Lectures on Several Reconstruction Schemes for Inverse Problems

Part 1: A Remark on the Uniqueness of the Inverse Scattering by Obstacles

Gen Nakamura

gnaka@math.sci.hokudai.ac.jp
Department of Mathematics, Hokkaido University, Japan

Joint work with N. Honda and M. Sini

ICMAT, Madrid, May 9, 2011

Plan of my lecture

· Part 1: Uniqueness of inverse scattering problem

· Part 2: Several reconstruction schemes other than the factorization method

(i) Part 2-1: One step enclosure method

(ii)Part 2-2: Equivalence of the schemes

· Part 3: Inverse boundary value problem for heat conductors

· Part 4: An example of real inverse problem (MRE)

Introduction

$D \subset \mathbb{R}^n$ $(n \geq 2)$: bounded open, $\mathbb{R}^n \setminus \overline{D}$: connected, ∂D : Lipschitz

Ω : bounded domain, $\overline{D} \subset \Omega$, $\partial\Omega$: C^2

$A = (a_{ij}) \in C^1(\mathbb{R}^n)$: $n \times n$ symmetric, positive matrix

$V \in L^\infty(\mathbb{R}^n)$: real valued

$$A = I,\, V = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}$$

$$P := \nabla \cdot A\nabla + V$$

Forward problem

${}^{\exists !}u \in H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{D})$:

$$\begin{cases} Lu := Pu + \kappa^2 u = 0 \text{ in } \ \mathbb{R}^n \setminus \overline{D} \\ u = 0 \text{ on } \partial D \,(\text{sound soft}) \text{ or } \partial_{\nu_A} u = 0 \text{ on } \partial D \ (\text{sound hard}) \\ u = u^i + u^s,\ u^i \approx e^{i\kappa d\cdot x} \text{ outside a large compact set}\,(\text{incident wave}), \\ \qquad d \in S^{n-1}\,(\text{unit sphere centered at } 0) \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r u^s - i\kappa u^s) = 0 \ (\text{Sommerfeld radiation condition}), \end{cases}$$

where $r = |x|,\ \ \kappa > 0$: wave number,

$\partial_{\nu_A} u = A\nabla u \cdot \nu,\ \nu\ :$ outer unit normal of ∂D.

far-field pattern and inverse problem

Further, $u^s(x) = u^s(x,d)$ admits an asymptotic expansion

$$u^s(x) = r^{-(n-1)/2} e^{i\kappa r} u^\infty(\hat{x}) + O(r^{-(n+1)/2}) \ (r \to \infty),$$

where $\hat{x} := |x|^{-1}x$,

$u^\infty(\hat{x}) = u^\infty(\hat{x}, d)$: far-field pattern associated to u^i.

Inverse problem

Let Ω, A, V be known and D be unknown. Then, reconstruct D from $u^\infty(\cdot, d)$ for finitely many $d \in S^{n-1}$.

Main Theorem

Theorem 1 (Main Theorem)

Let A, V be real analytic in a neighborhood of ∂D. Then, we have the followings.

(i) *(sound soft case) If ∂D consists of non-analytic hyper-surfaces, then for any given $d \in S^{n-1}$, we can reconstruct D from $u^\infty(\cdot, d)$.*

(ii) *(sound hard case) If ∂D consists of non-analytic C^1 hyper-surfaces and $V + \kappa^2 \neq 0$ near ∂D, then for any given linearly independent $d_j \in S^{n-1}\,(1 \le j \le n-1)$, we can reconstruct D from $u^\infty(\cdot, d_j)\,(1 \le j \le n-1)$*

Known results

Under the conditions $A = I$, $V = 0$,

uniqueness

(i) (sound soft case)
Colton-Sleeman ('83), Stefanov-Uhlmann ('04), Gintides ('05), $\cdots$

(ii) (polygonal or polyhedral D)
Cheng-Yamamoto ('05), Alessandrini-Rondi ('05),$\cdots$,
Liu-Zou('06,'07)

stability
Isakov ('91, '93), $\cdots$

reconstruction (polygonal D)
Ikehata ('04,'05) : reconstruction of the convex hull of D

