Microlocal analysis and inverse problems
Lecture 5: On the linearized local Calderón problem (a Watermelon approach)

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Outline

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The Calderón problem

In a seminal paper of 1980, A. Calderón asked whether it was possible to determine the electrical conductivity of a body by making current and voltage measurements at the boundary.

Mathematical formulation: let Ω be a bounded open set in $\mathbf{R}^n$, the electrical conductivity is represented by a positive bounded function γ. Given a potential f on the boundary, the induced potential on Ω satisfies

$$\operatorname{div}(\gamma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

The voltage to current map is given by

$$\Lambda_\gamma f = (\gamma \partial_\nu u)|_{\partial\Omega}$$

where ν is the exterior unit normal. The question raised by Calderón is whether the map $\gamma \mapsto \Lambda_\gamma$ is injective.

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The linearized Calderón problem

In fact, Calderón dealt with the linearized problem. We have

$$Q_\gamma(f,g) = \int_{\partial\Omega} \Lambda_\gamma f\, g\, ds = \int_\Omega \gamma \nabla u \nabla v\, dx$$

if u is a solutions to the former Dirichlet problem with boundary data f and v an harmonic extension of g.

Then the differential of the map $\gamma \mapsto Q_\gamma$ at $\gamma = 1$ is given by

$$D_\gamma Q|_{\gamma=1}(\delta\gamma)(f,g) = \int_\Omega \delta\gamma\, \nabla u \nabla v\, dx$$

if u and v are harmonic functions with trace f, g at the boundary.

The linearized problem is the injectivity of the former differential at $\gamma = 1$.

The linearized Calderón problem

The linearized problem can be reformulated in these terms: does the cancellation of the integral

$$\int_\Omega \delta\gamma \, \nabla u \nabla v \, dx = 0$$

for all couple of harmonic functions (u, v) imply $\delta\gamma = 0$?

The answer can easily seen to be yes: take u and v to be two conjugate harmonic exponentials

$$e^{-ix\cdot\zeta}, \quad \zeta \in \mathbf{C}^n, \quad \zeta^2 = 0$$

and one obtains

$$0 = |\zeta|^2 \int_\Omega \delta\gamma \, e^{-ix\cdot(\zeta+\bar{\zeta})} dx = |\zeta|^2 \, \widehat{1_\Omega \delta\gamma}(2 \operatorname{Re} \zeta)$$

hence $\delta\gamma = 0$ since any vector $\xi \in \mathbf{R}^n$ is the real part of a $\zeta \in \mathbf{C}^n$ such that $\zeta^2 = 0$.

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Inverse problem for the Schrödinger equation

We have

$$\operatorname{div}(\gamma\nabla u) = \gamma\Delta u + \nabla\gamma\cdot\nabla u = \frac{1}{\sqrt{\gamma}}(\Delta - q)(\sqrt{\gamma}u)$$

where $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. When γ is smooth enough, (using boundary determination) the Calderón problem is a particular case of the following inverse problem on the Schrödinger equation: does the equality of the Dirichlet-to-Neumann maps $\Lambda^s_{q_1} = \Lambda^s_{q_2}$ imply $q_1 = q_2$?

The Dirichlet-to-Neumann map Λ^s_q associated to the Schrödinger equation is

$$\Lambda^s_q : H^{\frac{1}{2}}(\partial\Omega) \ni f \mapsto \partial_\nu u$$

where u is a solution to the Dirichlet problem

$$-\Delta u + qu = 0, \quad u|_{\partial\Omega} = f.$$

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The linearized version

The linearized version of the inverse problem for the Schrödinger equation can be reformulated in these terms: does the cancellation of the integral

$$\int_\Omega \delta q \, uv \, dx = 0$$

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Local problem

The local problem or the problem with partial data are also of great interest.

Partial data:
Does the Dirichlet-to-Neumann map measured only on one part of the boundary uniquely determine the conductivity? More precisely, if

$$\Lambda_{\gamma_1} f|_\Sigma = \Lambda_{\gamma_2} f|_\Sigma \quad \text{for all } f \in H^{\frac{1}{2}}(\partial\Omega)$$

where Σ is, say, an open neighbourhood of a point x_0 of the boundary, do we have $\gamma_1 = \gamma_2$?
Local problem:
If for all functions $f \in H^{\frac{1}{2}}(\partial\Omega)$ supported in Σ we have

$$\Lambda_{\gamma_1} f|_\Sigma = \Lambda_{\gamma_2} f|_\Sigma$$

where Σ is, say, an open neighbourhood of a point x_0 of the boundary, do we have $\gamma_1 = \gamma_2$?

