Part 2-1 One Step Enclosure Method

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Introduction

$D \subset \mathbb{R}^3$: strictly convex bounded domain, ∂D : C^2

$d \in S^2$ (incident direction), $u^i(x,d) := e^{i\kappa x\cdot d}\,(x \in \mathbb{R}^n)$,

$\kappa > 0$ (wave number)

$u^s(x,d)\ (x \in \mathbb{R}^3 \setminus \overline{D}) \in H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$: solution to

$$\begin{cases} (\Delta + \kappa^2)u^s = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D} \\ u^s = -u^i \text{ on } \partial D \\ \lim_{r\to\infty} r(\frac{\partial u^s}{\partial r} - i\kappa u^s) = 0 \text{ with } r := |x|. \end{cases} \quad (1)$$

Inverse problem

$u^s(x,d)$ admits an asymptotic expansion:

$$u^s(x,d) = \frac{e^{i\kappa|x|}}{r} u^\infty(\hat{x},d) + O(\frac{1}{r^2}) \ (r \to \infty) \ \hat{x} := |x|^{-1}x. \qquad (2)$$

$u^\infty(\hat{x},d)\,(\hat{x},\ d \in S^2)$ is called the far-field pattern associated to

the inicident waves $u^i(x,d)\,(d \in S^2)$

Inverse problem Let D be unknown. Then, reconstruct D from the

far-field pattern u^∞.

Definition

Dirichlet to Neumann Λ_D is defined by

$$\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)\ f \to \Lambda_D f ::= \partial_\nu u^f,$$

where $u^f \in H^1(\Omega \setminus \overline{D})$ is the solution to

$$\begin{cases} (\Delta + \kappa^2)u = 0 \text{ in } \Omega \setminus \overline{D} \\ u = 0 \text{ on } \partial D,\ u = f \text{ on } \partial\Omega \end{cases} \tag{3}$$

provided that the above boundary value problem only has trivial solution when $f = 0$.

Theorem

It is equivalent to know either u^∞ or Λ_D.

So, many people consider the inverse scattering problem in terms of the inverse boundary value problem, that is to reconstruct D from Λ_D.

For the inverse boundary value problem, there is a method called enclosure method which uses complex geometric optic solutions.

Concerning the inverse scattering problem this is a two step method.

Roughly speaking, that is first to generate Λ_D from u^∞ and then try to identify D from Λ_D.

In this Part 2-1, we will show that one step enclosure method is possible under some conditions.

Complex geometric optic solution

Definition 1

(complex geometric optic solution=CGO solution)

Let $\tau \gg 1$, $t \in \mathbb{R}$, ω, $\omega^{\perp} \in S^2$, $\omega \perp \omega^{\perp}$. The CGO solution

$w = w(x, \tau, t, \omega)$ with linear phase

$\phi(x, \tau, t, \omega) := -\tau t + (x \cdot \omega + i\sqrt{\tau^2 + \kappa^2}\omega^{\perp})$ is defined by

$$w(x, \tau, t, \omega) = \exp(\phi(x, \tau, t, \omega)) \,. \tag{4}$$

For simplicity, we denote $w_0 = w_0(x, \tau, \omega) = w(x, \tau, 0, \omega)$.

Herglotz wave functions

Let $G \supset \overline{D}$ be a non-vibrating domain. That is ∂G is of C^2 smooth, $\mathbb{R}^3 \setminus \overline{G}$ is connected and $\Delta + \kappa^2$ in G with Dirichlet boundary condition on ∂G does not have 0 as its eigenvalue.

Then, there exists a sequence of functions $f_n = f_n(x, \tau, \omega) \in L^2(S^2)\,(n = 1, 2, \cdots)$ such that the Herglotz wave functions

$$Hf_n(x) := \int_{S^2} e^{i\kappa x\cdot d} f_n(x)\, ds(d) \to w_0(x)\ (n \to \infty) \text{ in } L^2(\partial G).$$

By taking G a little bit large and consider G as the one we obtain by shrinking it, the interior regularity of Helmholtz equation gives us

$$Hf_n \to w_0\ (n \to \infty) \text{ in } H^2(G).$$

Indicator function

Further, we clearly have for each $t \in \mathbb{R}$,

$$v^i_{f_n} = v_{f_n} := e^{-\tau t} H f_n \to w \ (n \to \infty) \text{ in } H^2(G)\,. \tag{5}$$

Now we define the indicator function $I(\tau, t, \omega)$ as follows.

