Microlocal analysis and inverse problems
Lecture 4 : Uniqueness results in admissible geometries

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Outline

Introduction

It is time now to state some precise uniqueness results. So far we have seen that:

1. In order to have Carleman estimates with opposite weights, one has to work with limiting Carleman weights. This is in order to comply with Hörmander's (necessary) criterium of solvability for non-selfadjoint operators.
2. On Riemannian manifolds, the existence of LCW is a limiting condition. It implies that manifolds have to be locally conformal to a product.
3. For reasons related to the global solvability of the transport equation, we will ask that the manifolds under scope be globally conformal to a product.
4. In fact, we need more conditions: for reasons related to the global solvability of the eikonal eqution, we ask the cutlocus of the manifold to be empty.

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Admissible geometries

The former remarks justify the introduction of:

Definition

A compact Riemannian manifold (M, g), with dimension $n \geq 3$ and with boundary ∂M, is called admissible if $M \Subset \mathbf{R} \times M_0$ for some $(n-1)$-dimensional simple manifold (M_0, g_0), and if $g = c(e \oplus g_0)$ where e is the Euclidean metric on $\mathbf{R}$ and c is a smooth positive function on M.

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Here, a compact manifold (M_0, g_0) with boundary is simple if for any $p \in M_0$ the exponential map $\exp_p$ with its maximal domain of definition is a diffeomorphism onto M_0, and if ∂M_0 is strictly convex (that is, the second fundamental form of $\partial M_0 \hookrightarrow M_0$ is positive definite).

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Main results (smooth potentials)

Theorem

Let (M, g) be admissible, and let q_1 and q_2 be two smooth functions on M. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.

(In fact, we have results for anisotropic magnetic Schrödinger operators).

Theorem

Let (M, g_1) and (M, g_2) be two admissible Riemannian manifolds in the same conformal class. If $\Lambda_{g_1} = \Lambda_{g_2}$, then $g_1 = g_2$.

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Reminder of the formal WKB expansion

Conjugated operator : $P_\varphi = \mathrm{e}^{\varphi/h} h^2 \Delta_g \mathrm{e}^{-\varphi/h}$ $\qquad P_\varphi^* = P_{-\varphi}$

Principal symbol : $p_\varphi = |\xi|^2 - |d\varphi|^2 + 2i\langle \xi, d\varphi\rangle$

We have

$$\Delta_g(\mathrm{e}^{\frac{1}{h}(\varphi+i\psi)}a) = \mathrm{e}^{\frac{1}{h}\varphi} P_\varphi(\mathrm{e}^{\frac{i}{h}\psi}a) = \mathrm{e}^{\frac{1}{h}(\varphi+i\psi)}\Big(h^0 p_\varphi(x, \mathrm{d}\psi) + 2h\Big[(\mathrm{grad}_g\varphi + i\mathrm{grad}_g\psi)a + \frac{1}{2}\Delta_g(\varphi+i\psi)a\Big] + h^2\Delta_g a\Big).$$

Eikonal equation: $p_\varphi(x, \mathrm{d}\psi) = 0$

Transport equation: $(\mathrm{grad}_g\varphi + i\mathrm{grad}_g\psi)a + \frac{1}{2}\Delta_g(\varphi+i\psi)a = 0$

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Complex geometrical optics (eikonal equation)

We suppose that the metric has the form

$$g(x) = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}$$

and we choose $\varphi(x) = x_1$. We have $\mathrm{d}\varphi = \mathrm{d}x_1$ and $\mathrm{grad}_g \varphi = c^{-1}\partial_{x_1}$.

Eikonal equation:

$$|\mathrm{d}\psi|_g^2 = |\mathrm{d}\varphi|_g^2 = c^{-1}$$
$$\langle \mathrm{d}\varphi, \mathrm{d}\psi \rangle_g = c^{-1}\partial_{x_1}\psi$$

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Eikonal equation: $|\mathrm{d}\psi|^2_{g_0} = 1$

In simple manifolds, it is easy to give explicit solutions of this eikonal equation

$$\psi(x) = d_{g_0}(x, \omega_0), \quad \omega_0 \in \tilde{M} \setminus M$$

where d_{g_0} is the geodesical distance ($\tilde{M}$ is a simple extension of M). We have

$$\mathrm{grad}_{g_0}\psi = (\mathrm{d}\psi)^\sharp = -\frac{\exp^{-1}_{\omega_0}(x')}{\psi(x')}$$

In fact, one can use geodesical polar coordinates

$$x' = \exp_{\omega_0}(r\theta), \quad r = d_{g_0}(x', \omega_0) > 0, \quad \theta \in S_{\omega_0}\tilde{M}.$$

In those coordinates, the metric reads

$$g_0 = \mathrm{d}r^2 + h_0(r)$$

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Transport equation: $c^{-1}\partial_{x_1} a + i\mathrm{grad}_g \psi a + \frac{1}{2}\Delta_g(\varphi + i\psi)a = 0$

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$$\Delta_{g_0}\psi = \frac{1}{|g_0|^{1/2}}\frac{\partial}{\partial r}\left(|g_0|^{1/2}\frac{\partial \psi}{\partial r}\right) = \frac{1}{2}\partial_r \log |g_0|$$

hence the transport equation reads

$$\partial_{x_1} a + \partial_r a + \frac{1}{4}\partial_r \log |g_0| \, a = 0$$

and can easily be solved

$$a = h(x_1 + ir)|g_0|^{-1/4} b(\theta)$$

where h is a holomorphic function.

