Part 2-2 Equivalence of Schemes

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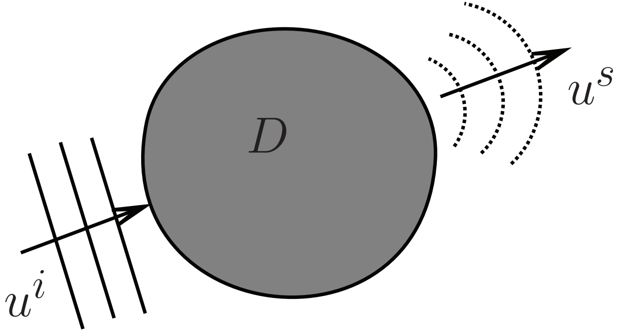
Contents

Inverse scattering problem



Set up

$D \subset \mathbb{R}^3$: a bounded domain (sound soft obstacle), with C^2 boundary ∂D, such that $\mathbb{R}^3 \setminus \overline{D}$ is connected

$u^i(x,d) := e^{i\kappa d\cdot x}$ (incident field or wave) with

$d \in \mathbb{S} := \{d \in \mathbb{R}^3 \,:\, |d| = 1\}$ (incident direction),

$\kappa > 0$ (wave number).

acoustical scattering problem by D for $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ (total field)

Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \tag{1}$$

Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D \tag{2}$$

Scattered field

Sommerfeld radiation condition

$u^s := u - u^i$ (scattered field or wave) satisfies

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - i\kappa u^s\right) = 0, \tag{3}$$

where $r = |x|$, $\theta := \frac{x}{|x|}$.

asymptotic of u^s

$$u^s(x) = u^s(x,d) = \frac{e^{i\kappa r}}{r} u^\infty(\theta,d) + O(r^{-2}) \quad (r \to \infty), \tag{4}$$

where $u^\infty(\cdot,d)$: far field pattern associated to $u^i(\cdot,d)$

Shape reconstruction problem

Shape reconstruction problem

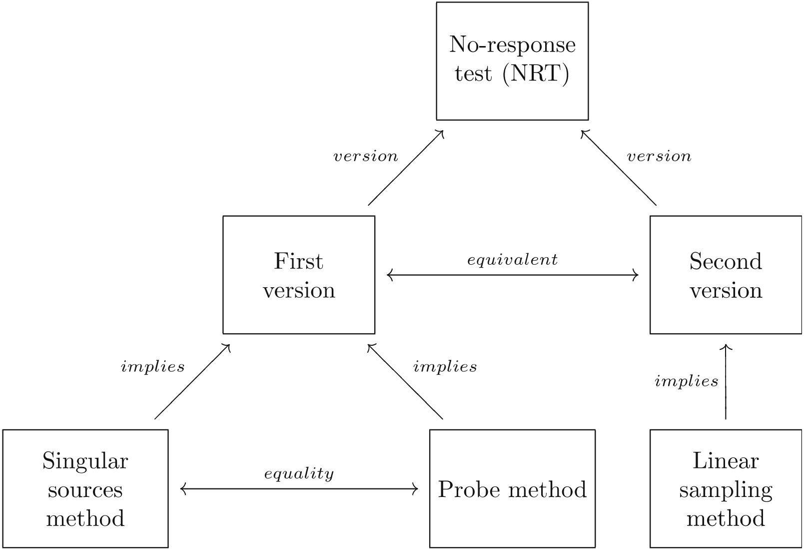
Given $u^{\infty}(\cdot,\cdot)$ on $\mathbb{S}\times\mathbb{S}$ for the scattering problem (1) - (3) find (reconstruct) the obstacle D.

Known non-iterative reconstruction methods

For example, they are

- *linear sampling method*: D. Colton and A. Kirsch (1996)
- *factorization method*: A. Kirsch (1998)
- *probe method*: M. Ikehata (1998)
- *singular sources method*: R. Potthast (1998)
- *no-response test*: D-R. Luke and R. Potthast (2003)
- *range test*: R. Potthast, J. Sylvester and S. Kusiak (2003)

No-response test view



Notations and terminologies

$$\Phi(x,y) := \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, x \neq y, x, y \in \mathbb{R}^3$$

$$\text{(fundamental solution of } \Delta + \kappa^2 \text{ on } \mathbb{R}^3\text{)}$$

$\Phi^s(\cdot, z)$: scattered field of $\Phi(\cdot, z)$, $z \notin \overline{D}$

$\Phi^\infty(\cdot, z)$: far-field pattern of $\Phi^s(\cdot, z)$

Definition 1

We call a bounded domain B, with C^2 regular boundary, such that $\mathbb{R}^3 \setminus \overline{B}$ is connected a non-vibrating domain if κ^2 is not a Dirichlet eigenvalue for $-\Delta$. Otherwise, we say B is vibrating.