Some references

A few dates:

1980 Calderon's seminal paper: linearization of the problem and uniqueness for conductivities close to a constant

1985 Kohn and Vogelius: boundary determination and real analytic case

1987 Sylvester and Uhlmann: resolution of the identifiability problem in dimension $n \geq 3$

1996 Nachmann: uniqueness in the 2D case

2002 Bukhgeim and Uhlmann: partial data on big subsets of the boundary

2004 Kenig, Sjöstrand and Uhlmann: partial data on possibly small subsets

2006 Astala, Päivärinta: resolution of the Calderón problem in 2D

2008 Bukhgeim: Schrödinger 2D

2008 Imanuvilov, Uhlmann and Yamamoto: partial data and local problem in 2D

Main Theorem

We investigate the linearized version of the local problem (on the Schrödinger equation, but the proof also works for the conductivity problem): does the cancellation of the integral

$$\int_\Omega f\, uv\, dx = 0$$

for all couple of harmonic functions (u, v) vanishing on some open subset Γ of the boundary imply $f = 0$?

Theorem

Let Ω be a connected bounded open set in $\mathbf{R}^n$, $n \geq 2$, with smooth boundary. The set of products of harmonic functions on Ω which vanish on a closed proper subset $\Gamma \subsetneq \partial\Omega$ of the boundary is dense in $L^1(\Omega)$.

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From local to global

In fact, it is enough to prove local uniqueness.

Theorem

Let Ω be a bounded open set in $\mathbf{R}^n$, $n \geq 2$, with smooth boundary. Let $x_0 \in \partial\Omega$ assume that we have the cancelation

$$\int_\Omega f\, uv\, dx = 0$$

for any couple of harmonic functions u and v vanishing on the complementary Γ of an open neighbourhood of x_0. Then there exists $\delta > 0$ such that f vanishes on $B(x_0, \delta) \cap \Omega$.

Using a conformal transformation, one can suppose without loss of generality that Ω is on one side of the tangent hyperplane to Ω at x_0.

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The setting

Our setting will therefore be as follows: $x_0 = 0$, $T_{x_0}(\Omega) : x_1 = 0$ and

$$\Omega \subset \{x \in \mathbf{R}^n : |x + e_1| < 1\}, \quad \Gamma = \{x \in \partial\Omega : x_1 \geq -2c\}.$$

We assume

$$\int_\Omega fuv\,dx = 0$$

for any couple of harmonic functions u and v on Ω satisfying

$$u|_\Gamma = v|_\Gamma = 0$$

The Laplacian on $\mathbf{R}^n$ has $p(\xi) = \xi^2$ as a principal symbol, we consider the (complex) characteristic set

$$p^{-1}(0) = \{\zeta \in \mathbf{C}^n : \zeta^2 = 0\}.$$

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Decomposition of the frequencies

In dimension $n = 2$, this set is the union of two complex lines

$$p^{-1}(0) = \mathbf{C}\gamma \cup \mathbf{C}\overline{\gamma}$$

where $\gamma = ie_1 + e_2 = (i, 1) \in \mathbf{C}^2$. The differential of the map

$$s : p^{-1}(0) \times p^{-1}(0) \to \mathbf{C}^n$$
$$(\zeta, \eta) \mapsto \zeta + \eta$$

at (ζ_0, η_0) is surjective

$$Ds(\zeta_0, \eta_0) : T_{\zeta_0}p^{-1}(0) \times T_{\eta_0}p^{-1}(0) \to \mathbf{C}^n$$
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provided $\mathbf{C}^n = T_{\zeta_0}p^{-1}(0) + T_{\eta_0}p^{-1}(0)$, i.e. provided ζ_0 and η_0 are linearly independent. This is the case if $\zeta_0 = \gamma$ and $\eta_0 = -\overline{\gamma}$; if $\varepsilon > 0$ is small enough, all $w \in \mathbf{C}^n$, $|w - 2ie_1| < 2\varepsilon$ may be decomposed under the form

$$w = \zeta + \eta, \quad \text{with } \zeta, \eta \in p^{-1}(0),\ |\zeta - \gamma| \lesssim \varepsilon,\ |\eta + \overline{\gamma}| \lesssim \varepsilon.$$

