Microlocal analysis and inverse problems
Lecture 3 : Carleman estimates

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Outline

Construction by complex geometrical optics

In the former lecture, we saw that for our purpose, that is construction of solutions to the Schrödinger equation by means of complex geometrical optics with opposite exponential behaviours as in Sylvester and Uhlmann, one needs to use limiting Carleman weights.

The purpose of this lecture is to indeed prove the corresponding Carleman estimates, both in the L^2 setting (which corresponds to bounded potentials) and in the L^p setting (which corresponds to unbounded potentials).

We will use alternatively h or $\tau = h^{-1}$ to denote the semiclassical parameter. Sorry for the change of notations! Recall that

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denotes the conjugated operator.

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Carleman estimates

Theorem

Let (U, g) be an open Riemannian manifold and (M, g) a compact Riemannian submanifold with boundary such that $M \Subset U$. Suppose that φ is a limiting Carleman weight on (U, g). Let q be a smooth function on M. There exist two constants $C > 0$ and $0 < h_0 \leq 1$ such that for all functions $u \in C_0^\infty(M^\circ)$ and all $0 < h \leq h_0$, one has the inequality

$$\|e^{\frac{\varphi}{h}} u\|_{H^1_{\mathrm{scl}}(M)} \leq Ch \|e^{\frac{\varphi}{h}}(-\Delta + q)u\|_{L^2(M)}.$$

We decompose P_φ into its self-adjoint and skew-adjoint parts

$$P_\varphi = A + iB, \quad A = -h^2 \Delta_g - |\mathrm{grad}_g \varphi|^2,\ B = -2i\langle \mathrm{grad}_g \varphi, h\mathrm{grad}_g \rangle - ih\Delta_g \varphi$$

and we have by integration by parts

$$\|P_\varphi v\|^2 = \|Av\|^2 + \|Bv\|^2 + i([A, B]v|v).$$

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Convexification

Convexification consists in replacing φ by $\tilde{\varphi} = f \circ \varphi$.
Add $\tilde{}$ to denote all the corresponding symbols. Note that

$$\begin{aligned} \operatorname{grad}_g(f \circ \varphi) &= (f' \circ \varphi) \operatorname{grad}_g \varphi \\ \nabla^2(f \circ \varphi) &= (f'' \circ \varphi)\, d\varphi \otimes d\varphi + \underbrace{(f' \circ \varphi)\, \nabla^2 \varphi}_{=0} \end{aligned}$$

therefore

$$\begin{aligned} \{\tilde{a}, \tilde{b}\}(x,\xi) &= 4(f'' \circ \varphi)\,(f' \circ \varphi)^2 |\operatorname{grad}_g \varphi|^4 + 4(f'' \circ \varphi)\,\langle \operatorname{grad}_g \varphi, \xi^\sharp \rangle^2 \\ &= 4(f'' \circ \varphi)\,(f' \circ \varphi)^2 + \underbrace{(f'' \circ \varphi)(f' \circ \varphi)^{-2}}_{=\beta}\, \tilde{b}^2. \end{aligned}$$

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Convexification

At the operator level, this gives

$$i[\tilde{A}, \tilde{B}] = 4h(f'' \circ \varphi)\,(f' \circ \varphi)^2 + h\tilde{B}\beta\tilde{B} + h^2 R$$

where R is a first order semiclassical differential operator. For the function f, we choose the following convex polynomial

$$f(s) = s + \frac{h}{2\varepsilon}s^2, \quad f'(s) = 1 + \frac{h}{\varepsilon}s, \quad f''(s) = \frac{h}{\varepsilon}.$$

We choose h/ε small enough so that $f' > \frac{1}{2}$ Note that the coefficients of R, as well as β, are uniformly bounded with respect to h and ε.

The estimate comes from the fact that the commutator is positive (by taking ε small enough to absorb error terms) and that $e^{\frac{1}{h}\tilde{\varphi}} \simeq e^{\frac{1}{h}\varphi}$.