Notations and terminologies continued

Let $g \in L^2(\mathbb{S})$.

$$v_g(x) := \int_{\mathbb{S}} e^{i\kappa x\cdot d} g(d)\, ds(d) \;\; (x \in \mathbb{R}^3) \tag{5}$$

(Herglotz wave function)

$$v_g^s(x) := \int_{\mathbb{S}} u^s(x,d) g(d)\; ds(d) \;\; (x \in \mathbb{R}^3 \setminus D) \tag{6}$$

(scattered field associated with Herglotz incident field $v_g^i := v_g$)

$$v_g^\infty(\theta) := \int_{\mathbb{S}} u^\infty(\theta,d) g(d) ds(d) \; (\theta \in \mathbb{S}) \tag{7}$$

(far-field pattern of v_g^s)

Representation

$$u^{\infty}(\theta, d) = \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial u^s(y,d)}{\partial \nu}e^{-i\kappa\theta\cdot y} - \frac{\partial e^{-i\kappa\theta\cdot y}}{\partial \nu}u^s(y,d)\right\}ds(y) \quad (8)$$

where the normal is directed into inside D.

$$\Phi^s(x,z) = \int_{\partial D}\left\{\frac{\partial \Phi^s(y,z)}{\partial \nu(y)}\Phi(x,y) - \Phi^s(y,z)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right\}\, ds(y), \quad x, z \in \mathbb{R}^3 \backslash \overline{D}. \quad (9)$$

$$0 = \int_{\partial D}\left\{\frac{\partial \Phi(y,z)}{\partial \nu(y)}\Phi(x,y) - \Phi(y,z)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\right\}\, ds(y), \quad x, z \in \mathbb{R}^3 \setminus \overline{D} \quad (10)$$

$$\Phi^s(x,z) = \int_{\partial D}\frac{\partial (\Phi^s + \Phi)(y,z)}{\partial \nu(y)}\Phi(x,y)\, ds(y), \quad x, z \in \mathbb{R}^3 \setminus \overline{D}. \quad (11)$$

Representation continued

$$
\begin{aligned}
&\int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta,d)f(\theta)g(d)\ ds(\theta)ds(d) \\
= \quad & \frac{1}{4\pi}\int_{\partial D}\left\{\int_{\mathbb{S}}\frac{\partial u^s(y,d)}{\partial \nu}g(d)\ ds(d)\cdot\int_{\mathbb{S}} e^{i\kappa\theta\cdot y}f(\theta)\ ds(\theta)\right. \\
& \left. -\int_{\mathbb{S}}\frac{\partial e^{i\kappa\theta\cdot y}}{\partial\nu}f(\theta)\ ds(\theta)\cdot\int_{\mathbb{S}}\underbrace{u^s(y,d)}_{=-u^i(y,d)}\ g(d)\ ds(d)\right\}\ ds(y) \\
= \quad & \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial v_g^s}{\partial\nu}(y)v_f^i(y)+\frac{\partial v_f^i}{\partial\nu}(y)v_g^i(y)\right\}\ ds(y). \qquad (12)
\end{aligned}
$$

1 st version of NRT

Definition 2 (1 st version of the no response method)

For any non-vibrating domain B, *we define the* indicator function *by*

$$I_1(B) := \lim_{\epsilon \to 0} \sup\{I_{1,\epsilon}(f,g) : \ ||v_f||_{L^2(\partial B)} < \epsilon, ||v_g||_{L^2(\partial B)} < \epsilon\} \quad (13)$$

with

$$I_{1,\epsilon}(f,g) := \Big| \int_{\mathbb{S}} \int_{\mathbb{S}} u^\infty(-\theta, d) f(\theta) g(d) \ ds(\theta) ds(d) \Big|. \quad (14)$$

For the set $\mathcal{G}$ of non-vabriting domains B no response test calculates the indicator function $I_1(B)$ and builds the intersection

$$D_{rec,1} := \bigcap_{B \in \mathbf{B}_1} B, \quad (15)$$

where

$$\mathbf{B}_1 := \{B \in \mathcal{G} : \ I_1(B) = 0\}. \quad (16)$$

Convergence result NRT

Theorem 3 (Convergence of the first version)

Let $\mathcal{G}$ as in Definition 2. We have

1. *if* $\overline{D} \subset B$ *then* $I_1(B) = 0$,
2. *if* $\overline{D} \not\subset B$ *then* $I_1(B) = \infty$.