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Harmonic exponentials

The exponentials with linear weights

$$e^{-\frac{i}{h}x\cdot\zeta}, \quad \zeta \in p^{-1}(0)$$

are harmonic functions. We need to add a correction term in order to obtain harmonic functions u satisfying the boundary requirement $u|_{\Gamma} = 0$. Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be a cutoff function which equals 1 on Γ, we consider the solution w to the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w|_{\partial\Omega} = -(e^{-\frac{i}{h}x\cdot\zeta}\chi)|_{\partial\Omega}. \end{cases}$$

The function

$$u(x,\zeta) = e^{-\frac{i}{h}x\cdot\zeta} + w(x,\zeta)$$

is harmonic and satisfies $u|_{\Gamma} = 0$.

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Exponential estimates

We have the following bound on w:

$$\begin{aligned} \|w\|_{H^1(\Omega)} &\leq C_1 \|e^{-\frac{i}{h}x\cdot\zeta}\chi\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C_2(1+h^{-1}|\zeta|)^{\frac{1}{2}}\, e^{\frac{1}{h}H_K(\operatorname{Im}\zeta)} \end{aligned} \tag{1}$$

where H_K is the supporting function of the compact subset $K = \operatorname{supp}\chi \cap \partial\Omega$ of the boundary

$$H_K(\xi) = \sup_{x\in K} x\cdot\xi, \quad \xi \in \mathbf{R}^n.$$

In particular, if we take χ to be supported in $x_1 \leq -c$ and equal to 1 on $x_1 \leq -2c$ then the bound (1) becomes

$$\|w\|_{H^1(\Omega)} \leq C_2(1+h^{-1}|\zeta|)^{\frac{1}{2}}\, e^{-\frac{c}{h}\operatorname{Im}\zeta_1}\, e^{\frac{1}{h}|\operatorname{Im}\zeta'|} \qquad \text{when } \operatorname{Im}\zeta_1 \geq 0. \tag{2}$$

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Using those exponential solutions, we obtain from the cancellation of the integral the following bound

$$\Big|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\Big| \leq \|f\|_{L^\infty(\Omega)}\big(\|e^{-\frac{i}{h}x\cdot\zeta}\|_{L^2(\Omega)}\|w(x,\eta)\|_{L^2(\Omega)} + \|e^{-\frac{i}{h}x\cdot\eta}\|_{L^2(\Omega)}\|w(x,\zeta)\|_{L^2(\Omega)} + \|w(x,\eta)\|_{L^2(\Omega)}\|w(x,\zeta)\|_{L^2(\Omega)}\big)$$

thus when $\operatorname{Im}\zeta_1 \geq 0, \operatorname{Im}\eta_1 \geq 0$ and $\zeta,\eta \in p^{-1}(0)$

$$\Big|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\Big| \leq C_3\|f\|_{L^\infty(\Omega)}(1+h^{-1}|\eta|)^{\frac12}(1+h^{-1}|\zeta|)^{\frac12} \times e^{-\frac{c}{h}\min(\operatorname{Im}\zeta_1,\operatorname{Im}\eta_1)}\,e^{\frac1h(|\operatorname{Im}\zeta'|+|\operatorname{Im}\eta'|)}$$

In particular if $|\zeta - a\gamma| < \varepsilon a$ and $|\eta + a\overline{\gamma}| < \varepsilon a$ with $\varepsilon \leq 1/2$ then

$$\Big|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\Big| \leq C_4 h^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{ca}{2h}}\,e^{\frac{2\varepsilon a}{h}}$$

We need to extrapolate the exponential decay.

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$$\left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \leq C_3\|f\|_{L^\infty(\Omega)}(1+h^{-1}|\eta|)^{\frac{1}{2}}(1+h^{-1}|\zeta|)^{\frac{1}{2}} \times e^{-\frac{c}{h}\min(\operatorname{Im}\zeta_1,\operatorname{Im}\eta_1)}\,e^{\frac{1}{h}(|\operatorname{Im}\zeta'|+|\operatorname{Im}\eta'|)}$$

In particular if $|\zeta - a\gamma| < \varepsilon a$ and $|\eta + a\overline{\gamma}| < \varepsilon a$ with $\varepsilon \leq 1/2$ then

$$\left|\int_\Omega f(x)e^{-\frac{i}{h}x\cdot(\zeta+\eta)}\,dx\right| \leq C_4h^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{ca}{2h}}\,e^{\frac{2\varepsilon a}{h}}$$

We need to extrapolate the exponential decay.