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Complex geometrical optics (L^p case)

Proposition

Assume that $q \in L^{n/2}(M)$. Let $\omega \in \tilde{M}_0 \setminus M_0$ be a fixed point, let $\lambda \in \mathbf{R}$ be fixed, and let $b \in C^\infty(S^{n-2})$ be a function. Write $x = (x_1, r, \theta)$ where (r, θ) are polar normal coordinates with center ω in $(\tilde{M}_0, g_0)$. For $|\tau|$ sufficiently large outside a countable set, there exists $u_0 \in H^1(M)$ satisfying

$$(-\Delta_g + q)u = 0 \quad \text{in } M,$$
$$u = \mathrm{e}^{-\tau x_1}(\mathrm{e}^{-i\tau r}|g|^{-1/4}\mathrm{e}^{i\lambda(x_1+ir)}b(\theta) + r)$$

where r_0 satisfies

$$|\tau|\|r\|_{L^2(M)} + \|r\|_{H^1(M)} + \|r\|_{L^{\frac{2n}{n-2}}(M)} \lesssim 1.$$

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Simple manifolds

Definition

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2. Simple manifolds are diffeomorphic to a ball.
3. A hemisphere is not simple.

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Attenuated X-ray transform

The unit sphere bundle :

$$SM_0 = \bigcup_{x \in M_0} S_x, \qquad S_x = \big\{(x,\xi) \in T_x M_0\,;\, |\xi|_g = 1\big\}.$$

Boundary: $\partial(SM_0) = \{(x,\xi) \in SM_0\,;\, x \in \partial M_0\}$ union of inward and outward pointing vectors:

$$\partial_\pm(SM_0) = \big\{(x,\xi) \in SM_0\,;\, \pm\langle \xi, \nu \rangle \leq 0\big\}.$$

Denote by $t \mapsto \gamma(t,x,\xi)$ the unit speed geodesic starting at x in direction ξ, and let $\tau(x,\xi)$ be the time when this geodesic exits M_0.
Godesic ray transform with constant attenuation $-\lambda$:

$$T_\lambda f(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi)) \mathrm{e}^{-\lambda t}\,\mathrm{d}t, \qquad (x,\xi) \in \partial_+(SM_0).$$

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Injectivity of the attenuated X-ray transform

Proposition

Let (M_0, g_0) be a simple manifold. There exists $\varepsilon > 0$ such that if λ is a real number with $|\lambda| < \varepsilon$ and if $f \in C^\infty(M)$, then the condition $T_\lambda f(x,\xi) = 0$ for all $(x,\xi) \in \partial_+(SM_0)$ implies that $f = 0$.

This was known when $\lambda = 0$. The proof in the attenuated case uses perturbation arguments.

What about nonsmooth functions ?

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Normal operator

Notations: $\mu(x,\xi) = -\langle \xi, \nu(x)\rangle$ and $\mathrm{d}N$ is the volume form on N.

Scalar product:

$$(h,\tilde{h})_{L^2_\mu(\partial_+(SM_0))} = \int_{\partial_+(SM_0)} h\tilde{h}\mu \,\mathrm{d}(\partial(SM_0))$$

Adjoint of the ray transform:

$$T_\lambda^* h(x) = \int_{S_x} \mathrm{e}^{-\lambda\tau(x,-\xi)} h(\varphi_{-\tau(x,-\xi)}(x,\xi)),\, \mathrm{d}S_x(\xi), \qquad x \in M_0.$$

where $\varphi_t(x,\xi) = (\gamma(t,x,\xi),\dot{\gamma}(t,x,\xi))$ is the geodesic flow.

Lemma

$T_\lambda^* T_\lambda$ is a self-adjoint elliptic pseudodifferential operator of order -1 in M_0^{int}.

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Injectivity of the attenuated X-ray transform (non-smooth case)

Lemma

Let (M_0, g_0) *be an* $(n-1)$*-dimensional simple manifold, and let* $f \in L^1(M_0)$*. Consider the integrals*

$$\int_{S^{n-2}} \int_0^{\tau(\omega,\theta)} f(r,\theta) \mathrm{e}^{-\lambda r} b(\theta) \,\mathrm{d}r \,\mathrm{d}\theta$$

where (r, θ) *are polar normal coordinates in* (M_0, g_0) *centered at some* $\omega \in \partial M_0$*, and* $\tau(\omega, \theta)$ *is the time when the geodesic* $r \mapsto (r, \theta)$ *exits* M_0*. If* $|\lambda|$ *is sufficiently small, and if these integrals vanish for all* $\omega \in \partial M_0$ *and all* $b \in C^\infty(S^{n-2})$*, then* $f = 0$*.*

Using Elliptic regularity

Preliminary step: extend (M_0, g_0) to a slightly larger simple manifold and f by zero. In this way we can assume that f is compactly supported in M_0^{int}.

