > T')$ with $0 < T' < T$, then $u(f, g)$ has the decaying property. That is $u(f, g)$ decays exponentially after $t = T'$. Hence, in this case, we can guarantee the exponential decay of the temperature after the experiment.

Proof of Theorem 6:

Consider only the case $n = 3$ in the rest of the arguments.

First, we recall the previous two facts.

(i) $w_{(y,s)} \in W(\Omega_T)$: solution to

$$\begin{cases} \mathcal{P}_D w_{(y,s)} = \mathrm{div}_x((\tilde{A} - A)\chi_D \nabla_x V_{(y,s)}) \text{ in } \Omega_T, \\ w_{(y,s)} = 0 \text{ on } \Gamma_T^D,\ \partial_A w_{(y,s)} = 0 \text{ on } \Gamma_T^N \\ w_{(y,s)}(x,0) = 0 \text{ for } x \in \Omega. \end{cases}$$

(ii)

$$I(y, s'; y, s) = -w_{(y,s)}(y', s') - \int_{\partial\Omega_T} w_{(y,s)} \partial_A \Gamma^*(\cdot, \cdot; y', s') d\sigma dt$$

If $y = c(\lambda) \notin \partial D$, it is easy to see the indicator function is finite at y.
So, let's consider the case $y \in \partial D$.

Setup

Note that

$$\mathcal{P}_D w_{(y,s)} = \mathrm{div}_x((\tilde{A} - A)\chi_D \nabla_x V_{(y,s)}) \quad \text{in } \Omega_T$$

Hence,

$E(x,t;y,s) := w_{(y,s)}(x,t) + V_{(y,s)}(x,t)$
($\Rightarrow$ fundamental solution for $\mathcal{P}_D$.)

Let $P = c(\lambda_0) \in \partial D$ for some λ_0

$x = y = c(\lambda_0 - \delta) \in C \setminus \overline{D}$ for $\delta > 0$.

$\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\Phi(P) = O$ ($C^{1,\alpha}$ diffeomorphism, $0 < \alpha \leq 1$),

$\Phi(D) \subset \mathbb{R}^3_- = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3; \xi_3 < 0\}$,

Jacobi matrix of Φ at P = identity matrix.

Let

$$
\begin{aligned}
E &: \partial_t - \nabla\cdot((A(x) + (\tilde{A}(x) - A(x))\chi_D)\nabla) \\
\Gamma_P &: \partial_t - \nabla\cdot((A(x) + (\tilde{A}(P) - A(x))\chi_D)\nabla) \\
\Gamma_- &: \partial_t - \nabla\cdot((A(\Phi^{-1}(\xi)) + (\tilde{A}(P) - A(\Phi^{-1}(\xi)))\chi_-)\nabla) \\
\Gamma_-^0 &: \partial_t - \nabla\cdot((A(P) + (\tilde{A}(P) - A(P))\chi_-)\nabla) \\
\Gamma^0 &: \partial_t - \nabla\cdot(A(P)\nabla) \\
\Gamma &: \partial_t - \nabla\cdot(A(x)\nabla).
\end{aligned}
$$

be the fund. sol. and corresponding operators, where χ_- is the characteristic function of the space $\mathbb{R}^3_-$.

Main part of the proof

Decompose $w_{(y,s)}$ as follows:

$$
\begin{aligned}
w_{(y,s)}(x,t) &= E(x,t;y,s) - \Gamma(x,t;y,s) \\
&= \{E(x,t;y,s) - \Gamma_P(x,t;y,s)\} + \{\Gamma_P(x,t;y,s) - \Gamma_-(\Phi(x),t;\Phi(y),s)\} \\
&\quad + \{\Gamma_-(\Phi(x),t;\Phi(y),s) - \Gamma_-^0(\Phi(x),t;\Phi(y),s)\} \\
&\quad + \{\Gamma_-^0(\Phi(x),t;\Phi(y),s) - \Gamma^0(\Phi(x),t;\Phi(y),s)\} \\
&\quad + \{\Gamma^0(\Phi(x),t;\Phi(y),s) - \Gamma^0(x,t;y,s)\} \\
&\quad + \{\Gamma^0(x,t;y,s) - \Gamma(x,t;y,s)\} + +\{\Gamma(x,t;y,s) - V_{(y,s)}(x,t)\},
\end{aligned}
$$

To show : $|w_{(y,s)}(y,s')| \to \infty$ as $s' \to s$, $y \to \partial D$

Let $\xi = \eta = \Phi(x) = \Phi(y) \to O$ $(\delta \downarrow 0)$ and consider the case, for example $n = 3$.

Behavior of each term

1.
$$\limsup_{\delta\to 0} |E(x,s+\varepsilon^2;y,s)-\Gamma_P(x,s+\varepsilon^2;y,s)| = O(\varepsilon^{\mu-3}),$$
as $\varepsilon\to 0$.

2.
$$\limsup_{\delta\downarrow 0} |(\tilde{\Gamma}_p-\Gamma_-)(\xi,s+\varepsilon^2;\eta,s)| = O(\varepsilon^{\alpha-3}) \quad \text{as} \quad \varepsilon\to 0.$$

3.
$$\limsup_{\delta\downarrow 0} |\Gamma_-(\xi,t+\varepsilon^2;\eta,s)-\Gamma_-^0(\xi,t+\varepsilon^2;\eta,s)| = O(\varepsilon^{\mu-3}) \quad \text{as} \quad \varepsilon\to 0.$$

(In 1,2,3, we used a pointwise space gradient estimate for a fundamental solution of parabolic equation with disconti. coeff..)

4. Put

$$W(\xi, t; \eta, s) := \Gamma^0_-(\xi, t; \eta, s) - \Gamma^0(\xi, t; \eta, s) \text{ (dominant)}$$

Denote $W(\xi, t; \eta, s)$ for $\pm\xi_n > 0$ by $W^\pm(\xi, t; \eta, s)$.

Then, there exist a constant $C > 0$ such that

$$\lim_{\delta \downarrow 0} |W^+(\eta, s + \varepsilon^2; \eta, s)| \geq C\varepsilon^{-3} \quad \text{as} \quad \varepsilon \to 0.$$

5.

$$\limsup_{\delta \downarrow 0} |\Gamma^0(\Phi(x), t; \Phi(y), s) - \Gamma^0(x, t; y, s)| = 0.$$

6. Let $G(x, t; y, s) = \Gamma^0(x, t; y, s) - \Gamma(x, t; y, s)$. Then,

$$\limsup_{\delta \downarrow 0} |G(y, s + \varepsilon^2; y, s)| = O(\varepsilon^{-2}) \quad \text{as} \quad \varepsilon \to 0.$$

7. It follows from the definitions of Γ and v that

$$\Gamma(x,t;y,s) - V_{(y,s)}(x,t)$$

is $C^1_t C^2_x$ at (y,s) and so bounded in some closed neighborhood of (y,s).

Remark for non-separated inclusions (open question)

The previous proof for the separated inclusions case works well except the estimate for $W(\xi, t; \eta, s)$.