Kang, Lim and Nakamura : reconstruction of some non-convex D

Properties of analytic sets

Let $Q(x, \partial_x)$ be an elliptic operator with analytic coefficients in a domain $U \subset \mathbb{R}^n$.

sound soft case

Theorem 2

Let $u \not\equiv 0$, $Qu = 0$ in U, S be a closed topological manifold in U, $\dim_{\mathbb{R}} S = n - 1$. Also, let

$$S \subset Z := \{x \in U\,;\, u(x) = 0\},\ Z_{\text{irr}} := \{x \in Z\,;\, \nabla u(x) = 0\}.$$

Then, we have the followings.

(i) $S \cap Z_{\text{irr}}$ *is nowhere dense in* S.

(ii) $S \setminus Z_{\text{irr}}$ *is real analytically smooth.*

sound soft case continued

Corollary 3

Let ∂D be a Lipschitz, non-analytic surface. Also, let $u \not\equiv 0$, $Lu = 0$ in $\Omega \setminus \overline{D}$, $u|_{\partial D} = 0$. Then, for each $x^0 \in \partial D$, u cannot be extended analytically across ∂D in any neighborhood of x_0.

sound hard case

Theorem 4

Let $U \subset \mathbb{R}^n$ be a domain, $f_j\,(1 \le j \le n-1)$ be real valued, analytic functions in U and A be an $n \times n$ matrix which is invertible and analytic in U. Also, let $S \subset U$ be a C^1 hyper-surface.

Assume that

(i) *there exists $x^0 \in U$ such that $\nabla f_j(x^0)\,(1 \le j \le n-1)$ are linearly independent,*

(ii) $\partial_{\nu_A} f_j = 0$ *on* S $(1 \le j \le n-1)$.

Then, S is real analytic in some dense subset of S.

sound hard case continued

Corollary 5

Let ∂D be a non-analytic C^1 surface.

Also, let $u_j \in H^1(\Omega \setminus \overline{D})$ $(1 \le j \le n-1)$ satisfy

$$Lu_j = 0 \quad \text{in} \quad \Omega \setminus \overline{D},\ \partial_{\nu_A} u_j = 0 \quad \text{on} \quad \partial D\ (1 \le j \le n-1).$$

Assume that $\nabla \mathrm{Re}\, u_j(x^0)$ $(1 \le j \le n-1)$ are linearly independent at some $x^0 \in \Omega \setminus \overline{D}$. Then, for each $x^1 \in \partial D$, one of u_j cannot be extended analytically across ∂D in any neighborhood of x^1.

Outline of proof of the result

Taylor coefficients of u^s

$G(x,y)$: outgoing Green function of L,

$G^\infty(\hat{x},y)$: far-field pattern of $G(x,y)$

By the Green formula and the asymptotic behaviors of u^s and G,

$$u^\infty(\hat{x},d) = -\int_{\partial D}\Big(\partial_{\nu_A}u^s(y,d)G^\infty(\hat{x},y) - \partial_{\nu_A}G^\infty(\hat{x},y)u^s(y,d)\Big)\,ds(y). \tag{1}$$

Hence,

$$\begin{aligned}\textstyle\int_{S^{n-1}} u^\infty(\theta,d)g(\theta)\,ds(\theta) = \\ -\textstyle\int_{\partial D}\Big(\partial_{\nu_A}u^s(y,d)v_g(y) - \partial_{\nu_A}v_g(y)u^s(y,d)\Big)\,ds(y),\end{aligned} \tag{2}$$

where

$$(Hg)(y) = v_g(y) = \int_{S^{n-1}} G^\infty(\hat{x},y)g(\hat{x})\,ds(\hat{x}). \tag{3}$$

Taylor coefficients of u^s continued

Let B be a non-vibrating domain, $\overline{D} \subset B$, $z \notin \overline{B}$.

Then,

$$H \,:\, L^2(S^{n-1}) \longrightarrow L^2(\partial B)$$

has a **dense range**.