Let b also depend on ω and change notations to write

$$\int_{S_x} \int_0^{\tau(x,\xi)} \mathrm{e}^{-\lambda t} f(\gamma(t,x,\xi)) b(x,\xi)\, \mathrm{d}t\, \mathrm{d}S_x(\xi) = 0.$$

Next we make the choice $b(x,\xi) = h(x,\xi)\mu(x,\xi)$ and integrate the last identity over ∂M_0 to obtain

$$\int_{\partial_+(SM_0)} \int_0^{\tau(x,\xi)} \mathrm{e}^{-\lambda t} f(\gamma(t,x,\xi)) h(x,\xi) \mu\, \mathrm{d}t\, \mathrm{d}(\partial(SM_0)) = 0.$$

By adjunction, we get

$$\int_{M_0} f(x) T_\lambda^* h(x)\, \mathrm{d}V(x) = 0$$

for all $h \in C_0^\infty((\partial_+(SM_0))^{\text{int}})$.

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Using Elliptic regularity

Choose $h = T_\lambda \varphi$ for $\varphi \in C_0^\infty(M_0^{\text{int}})$ so that

$$\int_{M_0} f(x) T_\lambda^* T_\lambda \varphi(x) \, \mathrm{d}V(x) = 0.$$

Since $T_\lambda^* T_\lambda$ is self-adjoint, we have

$$\int_{M_0} (T_\lambda^* T_\lambda f(x)) \varphi(x) \, \mathrm{d}V(x) = 0$$

for all test functions φ, so $T_\lambda^* T_\lambda f = 0$.

By ellipticity, since f was compactly supported in M_0^{int}, it follows that $f \in C_0^\infty(M_0^{\text{int}})$. One can now use the injectivity result for f smooth to conclude that $f = 0$.

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Outline

Using CGOs

Starting point: If $q = q_1 - q_2$

$$\int_M q u_1 u_2 \, \mathrm{d}V_g = 0$$

where

$$u_1 = \mathrm{e}^{-\tau(x_1 + ir)}(|g|^{-1/4}\mathrm{e}^{i\lambda(x_1+ir)}b(\theta) + r_1),$$
$$u_2 = \mathrm{e}^{\tau(x_1+ir)}(|g|^{-1/4} + r_2).$$

with $b \in C^\infty(S^{n-2})$ and

$$\|r_j\|_{L^{\frac{2n}{n-2}}(M)} = \mathcal{O}(1), \quad \|r_j\|_{L^2(M)} = o(1)$$

as $\tau \to \infty$.

Using CGOs

Noting that $\mathrm{d}V_g = |g|^{1/2}\mathrm{d}x_1\,\mathrm{d}r\,\mathrm{d}\theta$, we obtain that

$$\int_M q\mathrm{e}^{i\lambda(x_1+ir)}b(\theta)\,\mathrm{d}x_1\,\mathrm{d}r\,\mathrm{d}\theta = \int_M q(a_1r_2 + a_2r_1 + r_1r_2)\,\mathrm{d}V$$

The RHS converges to 0 as $\tau \to \infty$.

Taking the limit as $\tau \to \infty$, we obtain that

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_{S^{n-2}} q(x_1,r,\theta)\mathrm{e}^{i\lambda(x_1+ir)}b(\theta)\,\mathrm{d}x_1\,\mathrm{d}r\,\mathrm{d}\theta = 0.$$

This statement is true for all choices of $\omega \in \tilde{M}_0 \setminus M_0$, for all real numbers λ, and for all functions $b \in C^\infty(S^{n-2})$. Hence

$$\int_{S^{n-2}}\int_0^{\infty} f_\lambda(r,\theta)\mathrm{e}^{-\lambda r}b(\theta)\,\mathrm{d}r\,\mathrm{d}\theta = 0$$

where $f_\lambda \in L^1(M_0)$ is the function given by

$$f_\lambda(r,\theta) = \int_{-\infty}^{\infty}\mathrm{e}^{i\lambda x_1}q(x_1,r,\theta)\,\mathrm{d}x_1.$$

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End of the proof

If $|\lambda|$ is sufficiently small, it follows that $f_\lambda = 0$.

Since $q(\,\cdot\,, r, \theta)$ is a compactly supported function in $L^1(\mathbf{R})$ for a.e. (r, θ), the Paley-Wiener theorem shows that $q(\,\cdot\,, r, \theta) = 0$ for such (r, θ).

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