Outline

The Segal-Bargmann transform

The Segal-Bargmann transform of an L^∞ function f on $\mathbf{R}^n$ is given by the following formula

$$Tf(z) = \int_{\mathbf{R}^n} e^{-\frac{1}{2h}(z-y)^2} f(y)\, dy$$

with $z = x + i\xi \in \mathbf{C}^n$.

The analytic wave front set $\mathrm{WF}_a(f)$ of f is the complementary of the set of all covectors $(x_0, \xi_0) \in T^*\mathbf{R}^n \setminus 0$ such that there exists a neighbourhood V_{z_0} of $z_0 = x_0 - i\xi_0$ in $\mathbf{C}^n$, $\chi \in C_0^\infty(\mathbf{R}^n)$ with $\chi(x_0) = 1$, and $c > 0$ and $C > 0$ such that

$$|T(\chi f)(z)| \leq C e^{-\frac{c}{h} + \frac{1}{2h}|\operatorname{Im} z|^2}, \quad \forall z \in V_{z_0}, \quad \forall h \in (0,1].$$

The analytic wave front set $\mathrm{WF}_a(f)$ is a closed conic set and its image by the first projection $T^*\mathbf{R}^n \to \mathbf{R}^n$ is the analytic singular support of f, i.e. the set of points $x_0 \in \mathbf{R}^n$ for which there is no neighbourhood on which f is a real analytic function.

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$$|T(\chi f)(z)| \leq Ce^{-\frac{c}{h}+\frac{1}{2h}|\operatorname{Im} z|^2}, \quad \forall z \in V_{z_0}, \quad \forall h \in (0,1].$$

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Kashiwara's Watermelon theorem

If a distribution f is supported on one side of a plane H and if $x_0 \in \partial H \cap \operatorname{supp} f$ then f cannot be analytic at a x_0, so the analytic wave front set of f cannot be empty. The following result gives explicitly a covector which is in the wave front set.

Theorem

Let f be a distribution supported in a half-space H, let $x_0 \in \partial H \cap \operatorname{supp} f$ then the analytic wave front set of f contains all non-zero conormal vectors to the hyperplane at x_0.

In fact the microlocal version of Holmgren's uniqueness theorem is a consequence of a more general result due to Kashiwara

Watermelon Theorem

Let f be a distribution supported in a half-space H, if (x_0, ξ_0) belongs to the analytic wave front set of f, then so does $(x_0, \xi_0 + t\nu)$ where ν denotes a conormal to the hyperplane ∂H provided $\xi_0 + t\nu \neq 0$.

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Segal-Bargmann and Fourier transform

The kernel of the Segal-Bargmann transform of a function $f \in L^\infty$ can be written as a linear superposition of exponentials with linear weights

$$e^{-\frac{1}{2h}(z-y)^2} = e^{-\frac{z^2}{2h}}(2\pi h)^{-\frac{n}{2}} \int e^{-\frac{t^2}{2h}} e^{-\frac{i}{h} y\cdot(t+iz)}\, dt$$

therefore we get

$$Tf(z) = (2\pi h)^{-\frac{n}{2}} \iint e^{-\frac{1}{2h}(z^2+t^2)} e^{-\frac{i}{h} y\cdot(t+iz)} f(y)\, dt\, dy.$$

Note that there is an *a priori* exponential bound

$$|Tf(z)| \leq e^{\frac{1}{2h}|\operatorname{Im} z|^2} \|f\|_{L^\infty}.$$

If f is supported in the half-space $x_1 \leq 0$ then the former estimate can be improved into

$$|Tf(z)| \leq e^{\frac{1}{2h}(|\operatorname{Im} z|^2 - |\operatorname{Re} z_1|^2)} \|f\|_{L^\infty}$$

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The Watermelon approach

The idea of the proof of the Watermelon theorem is to propagate the exponential decay by use of the maximum principle. If f is supported in the half-space $x_1 \leq 0$, one works with the subharmonic function

$$\varphi(z_1) + \frac{1}{2}(\operatorname{Re} z_1)^2 - \frac{1}{2}(\operatorname{Im} z_1)^2 + h \log |Tf(z_0 + z_1 e_1)|$$

on a rectangle R.