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Spectral cluster estimates of Sogge

We denote by $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$ the sequence of eigenvalues of $-\Delta_{g_0}$ on M_0 and $(\psi_j)_{j\geq 0}$ the corresponding sequence of eigenfunctions

$$-\Delta_{g_0}\psi_j = \lambda_j \psi_j.$$

We denote by

$$\pi_j : L^2(M_0) \to L^2(M_0), u \mapsto (u, \psi_j)\psi_j$$

the projection on the linear space spanned by the eigenfunction ψ_j so that

$$\sum_{j=0}^{\infty} \pi_j = \mathrm{Id}, \quad \sum_{j=0}^{\infty} \lambda_j \pi_j = -\Delta_{g_0}$$

and by

$$\widehat{u}(j) = \int_{M_0} u \, \overline{\psi_j} \, \mathrm{d}V_{g_0}$$

the corresponding Fourier coefficients.

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Spectral cluster estimates of Sogge

We define the spectral clusters as

$$\chi_k = \sum_{k \leq \sqrt{\lambda_j} < k+1} \pi_j, \quad k \in \mathbf{N}.$$

Note that these are projection operators, $\chi_k^2 = \chi_k$, and they constitute a decomposition of the identity

$$\mathrm{Id} = \sum_{k=0}^{\infty} \chi_k.$$

The spectral cluster estimates of Sogge are

$$\|\chi_k u\|_{L^{\frac{2n}{n-2}}(M_0)} \leq C(1+k)^{\frac{1}{2}-\frac{1}{n}} \|u\|_{L^2(M_0)},$$

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Carleman estimates

Theorem

Let (M_0, g_0) *be an* $(n-1)$*-dimensional compact manifold without boundary, and equip* $\mathbf{R} \times M_0$ *with the metric* $g = e \oplus g_0$ *where* e *is the Euclidean metric. The Euclidean coordinate is denoted by* x_1*. For any compact interval* $I \subset \mathbf{R}$ *there exists a constant* $C_I > 0$ *such that if* $|\tau| \geq 4$ *and*

$$\tau^2 \notin \mathrm{Spec}(-\Delta_{g_0})$$

then we have

$$\|\mathrm{e}^{\tau x_1} u\|_{L^{\frac{2n}{n-2}}(\mathbf{R}\times M_0)} \leq C_I \|\mathrm{e}^{\tau x_1} \Delta_g u\|_{L^{\frac{2n}{n+2}}(\mathbf{R}\times M_0)}$$

when $u \in C_0^\infty(I \times M_0)$*.*

Short bibliography on L^p Carleman estimates on elliptic operators

These works are in relation with unique continuation of solutions to Schrödinger equation with unbounded potentials.

1985 Jerison-Kenig: first L^p Carleman estimates, logarithmic weights

1986 Jerison: simplification of the proof using spectral cluster estimates (see also Sogge's book)

1987 Kenig-Ruiz-Sogge: Elliptic operators with constant coefficients, linear weights

1989 Sogge: Elliptic operators with variable coefficients, non LCW

2001 Shen: Laplace operator on the torus

2005 Koch-Tataru: construction of parametrices in general context

Remarks

1. In unique continuation problems, one traditionally uses weights for which one has $\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} > 0$ on $p_\varphi^{-1}(0)$, i.e. of the form $x_1 + x_1^2/2$.
2. These estimates can be seen as the anisotropic analogue of the estimates of Jerison and Kenig (with $x_1 = s = \log r$ see Lecture 1).
3. These estimates can also be seen as the anisotropic analogue of the estimates of Kenig, Ruiz and Sogge (who proved L^p Carleman estimates for linear weights).
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Proof of Carleman estimates

Main goal:

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbf{R}\times M_0)} \leq C_I \|f\|_{L^{\frac{2n}{n+2}}(\mathbf{R}\times M_0)}$$

when $u \in C_0^\infty(I \times M_0)$ and

$$D_{x_1}^2 u + 2i\tau D_{x_1} u - \Delta_{g_0} u - \tau^2 u = f$$

with $D_{x_1} = -i\partial_{x_1}$.

$$(D_{x_1}^2 + 2i\tau D_{x_1} - \tau^2 + \lambda_j)\pi_j u = \pi_j f$$

Symbol of the operator: $\xi_1^2 + 2i\tau\xi_1 - \tau^2 + \lambda_j \neq 0$ if $\tau^2 \neq \lambda_j$

Inverse operator:

$$G_\tau f(x_1, x') = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} m_\tau\left(x_1 - y_1, \sqrt{\lambda_j}\right) \pi_j f(y_1, x')\, \mathrm{d}y_1$$

$$m_\tau(t, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{it\eta}}{\eta^2 + 2i\tau\eta - \tau^2 + \mu^2}\, \mathrm{d}\eta, \quad \mu > 0.$$

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Lemma

If $\tau > 0$, $\mu > 0$, $\tau \neq \mu$ *and* $t \in \mathbf{R}$ *then*

$$|m_\tau(t,\mu)| \leq \frac{1}{\mu} \mathrm{e}^{-|\tau-\mu|\,|t|}, \quad |m_\tau(t,0)| \leq |t| \mathrm{e}^{-\tau|t|}.$$

Proof.