Definition 2

$$I(\tau, t, \omega) = \lim_{n\to\infty} e^{-2\tau t} \int_{S^2} \int_{S^2} u^\infty(-\theta, d)\overline{f_n(\theta)} f_n(d)\, ds(d)\, ds(\theta). \tag{6}$$

Expressing indicator function by reflected solution

The well known identity

$$u^\infty(\theta,d) = \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial u^s(y,d)}{\partial\nu(y)}(y,d)\,e^{-i\kappa\theta\cdot y} - \frac{\partial e^{-i\kappa\theta\cdot y}}{\partial\nu(y)}u^s(y,d)\right\}ds(y) \tag{7}$$

with the unit normal vector ν of ∂D directed into D implies that

$$\begin{aligned} I(\tau,t,\omega) &= \frac{1}{4\pi}\lim_{n\to\infty}\int_{\partial D}\left\{\frac{\partial v^s_{f_n}}{\partial\nu}\overline{v^i_{f_n}} - v^s_{f_n}\overline{\frac{\partial v^i_{f_n}}{\partial\nu}}\right\}ds \\ &= \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial v}{\partial\nu}\overline{w} - v\overline{\frac{\partial w}{\partial\nu}}\right\}ds, \end{aligned} \tag{8}$$

where $v^s_{f_n}$ is the scattered wave corresponding to the incident wave $v^i_{f_n}$ and $v \in H^2_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{D})$ (reflected solution) is the solution to

Boundary value problem for the reflected solution

$$\begin{cases} (\Delta + \kappa^2)v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D} \\ v = -w \text{ on } \partial D \\ \lim_{r\to\infty} r\Big(\dfrac{\partial v}{\partial \nu} - i\kappa v\Big) = 0. \end{cases} \tag{9}$$

We note that if $t = 0$, the above w becomes w_0 and the Dirichlet data on ∂D for v becomes $-w_0$.

What we actually going to reconstruct is the so called supporting function $h_D(\omega) := \sup_{x\in D} x\cdot\omega$ for each $\omega \in S^2$. This is enough, because D can be obtained as the region enclosed by the hyperplanes $\{x\cdot\omega = h_D(\omega)\ :\ \omega \in S^2\}$.

Main theorem

The next theorem gives how to identify each $h_D(\omega)$ in terms of the behaviour of $I(\tau, t, \omega)$.

Theorem 3

(i) *If* $\tau \to \infty$,

$$|I(\tau,t,\omega)| : \begin{cases} \text{exponentially decaying if } t > h_D(\omega) \\ \text{bounded if } t = h_D(\omega) \\ \text{exponentially increasing if } t < h_D(\omega)\,. \end{cases} \quad (10)$$

(ii)

$$2(t - h_D(\omega)) = -\lim_{\tau\to\infty} \frac{\log|I(\tau,t,\omega)|}{\tau}. \quad (11)$$

Idea of proof

Due to the fact

$$I(\tau, t, \omega) = e^{-2(t-h_D(\omega))\tau} I(\tau, h_D(\omega), \omega), \quad (12)$$

it is enough to prove that there exist positive constants C_1, C_2 such that

$$C_1 \leq |I(\tau, h_D(\omega), \omega)| \leq C_2. \quad (13)$$

By the assumption that ∂D is strictly convex C^2 surface, the hyperplane $x \cdot \omega = h_D(\omega)$ intersect with ∂D at one point P.

Let B, B' be cocentric balls with center P such that $\overline{B'} \subset B$.

We will construct $z \in H^2(B \setminus \overline{D})$ which satisfies

$$\begin{cases} (\Delta + \kappa^2) z = 0 \text{ in } B \setminus \overline{D} \\ z = -w \text{ on } B \cap \partial D \\ z \text{ is exponentially decreasing in } (B \setminus B') \cap (\mathbb{R}^3 \setminus \overline{D}) \text{ as } \tau \to \infty. \end{cases} \quad (14)$$

Then, z is the dominant term of v. In fact, taking

$\chi \in C_0^\infty(B)$, $\chi = 1$ in B', consider $q := v - \chi z \in H^2_{\rm loc}(\mathbb{R}^3 \setminus \overline{D})$.

We easily see that q satisfies

$$\begin{cases} (\Delta + \kappa^2)q = -2\nabla\chi \cdot \nabla z - (\Delta\chi)z \text{ in } \mathbb{R}^3 \setminus \overline{D} \\ q = (\chi - 1)w \text{ on } \partial D \\ \lim_{r\to\infty} r\Big(\dfrac{\partial q}{\partial \nu} - i\kappa q\Big) = 0\,. \end{cases} \tag{15}$$

Here the right hand side of the equation and boundary data in (15) are all exponentially decaying as $\tau \to \infty$. Hence, by the well-posedness of (15), q will exponentially decrease as $\tau \to \infty$.