Hence, for any $\rho > 0$ and $\alpha \in Z_+^n := (\mathbb{N} \cup \{0\})^n$,

$$^{\exists} g_m = g_m^{z,\rho,\alpha} \in L^2(S^{n-1})\,(m \in \mathbb{N})\ :$$

$$v_{g_m} \to \psi(x,z,\rho,\alpha) \,:=\, \tfrac{\rho^{|\alpha|}}{\alpha!}\partial_z^\alpha G(x,z)\ (x \in \overline{B}) \quad \text{in} \quad C^1(\overline{B})$$

Taylor coefficients of u^s continued

Therefore,

$$\lim_{m\to\infty} \int_{S^{n-1}} u^\infty(\theta, d) g_m^{z,\rho,\alpha}(\theta)\, ds(\theta)$$

$$= -\int_{\partial D} \Big(\partial_{\nu_A} u^s(y,d)\psi(y,z,\rho,\alpha) - \partial_{\nu_A}\psi(y,z,\rho,\alpha) u^s(y,d) \Big)\, ds(y)$$

$$= \frac{\rho^{|\alpha|}}{\alpha!} \partial_z^\alpha u^s(z,d). \qquad (4)$$

Reconstruction of D for the sound soft case

Let

$$I(z,\rho,\alpha) := \lim_{m\to\infty}\int_{S^{n-1}} u^\infty(\theta,d) g_m^{z,\rho,\alpha}(\theta)\, ds(\theta). \tag{5}$$

Theorem 6

Let $C=\{c(t) : 0\le t\le 1\}$ be a non-self intersecting continuous curve in $\overline{\Omega}$ which joins the two different points $c(0)$, $c(1)\in\partial\Omega$, and for $0<t_0<1$, $c(t_0)\in\partial D$, $c(t)\notin\overline{D}\,(0\le t<t_0)$. Then, we have the followings.

$$t_0=\sup\{t\in[0,1] : \sup_{0<t'<t} R(c(t');C)<\infty\}, \tag{6}$$

where

$$R(c(t);C)=\limsup_{\rho\to 0}\limsup_{|\alpha|\to\infty}\Big|I(c(t),\rho,\alpha)\Big|. \tag{7}$$

Reconstruction of D for the sound hard case

If $x^0 \notin \overline{D}, x^0$ is near ∂D, we can show the assumption of Corollary 5 for $\mathrm{Re}\, u_j = \mathrm{Re}\, u(\cdot, d_j)$ $(1 \le j \le n-1)$ is OK :

$$\nabla \mathrm{Re}\, u_j(x^0)\ (1 \le j \le n-1)\ :\ \text{linearly independent}$$

by using the assumption $V + \kappa^2 \neq 0$ near ∂D.

$$\begin{cases} I_j(z,\rho,\alpha) := \lim_{m\to\infty} \int_{S^{n-1}} u^\infty(\theta, d_j) g_m^{z,\rho,\alpha}(\theta)\, ds(\theta)\ (1 \le j \le n-1) \\ J(z,\rho,\alpha) := \sum_{j=1}^{n-1} \Big| I_j(z,\rho,\alpha) \Big|. \end{cases} \tag{8}$$

Reconstruction of D for the sound hard case continued

Theorem 7

Let $C = \{c(t) \,:\, 0 \leq t \leq 1\}$ *be a non-self intersecting continuous curve in* $\overline{\Omega}$ *which joins the two different points* $c(0)$, $c(1) \in \partial\Omega$, *and for* $0 < t_0 < 1$, $c(t_0) \in \partial D$, $c(t) \notin \overline{D}\,(0 \leq t < t_0)$. *Then, we have the followings.*

$$t_0 = \sup\{t \in [0,1] : \sup_{0<t'<t} \tilde{R}(c(t');C) < \infty\}, \tag{9}$$

where

$$\tilde{R}(c(t);C) = \limsup_{\rho\to 0}\limsup_{|\alpha|\to\infty} \Big| J(c(t),\rho,\alpha) \Big|. \tag{10}$$