Thus the unknown scatterer is given by the intersection of all test domains B *for which* $I_1(B)$ *is zero, i.e*

$$\overline{D} = D_{rec,1}.$$

Proof: $\overline{D} \subset B$ case

$||v_g||_{L^2(\partial B)} = ||v_g^i||_{L^2(\partial B)} < \epsilon \Longrightarrow$

$||v_g||_{L^2(B)} < c\epsilon \Longrightarrow ||v_g||_{H^2(D)} < c'\epsilon \Longrightarrow$

$$||v_g^s||_{L^2(\partial D)} < c_1\epsilon, \qquad \left|\left|\frac{\partial v_g^s}{\partial \nu}\right|\right|_{L^2(\partial D)} < c_1\epsilon \tag{17}$$

together with $||v_f||_{H^2(D)} = ||v_f^i||_{H^2(D)} < \tilde{c}\epsilon$

$\Longrightarrow$

$$|I_{1,\epsilon}(f,g)| \leq C\epsilon^2 \tag{18}$$

$\Longrightarrow$

$$I_1(B) = \lim_{\epsilon \to 0} \sup\{I_{1,\epsilon}(f,g) : \ ||v_g||_{L^2(\partial B)} < \epsilon, ||v_f||_{L^2(\partial B)} < \epsilon\} = 0. \tag{19}$$

Proof: $\overline{D} \not\subset B$ case

$z \in \partial D$: z is on the boundary of the unbounded component of

$\mathbb{R}^3 \setminus \overline{D \cup B}$,

$(z_p)_{p\in\mathbb{N}} \subset \mathbb{R}^3 \setminus (\overline{B \cup D})$: $z_p \to z\,(p \to \infty)$

$B_p\,(p \in \mathbb{N})$: non-vibrating domains, $\overline{B \cup D} \subset B_p$, $z_p \in \mathbb{R}^n \setminus \overline{B_p}$

$$\left|\left|v_{g_n^p} - \frac{\epsilon}{2}\alpha_p \Phi(\cdot, z_p)\right|\right|_{L^2(\partial B_p)} \to 0\ (n \to \infty) \tag{20}$$

$$\alpha_p := ||\Phi(\cdot, z_p)||^{-1}_{L^2(\partial B)}. \tag{21}$$

Then,

$$||v_{g_n^p}||_{L^2(\partial B)} < \epsilon\ \ (n \gg 1), \tag{22}$$

$$\lim_{n\to\infty}\int_{\mathbb{S}}\int_{\mathbb{S}} u^{\infty}(-\theta,d)\ g_n^p(\theta)\ g_n^p(d)\ ds(\theta)ds(d)$$

$$= \frac{\epsilon^2\alpha_p^2}{16\pi}\int_{\partial D}\frac{\partial(\Phi^s+\Phi)(y,z_p)}{\partial\nu(y)}\cdot\Phi(y,z_p)\ ds(y) \tag{23}$$

$$= \frac{\epsilon^2\alpha_p^2}{16\pi}\Phi^s(z_p,z_p). \tag{24}$$

$$|\Phi^s(z_p,z_p)| \geq c_1[d(z_p,\partial D)]^{-1} \tag{25}$$

$$\alpha_p^2 := ||\Phi(\cdot,z_p)||^{-2}_{L^2(\partial B)} \geq c_2[\ln(d(z_p,\partial B))]^{-1} \tag{26}$$

$$\lim_{p\to\infty}\lim_{n\to\infty}\int_{\mathbb{S}}\int_{\mathbb{S}} u^{\infty}(-\theta,d)g_n^p(\theta)g_n^p(d)d\theta ds(d) = \infty. \tag{27}$$

Hence $I_1(B) = \infty$.