One of the edges of R is contained in the neighbourhood V_{z_0} where there is the additional exponential decay of the Segal-Bargmann transform and one chooses φ to be a non-negative harmonic function vanishing on the boundary of R except for the segment where there is the exponential decay. The fact that φ is positive on the interior of the rectangle R allows to propagate the exponential decay of the Segal-Bargmann transform and this translates into the propagation of singularities described in the Watermelon theorem.

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Estimates on the Segal-Bargmann transform

If $|t| < \varepsilon a$ and $|z - 2ae_1| < \varepsilon a$ with $\varepsilon \ll 1$, the decomposition of frequencies gives

$$t + iz = \zeta + \eta, \quad \zeta, \eta \in p^{-1}(0),\ |\zeta - a\gamma| < \varepsilon a,\ |\eta + a\overline{\gamma}| < \varepsilon a$$

in that setting the estimate that we have established reads

$$\left| \int_\Omega f(y) e^{-\frac{i}{h} y\cdot(t+iz)}\, dy \right| \leq C_4 \|f\|_{L^\infty(\Omega)} e^{-\frac{ca}{2h}}\, e^{\frac{2\varepsilon a}{h}}$$

thus cutting in two the integral (in t) giving $Tf(z)$ as a linear superposition we get

$$|Tf(z)| \lesssim h^{-1} \|f\|_{L^\infty(\Omega)} e^{\frac{1}{2h}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2)}\, e^{\frac{2\varepsilon a}{h}} \left(e^{-\frac{ca}{2h}} + e^{-\frac{\varepsilon^2 a^2}{4h}} \right)$$

provided $|z - 2ae_1| < \varepsilon a$ and $\operatorname{Re} z_1 \geq 0$. Now choosing $\varepsilon \ll 1$ and $a \lesssim \varepsilon$

$$|Tf(z)| \lesssim h^{-1} \|f\|_{L^\infty(\Omega)} e^{\frac{1}{2h}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2 - \frac{ca}{2})}.$$

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Back to the Watermelon approach

To sum-up we have obtained the following bounds on the Segal-Bargmann transform of f

$$e^{-\frac{\Phi(z_1)}{2h}}|Tf(z_1,x')| \lesssim h^{-1}\|f\|_{L^\infty(\Omega)} \begin{cases} 1 & \text{when } z_1 \in \mathbf{C} \\ e^{-\frac{a}{2h}} & \text{when } \operatorname{Re} z_1 = 2a,\ |\operatorname{Im} z_1|^2 + |x'|^2 \leq \varepsilon a \end{cases}$$

where the weight Φ is given by the following expression

$$\Phi(z_1) = \begin{cases} |\operatorname{Im} z_1|^2 & \text{when } \operatorname{Re} z_1 \leq 0 \\ |\operatorname{Im} z_1|^2 - |\operatorname{Re} z_1|^2 & \text{when } \operatorname{Re} z_1 \geq 0. \end{cases}$$

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The central lemma

Lemma

Let F be an entire function satisfying the following bounds

$$e^{-\frac{\Phi(s)}{2h}}|F(s)| \leq \begin{cases} 1 & \textit{when } s \in \mathbf{C} \\ e^{-\frac{c}{2h}} & \textit{when } \operatorname{Re} s = L, |\operatorname{Im} s| \leq b. \end{cases}$$

then there exist $c', \delta > 0$ such that F satisfies

$$|F(s)| \leq e^{-\frac{c'}{2h}}, \qquad \textit{when } |\operatorname{Re} s| \leq \delta \textit{ and } |\operatorname{Im} s| \leq b/2.$$

Applying the former lemma we obtain

$$|Tf(x)| \leq Ch^{-1}\|f\|_{L^\infty(\Omega)}e^{-\frac{c'}{2h}}$$

for all $x \in \Omega, |x_1| \leq \delta \ll 1$. Letting h tend to 0 we deduce

$$f(x) = 0, \quad \forall x \in \Omega, \quad 0 \geq x_1 \geq -\delta.$$

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