This follows by writing

$$\frac{1}{(i\eta-(\tau+\mu))(i\eta-(\tau-\mu))} = \frac{1}{2\mu}\left[\frac{1}{i\eta-(\tau+\mu)} - \frac{1}{i\eta-(\tau-\mu)}\right]$$

and by noting that for $\alpha > 0$

$$\mathcal{F}_\eta^{-1}\left\{\frac{1}{i\eta+\alpha}\right\}(t) = \begin{cases} 0, & t < 0 \\ \mathrm{e}^{-\alpha t}, & t > 0, \end{cases}.$$

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Proof of Carleman estimates

Using the spectral cluster estimates we get

$$\|u\|_{L^{\frac{2n}{n-2}}(M_0)} = \Big\| \sum_{k=0}^{\infty} \chi_k^2 u \Big\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty} (1+k)^{\frac{1}{2}-\frac{1}{n}} \|\chi_k u\|_{L^2(M_0)}.$$

Apply the estimate to $u = G_\tau f(x_1, \cdot)$,

$$\|G_\tau f(x_1,\cdot)\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty} (1+k)^{\frac{1}{2}-\frac{1}{n}} \times \Bigg(\sum_{k \leq \sqrt{\lambda_j} < k+1} \Big| \int_{-\infty}^{\infty} m_\tau(x_1 - y_1, \sqrt{\lambda_j}) \widehat{f}(y_1, j)\, \mathrm{d}y_1 \Big|^2 \Bigg)^{\frac{1}{2}}.$$

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Proof of Carleman estimate

By Minkowski's inequality, we have

$$\|G_\tau f(x_1,\cdot)\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty}(1+k)^{\frac{1}{2}-\frac{1}{n}} \times \int_{-\infty}^{\infty}\Big(\sum_{k\leq\sqrt{\lambda_j}<k+1}\Big|m_\tau\big(x_1-y_1,\sqrt{\lambda_j}\big)\widehat{f}(y_1,j)\Big|^2\Big)^{\frac{1}{2}}\,\mathrm{d}y_1$$

and since

$$\sum_{k\leq\sqrt{\lambda_j}<k+1}\Big|m_\tau\big(x_1-y_1,\sqrt{\lambda_j}\big)\widehat{f}(y_1,j)\Big|^2 \leq \sup_{k\leq\sqrt{\lambda_j}<k+1}\Big|m_\tau\big(x_1-y_1,\sqrt{\lambda_j}\big)\Big|^2 \times \|\chi_k f(y_1,\cdot)\|^2_{L^2(M_0)}$$

Proof of Carleman estimate

using once again the spectral cluster estimate we finally get

$$\|G_\tau f(x_1,\cdot)\|_{L^{\frac{2n}{n-2}}(M_0)} \lesssim \sum_{k=0}^{\infty}(1+k)^{1-\frac{2}{n}}$$

$$\times \int_{-\infty}^{\infty} \sup_{k\leq\sqrt{\lambda_j}<k+1} \left|m_\tau(x_1-y_1,\sqrt{\lambda_j})\right| \times \left\|f(y_1,\cdot)\right\|_{L^{\frac{2n}{n+2}}(M_0)} \mathrm{d}y_1.$$

Using the Lemma, we estimate

$$\sup_{k\leq\sqrt{\lambda_j}<k+1} \left|m_\tau(t,\sqrt{\lambda_j})\right| \leq \frac{1}{k}\begin{cases} \mathrm{e}^{-(k-\tau)|t|} & \text{when } \tau<k \\ 1 & \text{when } k\leq\tau<k+1 \\ \mathrm{e}^{-(\tau-k-1)|t|} & \text{when } \tau\geq k+1 \end{cases}$$