Singular sources method (SSM)

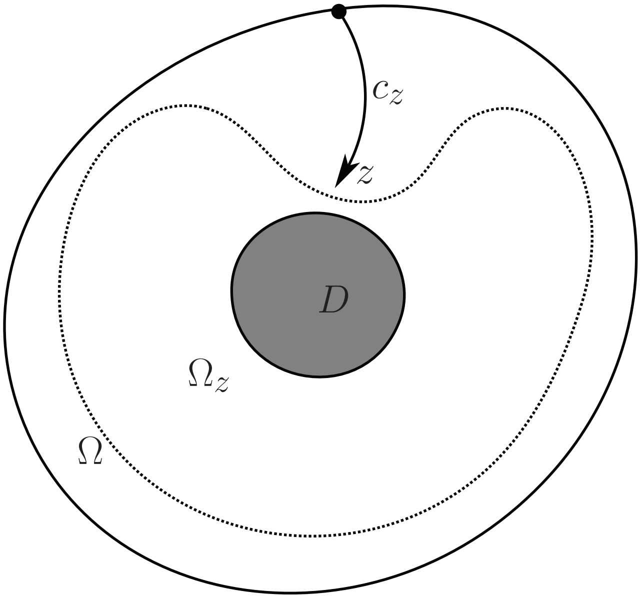
Let

$z_p = z,\ \frac{\epsilon}{2}\alpha_p\Phi(\cdot, z_p) = \Phi(\cdot, z),\ (g_n^p)_{n\in\mathbb{N}} = (g_n^z)_{n\in\mathbb{N}} \subset L^2(\mathbb{S})$

as in the proof of Theorem 3.

Proposition 4

$$\Phi^s(z,z) = 4\pi \lim_{n\to\infty} \int_{\mathbb{S}} \int_{\mathbb{S}} u^\infty(-\theta, d)\ g_n^z(\theta)\ g_n^z(d)\ ds(\theta) ds(d). \qquad (28)$$


c_z
z
D
Ω_z
Ω

Indicator function of SSM

Definition 5 (indicator function)

$z \in \Omega \Supset D$: *bounded domain*

$\mathcal{C}_z$: *set of continuous curves* c_z *joining* z *to* $\partial\Omega$. *For any curve* $c_z \in \mathcal{C}_z$, $z' \in c_z$, *let* $\Omega_{z'} \Subset \Omega \setminus c_{z'}$ *be a non-vibrating domain.*

$\{v_{g_n^{z'}}\}$: *sequence of Herglotz waves which approximate* $\Phi(\cdot, z')$ *in* Ω_z,

$$I_{ssm}(z, z', c_{z'}, \Omega_{z'}) \quad := 4\pi \limsup_{n\to\infty} \Big| \int_{\mathbb{S}} \int_{\mathbb{S}} u^{\infty}(-\theta, d) g_n^{z'}(\theta) g_n^{z'}(d) ds(\theta) ds(d) \Big|$$

$$(= \Phi^s(z', z') \text{ if } c_{z'} \subset \mathbb{R}^3 \setminus \overline{D}).$$

Then, we define the indicator function $I(z)$ *by*

$$I(z) := \inf_{c_z \in \mathcal{C}_z} \sup_{z' \in c_z} \inf_{\Omega_{z'}} I_{ssm}(z, z', c_{z'}, \Omega_{z'}).$$

Convergence result for SSM

Theorem 6

The indicator function $I(\cdot)$ satisfies the two properties:
1.) If $z \in \Omega \setminus \overline{D}$, then

$$I(z) < \infty.$$

2.) If $z \in \overline{D}$, then

$$I(z) = \infty.$$

As a conclusion, the obstacle D is characterized by the indictor function $I(\cdot)$ as follows:

$$\overline{D} = \{z \in \Omega\,;\, I(z) = \infty\}.$$

Probe method

$\Omega \Supset D$: non-vibrating domain

Dirichlet-Neumann map:

$$\Lambda_D : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega),$$

where

$\Lambda_D f := \frac{\partial u^f}{\partial \nu} |_{\partial\Omega}$, ν : unit normal of $\partial\Omega$ directed outside of Ω,

$u^f \in H^1(\Omega \setminus \overline{D})$: solution to

$$\begin{cases} \Delta u^f + k^2 u^f = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ u^f = f \quad \text{on} \quad \partial\Omega, \\ u^f = 0 \quad \text{on} \quad \partial D. \end{cases} \tag{29}$$

Indicator function of probe method

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f(x) \cdot f(x) ds(x), \tag{30}$$

where

$\Lambda_\emptyset$: Dirichlet-Neumann map for (29) when $D = \emptyset$.

For $z \in \Omega$, we take c_z, $z' \in c_z$ and $\Omega_{z'}$ as before concerning the singular sources method.

Approximate $\Phi(\cdot, z')$ on $\partial\Omega_{z'}$ by a sequence of Herglotz waves $v_{g^n_{z'}}$.