The rest of the proof is to show (13) using the explicit form of the dominant part of z.

Outline of constructing z

By a translation and rotation, we can assume

$P = (0,0,0)$, $-\nu\Big|_P = \omega = (0,0,1)$, $\omega^{\perp} = (1,0,0)$, $h_D(\omega) = 0$.

Further, by the assumption on ∂D, there exists a C^2 class function $\varphi(x')$ defined near $x' := (x_1, x_2) = (0,0)$ such that the followings hold locally near the origin.

$$\begin{aligned} &\partial D : x_3 = \varphi(x'),\ D : x_3 < \varphi(x') \\ &\varphi(x') = -a|x'|^2 + O(|x'|^3) \text{ for some constant } a > 0 \end{aligned} \tag{16}$$

In terms of the local coordinates $(y_1, y_2, y_3) = (x_1, x_2, x_3 - \varphi(x'))$ in the neighborhood of $\overline{B}$ by taking B small, the Helmholtz equation for z becomes

$$Lz := g^{jk}\partial_j\partial_k z + b^j\partial_j + \kappa^2 z = 0 \text{ in } y_3 > 0. \tag{17}$$

with the Einstein summation convention, notations $\partial_j = \frac{\partial}{\partial y_j}$ and

$$g^{jk} := \frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}, \quad b^k := \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}\Big(\frac{\partial y_k}{\partial x_i}\Big). \tag{18}$$

Now let $\eta = \eta(\tau, \kappa) := \sqrt{\tau^2 + \kappa^2}$ and conjugate L by $e^{i\eta y_1}$ which yields the operator M

$$M\cdot = e^{-i\eta y_1}L(e^{i\eta y_1}\cdot). \tag{19}$$

We note that for large τ the behaviour of η is equivalent to that of τ.

Further, we define a concept of orders for operators as follows. That is the operators ∂_3, the multiplication by η or τ are considered as operators of order 1, the multiplication by y_3 is an operator of order -1 and $\partial_j\,(j=1,2)$ are operators of order $1/2$. We denote the order of an operator A in terms of the concept of orders by $\mathrm{ord}A$. More precisely, if $\mathrm{ord}A = m$, A is just the sum of operators of order m.

Expanding the coefficients of M around $y_3 = 0$ up to their regularity, we have

$$M = \sum_{j=0}^{3} M_{2-j} \text{ with } \mathrm{ord}\, M_{2-j} = 2 - j/2. \tag{20}$$

The explicit forms of $M_{2-j}\,(0 \le j \le 3)$ are given as follows.

$$
\begin{cases}
M_2 = \overset{\circ}{g}^{33}\,\partial_3^2 + 2i\eta\,\overset{\circ}{g}^{13}\,\partial_3 - \eta^2\,\overset{\circ}{g}^{11} \\[2ex]
M_1 = 2i\,\overset{\circ}{g}^{11}\,\partial_1 + 2\,\overset{\circ}{g}^{13}\,\partial_1\partial_3 \\[2ex]
M_0 = i\eta\,\overset{\circ}{b}^{1} + \overset{\circ}{b}^{3}\,\partial_3 + y_3\Big(\dot{g}^{33}\partial_3^2 + 2i\eta\dot{g}^{13}\partial_3 - \eta^2\dot{g}^{11}\Big) + \overset{\circ}{g}^{11}\,\partial_1^2 \\[2ex]
M_{-1} = 2i\eta y_3\dot{g}^{11}\partial_1 + 2y_3\dot{g}^{13}\partial_1\partial_3 + \overset{\circ}{b}^{1}\,\partial_1,
\end{cases}
\tag{21}
$$

where $\overset{\circ}{g}^{jk} = g^{jk}\Big|_{y_3=0}$, $\overset{\circ}{b}^{j} = b^j\Big|_{y_3=0}$, $\dot{g}^{jk} = \dot{g}^{jk}(y) = \int_0^1 (\partial_3 g^{jk})(y', \theta y_3)\,d\theta$.