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This allows us to estimate the series

$$\sum_{k=0}^{\infty}(1+k)^{1-\frac{2}{n}} \sup_{k\leq\sqrt{\lambda_j}<k+1} \left|m_\tau(t,\sqrt{\lambda_j})\right|$$

$$\lesssim \sum_{1\leq k\leq \tau-2} k^{-\frac{2}{n}} \mathrm{e}^{-(\tau-k-1)|t|} + \tau^{-\frac{2}{n}} + \sum_{k>\tau+1} k^{-\frac{2}{n}} \mathrm{e}^{-(k-\tau)|t|} + \mathrm{e}^{-(\tau/2)|t|}$$

$$\lesssim \int_0^{\tau-2} r^{-\frac{2}{n}} \mathrm{e}^{-(\tau-r-2)|t|}\,\mathrm{d}r + 1 + \int_\tau^\infty r^{-\frac{2}{n}} \mathrm{e}^{-(r-\tau)|t|}\,\mathrm{d}r.$$

Whence

$$\sum_{k=0}^{\infty}(1+k)^{1-\frac{2}{n}} \sup_{k\leq\sqrt{\lambda_j}<k+1} \left|m_\tau(t,\sqrt{\lambda_j})\right| \lesssim 1+|t|^{-1+\frac{2}{n}}.$$

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We obtain

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$$\lesssim \int_{-\infty}^{\infty} |x_1-y_1|^{-1+\frac{2}{n}}\|f(y_1,\cdot)\|_{L^{\frac{2n}{n+2}}(M_0)}\,\mathrm{d}y_1 + |I|^{\frac{1}{2}-\frac{1}{n}}\|f\|_{L^{\frac{2n}{n+2}}(I\times M_0)}$$

and we conclude using the Hardy-Littlewood-Sobolev inequality

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Outline

Relation Carleman estimates / Resolvent estimates

Here we follow Kenig, Ruiz and Sogge and relate our Carleman estimates in the product context to resolvent estimates by freezing derivatives.

Inspired by Hähner's proof, we further conjugate the operator by an harmless oscillating factor

$$-e^{\tau x_1 - \frac{i}{2}x_1} P e^{-\tau x_1 + \frac{i}{2}x_1} = \left(D_{x_1} + \frac{1}{2}\right)^2 + 2i\tau\left(D_{x_1} + \frac{1}{2}\right) - \tau^2 - \Delta_{g_0}.$$

After translation and scaling $I = [0, 2\pi]$ and use Fourier series.

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After translation and scaling $I = [0, 2\pi]$ and use Fourier series.

Kenig-Ruiz-Sogge approach

We denote by $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the sequence of eigenvalues of $-\Delta_{g_0}$ on M_0 and $(\psi_k)_{k\geq 0}$ the corresponding sequence of eigenfunctions

$$-\Delta_{g_0}\psi_k = \lambda_k \psi_k.$$

We denote by $\pi_k : L^2(M_0) \to L^2(M_0)$ the projection on the linear space spanned by the eigenfunction ψ_k so that

$$\sum_{k=0}^{\infty} \pi_k = \mathrm{Id}, \quad \sum_{k=0}^{\infty} \lambda_k \pi_k = -\Delta_{g_0}.$$

Eigenvalues of the Laplacian Δ_g: $-(j^2 + \lambda_k)$

Eigenfunctions: $\mathrm{e}^{ijx_1}\psi_k$.

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$$\pi_{j,k} f(x) = \left(\int_0^{2\pi} \mathrm{e}^{-ijy_1} \pi_k f(y_1, x') \, \mathrm{d}y_1 \right) \mathrm{e}^{ijx_1},$$

and define the spectral clusters as

$$\chi_m = \sum_{m \leq \sqrt{j^2+\lambda_k} < m+1} \pi_{j,k}, \quad m \in \mathbf{N}.$$

Note that these are projectors $\chi_m^2 = \chi_m$.

Spectral cluster estimates of Sogge:

$$\|\chi_m u\|_{L^{\frac{2n}{n-2}}(M)} \leq C(1+m)^{\frac{1}{2}} \|u\|_{L^2(M)}$$

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We are now ready to reduce the proof of Carleman estimates to resolvent estimates.