When $z \in \Omega \setminus \overline{D}$, $z' \in c_z$, $c_{z'} \subset \mathbb{R}^3 \setminus \overline{D}$, $\Omega_{z'} \Subset \Omega \setminus c_{z'}$,

$$\begin{aligned} I_{pb}(z, z', c_{z'}, \Omega_{z'}) &:= \lim_{n\to\infty} \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) v_{g_n^{z'}}(x) \cdot v_{g_n^{z'}}(x) ds(x) \\ &:= -\lim_{n\to\infty} \int_{\partial D} \left(\frac{\partial v^s_{g_n^{z'}}}{\partial \nu}(x) - \frac{\partial v_{g_n^{z'}}}{\partial \nu}(x)\right) \cdot v_{g_n^{z'}}(x) ds(x) \end{aligned},$$

$$I_{pb}(z, z', c_{z'}, \Omega_{z'}) = -\int_{\partial D} \left\{ \frac{\partial \tilde{\Phi}_\Omega}{\partial \nu}(x, z') - \frac{\partial \Phi}{\partial \nu}(x, z') \right\} \cdot \Phi(x, z') ds(x),$$

where

$v^s_{g^n_{z'}}$ and $\tilde{\Phi}_\Omega$: solutions of (29) replacing f by $v_{g^n_{z'}}|_{\partial\Omega}$ and $\Phi(\cdot, z')|_{\partial\Omega}$, respectively.

$\Phi^s_\Omega(x, z') := \tilde{\Phi}_\Omega(x, z') - \Phi(x, z')$: reflected solution for the problem (29) i.e. solution to

$$\begin{cases} \Delta \Phi^s_\Omega + k^2 \Phi^s_\Omega = 0 & \text{in} \quad \Omega \setminus \overline{D}, \\ \Phi^s_\Omega(\cdot, z') = 0 & \text{on} \quad \partial\Omega, \\ \Phi^s_\Omega(\cdot, z') = -\Phi(\cdot, z') & \text{on} \quad \partial D. \end{cases} \tag{31}$$

$$I_{pb}(z, z', c_{z'}, \Omega_{z'}) = -\Phi^s_\Omega(z', z') + \int_{\partial\Omega} \frac{\partial}{\partial \nu} \Phi^s_\Omega(x, z')\Phi(x, z')ds(x),$$

where

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu} \Phi^s_\Omega(x, z')\Phi(x, z')ds(x)$$

is bounded with respect to z.

$$|I_{pb}(z, z', c_{z'}, \Omega_{z'}) + \Phi^s_\Omega(z', z')| = O(1)$$
$$\text{for } z \text{ in } \Omega \setminus \overline{D},\ z' \in c_z,\ z' \in c_z,\ c_{z'} \subset \mathbb{R}^3 \setminus \overline{D},\ \Omega_{z'} \Subset \Omega \setminus c_{z'}. \quad (32)$$

$$|\Phi^s_\Omega(z', z') + \Phi^s(z', z')| = O(1)$$
$$\text{for } z \text{ in } \Omega \setminus \overline{D} \text{ near } \partial D,\ z' \in c_z,\ z' \in c_z,\ c_{z'} \subset \mathbb{R}^3 \setminus \overline{D},\ \Omega_{z'} \Subset \Omega \setminus c_{z'}. \quad (33)$$

Proposition 7

$$|I_{pb}(z, z', c_{z'}, \Omega_{z'}) - I_{ssm}(z, z', c_{z'}, \Omega_{z'})| = O(1),$$

for z *in* $\Omega \setminus \overline{D}$ *near* ∂D, $z' \in c_z$, $c_{z'} \subset \mathbb{R}^3 \setminus \overline{D}$ *and* $\Omega_{z'} \Subset \Omega \setminus c_{z'}$.

Theorem 8

The natural far field versions of the probe method and the singular sources method are identical. This common version is given by Definition 5.

2nd version of NRT

$B \subset \mathbb{R}^3$: non-vibration domain

The basic idea of the range test is to test the solvability of the equation

$$\frac{1}{4\pi}\int_{\partial B} e^{-i\kappa\theta\cdot x}\psi(x)ds(x) = u^\infty(\theta, d), \quad (\theta \in \mathbb{S}),\ d \in \mathbb{S}. \tag{34}$$

We will apply this technique to the case $u^\infty = v_g^\infty$.