As for $\tilde{z} := e^{-i\eta y_1} z$, we will look for its dominant part $\tilde{z}^{(-1)}$ in the form

$$\tilde{z}^{(-1)} = \sum_{j=0}^{2} \tilde{z}_{-j} \tag{22}$$

with the estimates such that each term $\tilde{z}_{2-j}$ and $M\tilde{z}^{(-1)}$ satisfy

$$\partial_3^k \partial_{y'}^{\alpha} \tilde{z}_{-j} = O(\eta^{-j/2+k+|\alpha|/2} e^{\mu y_3 + \eta\varphi(y')})\ (\tau \to \infty) \tag{23}$$

and

$$\partial_3^k \partial_{y'}^{\alpha} M\tilde{z}^{(-1)} = O(\eta^{(1+|\alpha|)/2+k} e^{\mu y_3 + \eta\varphi(y')})\ (\tau \to \infty), \tag{24}$$

respectively, for any $k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and multi-index $\alpha \in \mathbb{Z}_+^2$, where

$$\partial_{y'} := (\partial_1, \partial_2) \quad \text{and} \quad \mu := -(\overset{\circ}{g}{}^{33})^{-1}\eta\sqrt{\overset{\circ}{g}{}^{11}\overset{\circ}{g}{}^{33} - (\overset{\circ}{g}{}^{13})^2}.$$

By

$$M\tilde{z}^{(-1)} = M_2\tilde{z}_0 + \Big(M_2\tilde{z}_{-1} + M_1\tilde{z}_0\Big) + \Big(M_2\tilde{z}_{-2} + M_1\tilde{z}_{-1} + M_0\tilde{z}_0\Big) + r$$

$$r = \Big(M_1\tilde{z}_{-2} + M_0\tilde{z}_{-1} + M_{-1}\tilde{z}_0\Big) + (M_0\tilde{z}_{-2} + M_{-1}\tilde{z}_{-1}) + M_{-1}\tilde{z}_{-2}, \tag{25}$$

we require $z_{-j}\,(0 \le j \le 2)$ to satisfy

$$\begin{cases} M_2\tilde{z}_0 = 0 \text{ in } 0 < y_3 < y_3^0,\ \tilde{z}_0 = -e^{\eta\varphi(y')} \text{ on } y_3 = 0 \\ M_2\tilde{z}_{-1} = -M_1\tilde{z}_0 \text{ in } 0 < y_3 < y_3^0,\ z_{-1} = 0 \text{ on } y_3 = 0 \\ M_2\tilde{z}_{-2} = -M_1\tilde{z}_{-1} - M_0\tilde{z}_0 \text{ in } 0 < y_3 < y_3^0,\ z_0 = 0 \text{ on } y_3 = 0, \end{cases} \tag{26}$$

where y_3^0 is chosen such that in the push forward of $B \setminus \overline{D}$ to y-space is in $\{y \in \mathbb{R}^3 \,:\, 0 < y_3 < y_3^0\}$.

We easily see that $\tilde{z}_0$ is given by

$$\tilde{z}_0 = -e^{\lambda y_3 + \eta\varphi(y')} \text{ with } \lambda := (\overset{\circ}{g}{}^{33})^{-1}\eta(-i\,\overset{\circ}{g}{}^{13} - \sqrt{\overset{\circ}{g}{}^{11}\overset{\circ}{g}{}^{33} - (\overset{\circ}{g}{}^{13})^2}). \quad (27)$$

The estimates (23) and

$$\partial_{y'}^{\alpha}\partial_{y_3}^{m} r = O(\eta^{m+(|\alpha|+1)/2} e^{\mu y_3 + \eta\varphi}) \ (\tau \to \infty) \quad (28)$$

which is the same as (24) can be shown by using the following lemma.

Lemma 4

Let $\tilde{z} = \tilde{z}(y_3, y', \tau)$ *satisfy*

$$\begin{cases} M_2\tilde{z} = y_3^j \eta^k e^{\lambda y_3 + \eta\varphi(y')} h(y') \text{ in } y_3 > 0 \\ \tilde{z} = 0 \text{ on } y_3 = 0 \\ \tilde{z} = O(1) \ (\tau \to \infty) \end{cases} \tag{29}$$

in a neighborhood of $y' = 0$, *where* $h = h(y')$ *satisfies* $\partial_{y'}^{\alpha} = O(1) \ (\tau \to \infty)$. *Then, we have the estimate*

$$\partial_{y'}^{\alpha}\partial_{y_3}^{m}\tilde{y} = O(\eta^{m+k-(j+1)+|\alpha|/2} e^{\mu y_3 + \eta\varphi(y')}) \ (\tau \to \infty). \tag{30}$$

Now, let δ, $\gamma > 0$ be constants and $\zeta \in C_0^{\infty}(B)$ such that

$$\varphi(y') \leq -\delta|y'|^2, \ \mu(y') \leq -\gamma \ (y' \in \text{supp}\zeta).$$

Then, we have

$$L(\zeta z^{(-1)}) = \zeta e^{i\eta y_1} r + \tilde{r}$$

with $\tilde{r} \in C_0^\infty(B \setminus D)$ which is exponentially decaying as $\tau \to \infty$.