We use *Tikhonov regularization scheme* for the regularization of (34):

$$\psi_\alpha := (\alpha + S^{\infty,*}S^\infty)^{-1}S^{\infty,*}u^\infty \tag{35}$$

with $\alpha > 0$ (*regularization parameter*),

$S^\infty : L^2(\partial B) \to L^2(\mathbb{S})$ (*far field operator*):

$$(S^\infty\varphi)(\theta) := \frac{1}{4\pi}\int_{\partial B} e^{-i\kappa\theta\cdot y}\varphi(y)\,ds(y)\ \ (\theta \in \mathbb{S}). \tag{36}$$

Definition 9 (2nd version of NRT)

For any non-vibrating domain B we define the indicator function

$$I_2(B) := \limsup_{\epsilon \to 0} \Big\{ \lim_{\alpha \to 0} ||\psi_g^\alpha||_{L^2(\partial B)} : \ \psi_g^\alpha \text{ is the regularized solution (35) of } (34) \text{ with } u^\infty = v_g^\infty, \ g \in L^2(\mathbb{S}) \text{ and } ||v_g||_{L^2(\partial B)} < \epsilon \Big\} \quad (37)$$

For the set $\mathcal{G}$ of non-vibrating domains B, the 2nd version of NRT calculates the indicator function $I_2(B)$ and builds the intersection

$$D_{rec,2} := \bigcap_{B \in \mathbf{B}_2} B, \quad (38)$$

where

$$\mathbf{B}_2 := \{B \in \mathcal{G} : \ I_2(B) = 0\}. \quad (39)$$

Theorem 10 (Convergence of 2nd NRT)

Let $\mathcal{G}$ be as in Definition 9. We have

1. *if* $\overline{D} \subset B$ *then* $I_2(B) = 0$*,*
2. *if* $\overline{D} \not\subset B$ *then* $I_2(B) = \infty$*.*

Thus, the obstacle D *can be characterized by*

$$\overline{D} = D_{rec,2}.$$

The tools to prove Theorem 10.

Lemma 11

B : a domain with C^2 boundary,

$A : L^2(\partial B) \to L^2(\mathbb{S})$: injective integral operator with continuous kernel and dense range

$\psi_\alpha := (\alpha I + A^* A)^{-1} A^* f$: Tikhonov regularized solution of $A\psi = f$,

where $\alpha > 0$: regularization parameter, A^* : adjoint of A

Then,

$$\lim_{\alpha \to 0} \|\psi_\alpha\|_{L^2(\partial B)} = \begin{cases} \infty, & \text{if } f \text{ is not in } A(L^2(\partial B)), \\ \|\psi^*\|_{L^2(\partial B)}, & \text{if } A\psi^* = f. \end{cases} \qquad (40)$$

Lemma 12 (⇐ Rellich's lemma)

If there exists $\psi \in L^2(\partial B)$ *such that* $S^\infty \psi = u^\infty$*, then the scattered field* u^s *of* u^∞ *is given by*

$$u^s = \int_{\partial B} \Phi(\cdot, y)\psi(y)ds(y)$$

in the unbounded component of $\mathbb{R}^3 \setminus (B \cup D)$*.*

Lemma 13

The operator $S : L^2(\partial B) \to H^1(\partial B)$ *defined by*

$$(S\psi)(x) := \int_{\partial B} \Phi(x, y)\psi(y)ds(y) \quad (x \in \partial B) \tag{41}$$

is an isomorphism and the operator $S^\infty : L^2(\partial B) \to L^2(\mathbb{S})$ *has a dense range if* B *is a non-vibrating domain.*

Proof of Theorem 10

We will only show the case $\overline{D} \subset B$ to see the idea behind the second version of NRT. There we only use Lemma 13. Later the convergence will be shown by giving the equivalence of the two versions of NRT.

Let $\overline{D} \subset B$.

$g \in L^2(\mathbb{S})$: $||v_g||_{L^2(\partial B)} < \epsilon \Longrightarrow ||v_g^s||_{H^1(\partial B)} < C\epsilon$.

By Lemma 13,

$$S\psi = v_g^s \text{ on } \partial B \tag{42}$$

has a solution $\psi \in L^2(\partial B)$,

$$||\psi||_{L^2(\partial B)} < \tilde{c}\epsilon. \tag{43}$$

Hence, ψ is the solution of (34) with $u^\infty = v_g^\infty$, which implies $I_2(B) = 0$.

Convergence of the linear sampling method implies the convergence of the no response method

The linear sampling method

For $z \in \mathbb{R}^3$, observe that the far-field of $\Phi(\cdot, z)$ is given by

$$\Phi_\infty(\theta, z) = \frac{1}{4\pi} e^{-i\kappa\theta \cdot z} \ (\theta \in \mathbb{S}).$$

Consider

$$F g_z = \Phi_\infty(\cdot, z), \tag{44}$$

where the far-field opertor $F : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ is given by

$$F g(\theta) := \int_{\mathbb{S}} u^\infty(\theta, d) g(d) ds(d) \ (\theta \in \mathbb{S}).$$

The idea of the linear sampling method is to approximately solve this equation and look at the behavior of the norms of g_z.

Theorem 14

Assume that D is non-vibrating.
1) If $z \in D$, then for every $\epsilon > 0$ there exists a solution $g^\epsilon(\cdot, z)$ in $L^2(\mathbb{S})$ of the inequality

$$\|Fg^\epsilon(\cdot, z) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} < \epsilon$$

such that

$$\lim_{z \to \partial D} \|g^\epsilon(\cdot, z)\|_{L^2(\mathbb{S})} = \lim_{z \to \partial D} \|v_{g^\epsilon}(\cdot, z)\|_{H^1(D)} = \infty.$$

2) If $z \in \mathbb{R}^3 \setminus \overline{D}$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g^{\epsilon,\delta}(\cdot, z)$ in $L^2(\mathbb{S})$ of the inequality

$$\|Fg^{\epsilon,\delta}(\cdot, z) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} < \epsilon + \delta$$

such that $\lim_{\delta \to 0} \|g^{\epsilon,\delta}(\cdot, z)\|_{L^2(\mathbb{S})} = \lim_{\delta \to 0} \|v_{g^{\epsilon,\delta}}(\cdot, z)\|_{H^1(D)} = \infty.$

Using the density of the linear sampling method for the 2nd version of NRT

Assume that D is a non-vibrating domain and

g^ϵ of (44) coincides with the particular solution given by

Colton-Cakoni :
On the mathematical basis of the linear simpling method,

Georgian Math. J. **10** (2003), 95-104.

Theorem 15

Consider a non-vibrating domain B such that $\overline{D} \not\subset B$. Given the densities $g^\epsilon(\cdot, z)$ provided by Theorem 14, there is a density $\tilde{g}(\cdot, z)$ such that the Herglotz wave function $v_{\tilde{g}(\cdot,z)}$ is bounded in a neighbourhood of z and the density $g_{RT}(\cdot, z) := g^\epsilon(\cdot, z) + \tilde{g}(\cdot, z)$ leads to a blow-up of the functional I_2 of the 2nd version of NRT.

Several other equivalences

Theorem 16

The two versions of NRT are equivalent. More precisely, let for any non-vibrating domain B, then $\mu_1(B) \sim \mu_2(B)$, where $\mu_1(B)$ and $\mu_2(B)$ are $I_1(B)$ and $I_2(B)$ without $\lim_{\epsilon \to 0}$ and setting $\epsilon = 1$, respectively.

Proof. Let B be a non-vibrating domain. Then,

$$\mu_2(B) = \lim_{\alpha \to 0} \|\psi_g^\alpha\|_{L^2(\partial B)} = \lim_{\alpha \to 0} \sup_{\|v\|_{L^2(\partial B)} \leq 1} \left| (v, \psi_g^\alpha)_{L^2(\partial B)} \right| \tag{45}$$

$$= \lim_{\alpha \to 0} \sup_{\|v\|_{L^2(\partial B)} \leq 1} \left| (v, (\alpha + S^{\infty,*} S^\infty)^{-1} S^{\infty,*} v_g^\infty)_{L^2(\partial B)} \right| \tag{46}$$

$$(\mu_2(B)) = \lim_{\alpha\to 0} \sup_{\|v\|_{L^2(\partial B)}\le 1} \left|(((\alpha + S^{\infty,*}S^{\infty})^{-1}S^{\infty,*})^*v, v_g^\infty)_{L^2(S^2)}\right| \tag{47}$$

Here, by H: $L^2(S^2) \to L^2(\partial B)$ has a dense range, $S^{\infty,*} = \gamma H$ and $\gamma = 1/4\pi$,

$$= \lim_{\alpha\to 0} \sup_{f\in L^2(S^2), \|Hf\|_{L^2(\partial B)}\le 1} \tag{48}$$

$$\left|(((\alpha + (\gamma H)(\gamma H)^*)^{-1}(\gamma H))^*Hf, v_g^\infty)_{L^2(S^2)}\right| \tag{49}$$

$$= \lim_{\alpha\to 0} \sup_{f\in L^2(S^2), \|Hf\|_{L^2(\partial B)}\le 1} \tag{50}$$

$$\left|\gamma^{-1}(((\alpha\gamma^{-2} + HH^*)^{-1}H)^*Hf, v_g^\infty)_{L^2(S^2)}\right| \tag{51}$$