Then by a direct computation, we have

$$\|L(\zeta z^{(-1)})\|_{L^2(B \setminus \overline{D})} \le C \|L(\zeta z^{(-1)})\|_{L^2(\mathbb{R}^3_+)} = O(\eta^{-3/4}) \ (\tau \to \infty)$$

for some constant $C > 0$.

Finally, we define z by

$$z = \zeta z^{(-1)} + z' \tag{31}$$

where $z' \in H^2(B \setminus \overline{D})$ is the solution to

$$Lz' = -L(\zeta z^{(-1)}) \text{ in } B \setminus \overline{D},\ z' = 0 \text{ on } \partial(B \setminus \overline{D}). \tag{32}$$

By replacing B by an another B if necessary to avoid having the non-uniqueness for (32), the well-posedness of the boundary value problem implies the estimate

$$\|z'\|_{H^2(B\setminus\overline{D})} = O(\eta^{-3/4})\ (\tau \to \infty). \tag{33}$$

Therefore, we have constructed z and shown its dominant part is $\zeta z^{(-1)}$.

Estimate of the indicator function

The estimate for $I(\tau, h_D(\omega), \omega)$ is the key and we will see later that the estimate for $I(\tau, t, \omega)$ $(t \neq h_D(\omega))$ easily follows from the key estimate. We continue to have the same setting as before in the previous section. Then, since $h_D(\omega) = 0$, $I(\tau, 0, \omega) = I(\tau, h_D(\omega), \omega)$. By taking the previous ζ to satisfy $\zeta = 1$ on $\mathrm{supp}\chi$, we have

$$I(\tau, 0, \omega) = J(\eta) + R(\eta), \tag{34}$$

with

$$J(\eta) = \frac{1}{4\pi} \int_{\partial D} \Big\{ \frac{\partial(\chi z^{(-1)})}{\partial \nu} \overline{w_0} - (\chi z^{(-1)}) \overline{\frac{\partial w_0}{\partial \nu}} \Big\} \, ds, \tag{35}$$

$$R(\eta) = R_1(\eta) + R_2(\eta)$$

$$R_1(\eta) = \frac{1}{4\pi} \int_{\partial D} \left\{ \frac{\partial(\chi z')}{\partial\nu} \overline{w_0} - (\chi z') \overline{\frac{\partial w_0}{\partial\nu}} \right\} ds \tag{36}$$

$$R_2(\eta) = \frac{1}{4\pi} \int_{\partial D} \left\{ \frac{\partial q}{\partial\nu} \overline{w_0} - q \overline{\frac{\partial w_0}{\partial\nu}} \right\} ds.$$

Since $R_2(\eta)$ is exponentially decaying as $\tau \to \infty$ and the dominant part of $J(\eta)$ is

$$J_1(\eta) = \frac{1}{4\pi} \int_{\partial D} \left\{ \frac{\partial(\chi z_0)}{\partial\nu} \overline{w_0} - (\chi z_0) \overline{\frac{\partial w_0}{\partial\nu}} \right\} ds \tag{37}$$

with $z_0 = e^{i\eta y_1} \tilde{z}_0$, by comparing the behaviours of $J_1(\eta)$, $R_1(\eta)$ as $\tau \to \infty$, we can show that $J_1(\eta)$ is the dominant term and as a consequence we have (i) for $t = h_D(\omega)$ in our theorem.

For the case $t \neq h_D(\omega)$, by the definition of the indicator function, we first observe that

$$I(\tau, t, \omega) = e^{2(t-h_D(\omega))\tau} I(\tau, h_D(\omega)\omega) \tag{38}$$

for any $t \in \mathbb{R}$. This immediately proves the rest of (i) in the theorem.

From (6),

$$\log|I(\tau, t, \omega)| = -2(t - h_D(\omega))\tau + \log|I(\tau, h_D(\omega), \omega)|.$$

Hence,

$$\begin{aligned} -2(t - h_D(\omega)) &= \frac{\log|I(\tau,t,\omega)|}{\tau} - \frac{\log|I(\tau,h_D(\omega),\omega)|}{\tau} \\ &= \frac{\log|I(\tau,t,\omega)|}{\tau} + O(\tau^{-1}) \ (\tau \to \infty), \end{aligned} \tag{39}$$

which proves (ii) in the theorem.

Open problem

Recently M. Sini and K. Yoshida proved the enclsure method without assuming any strict convexity (more precisely the finiteness of the touching points and curvature condition) for the near field measurements. So, how about the case for one step enclosure method ?