If we put $K = H^*\colon L^2(\partial B) \to L^2(S^2),\ \beta = \alpha\gamma^{-2}$, we have

$$((\alpha\gamma^{-2} + HH^*)^{-1}H)^*Hf = K(\beta + K^*K)^{-1}K^*f \tag{52}$$

Now let (μ_n, φ_n, g_n) be the singular system of K. Then, because the injectivity of K follows from H having the dense range, we have

$$\begin{aligned} &(\beta + K^*K)^{-1}K^*f = \sum_n \tfrac{\mu_n}{\beta+\mu_n^2}(f, g_n)\varphi_n \\ &\therefore K(\beta + K^*K)^{-1}K^*f = \sum_n \tfrac{\mu_n^2}{\beta+\mu_n^2}(f, g_n)g_n \underset{\beta\to 0}{\to} \sum_n (f, g_n)g_n. \end{aligned}$$

Here, by $K^* = (H^*)^* = H$ is injective because B is non-vibrating, $K\colon L^2(\partial B) \to L^2(S^2)$ has a dense range.

Hence, by $g_n = \frac{1}{\mu_n} K\varphi_n$, and $\{\varphi_n\} \subset L^2(\partial B)$ is a complete orthonormal, system, $\{g_n\}$ is complete and hence is a complete orthonormal system.

Therefore, $\sum_n (f, g_n) g_n = f$ and hence

$$\mu_2(B) := \gamma^{-1} \sup_{f \in L^2(S^2), \|Hf\|_{L^2(\partial B)} \leq 1} \left| (f, v_g^\infty)_{L^2(S^2)} \right| \tag{53}$$

$$= \gamma^{-1} \sup_{f \in L^2(S^2), \|v_f\|_{L^2(\partial B)} \leq 1} \left| \int_{S^2} \int_{S^2} \overline{v_g^\infty(\theta, d)} f(\theta) \overline{g(d)} \, dS(\theta) \, dS(d) \right| \tag{54}$$

$$= \gamma^{-1} \sup_{f \in L^2(S^2), \|v_f\|_{L^2(\partial B)} \leq 1} \left| \int_{S^2} \int_{S^2} v_g^\infty(\theta, d) \overline{f(\theta)} g(d) \, dS(\theta) \, dS(d) \right| \tag{55}$$

$$= \gamma^{-1} \mu_1(B) \tag{56}$$

Some formal arguments in the above proof can be justified rigorously.

We can also prove the equivalence of the modified SSM and first version of the no-response test.

Definition 17

Let B *be a non-vibrating domain such that* $\bar{D} \subset B$. *Also, for* $0 < \varepsilon << 1 \quad (\mathrm{fix})$, *let* $B_\varepsilon := \left\{x \in \mathbb{R}^3 \colon \mathrm{dist}\,(x, B) < \varepsilon\right\}$: *non-vibrating. Then, we define* $\mu_{SSM}(B)$ *by*

$$\mu_{\mathsf{SSM}}(B) := \sup_{z,\zeta \in \partial B_\varepsilon} \left| \int_{S^2} \int_{S^2} u^\infty(-\hat{x}, d) g_z(d) g_\zeta(\hat{x})\, dS(d)\, dS(\hat{x}) \right|, \quad (57)$$

where $v_{g_z} \approx \alpha_z \Phi(\cdot, z)$ *in* $L^2(\partial B)$, $v_{g_\zeta} \approx \alpha_\zeta \Phi(\cdot, \zeta)$ *in* $L^2(\partial B)$ *with* $\alpha_z = \|\Phi(\cdot, z)\|^{-1}_{L^2(\partial B)}$ *etc.* $(z, \zeta \in \partial B_\varepsilon)$.

Theorem 18

We have the equivalence:

$\mu_{SSM}(B) \sim \mu_1(B)$ *for* $^\forall B$*: non-vibrating domain,* $|\partial B_\varepsilon|(= \text{area of}\, \partial B_\varepsilon) \leq M$,

where we have slightly modified $\mu_1(B)$ *as follows.*

$$\mu_1(B) := \sup \left\{ \left| \int_{S^2} \int_{S^2} u^\infty(-\hat{x}, d) g(d) \tilde{g}(\hat{x})\, dS(d)\, dS(\hat{x}) \right| : \right. \tag{58}$$

$$\left. \|v_g\|_{L^2(\partial B)} \leq K, \quad \|v_{\tilde{g}}\|_{L^2(\partial B)} \leq K \right\} \; (K > 1\, \text{fix}) \tag{59}$$