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C\|f\|_{L^{\frac{2n}{n+2}}(M)}$$

when

$$\left(D_{x_1}+\frac{1}{2}\right)^2 u + 2i\tau\left(D_{x_1}+\frac{1}{2}\right)u - \Delta_{g_0}u - \tau^2 u = f.$$

Inverse operator:

$$\tilde{G}_\tau f = \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{\pi_{j,k}f}{\left(j+\frac{1}{2}\right)^2+2i\left(j+\frac{1}{2}\right)\tau+\lambda_k-\tau^2}.$$

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Kenig-Ruiz-Sogge approach

Use Littlewood-Paley theory to localize in frequency with respect to the Euclidean variable x_1;

$$u = \sum_{\nu=0}^{\infty} u_\nu, \quad f = \sum_{\nu=0}^{\infty} f_\nu$$

with

$$u_0 = \left(\int_0^{2\pi} u(y_1, x')\,\mathrm{d}y_1\right),$$

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and similarly for f. It suffices to prove

$$\|u_\nu\|_{L^{\frac{2n}{n-2}}} \leq C\|f_\nu\|_{L^{\frac{2n}{n+2}}}.$$

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The conjugated operator and the localization in frequency commute

$$u_\nu = \tilde{G}_\tau f_\nu.$$

We denote

$$R(z) = \left(\left(D_{x_1} + \frac{1}{2}\right)^2 - \Delta_{g_0} + z\right)^{-1}$$

the resolvent. The error made by replacing $\tilde{G}_\tau$ with the resolvent is

$$\left(R(-\tau^2 + i(2^\nu + 1)\tau) - G_\tau\right) f_\nu = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} a^\nu_{jk}(\tau)\, \pi_{j,k} f_\nu$$

$$a^\nu_{jk}(\tau) = \frac{i\tau(2^\nu - 2j) 1_{[2^{\nu-1}, 2^\nu)}(j)}{\left(\tilde{j}^2 + 2i\tau\tilde{j} - \tau^2 + \lambda_k\right)\left(\tilde{j}^2 + i(2^\nu + 1)\tau - \tau^2 + \lambda_k\right)}.$$

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Using the spectral cluster estimates

$$\left\| \left(R(-\tau^2 + i2^\nu\tau) - \tilde{G}_\tau \right) f_\nu \right\|_{L^{\frac{2n}{n-2}}(M)}$$

$$\lesssim \sum_{m=0}^{\infty} (1+m)^{\frac{1}{2}} \left\| \chi_m \left(R(-\tau^2 + i(2^\nu+1)\tau) - \tilde{G}_\tau \right) f_\nu \right\|_{L^2(M)}$$

$$\leq \sum_{m=0}^{\infty} (1+m)^{\frac{1}{2}} \sup_{m \leq \sqrt{j^2+\lambda_k} < m+1} \left| a^\nu_{jk}(\tau) \right| \times \|\chi_m f_\nu\|^2_{L^2(M)}$$

and furthermore

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Kenig-Ruiz-Sogge approach

The above series converge and is uniformly bounded with respect to τ and ν

$$\sup_{m\leq\sqrt{j^2+\lambda_k}<m+1} \left|a^{\nu}_{jk}(\tau)\right| \lesssim \frac{2^{\nu}|\tau|}{(m^2-\tau^2)^2+4^{\nu+1}\tau^2}$$

as well as

$$\sum_{m=0}^{\infty} \frac{2^{\nu}|\tau|(1+m)}{(m^2-\tau^2)^2+4^{\nu+1}\tau^2} \lesssim \int_0^{\infty} \frac{2^{\nu}|\tau|t}{(t^2-\tau^2)^2+4^{\nu+1}\tau^2}\,\mathrm{d}t$$

and if we perform the change of variables $s=4^{-\nu-1}\tau^{-2}(t^2-\tau^2)$ in the right-hand side integral, we obtain the bound

$$\sum_{m=0}^{\infty} \frac{2^{\nu}|\tau|(1+m)}{(m^2-\tau^2)^2+4^{\nu+1}\tau^2} \lesssim \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{s^2+1}.$$

Kenig-Ruiz-Sogge approach

Summing up our computations, we have the error estimate

$$\left\|\left(R(-\tau^2+i(2^\nu+1)\tau)-\tilde{G}_\tau\right)f_\nu\right\|_{L^{\frac{2n}{n-2}}(M)} \lesssim \|f_\nu\|_{L^{\frac{2n}{n+2}}(M)},$$

this means that it is enough to prove the resolvent estimate

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Carleman estimates reduce to resolvent estimates of the form

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \lesssim \left\|\left(\left(D_1+\frac{1}{2}\right)^2-\Delta_{g_0}+z\right)u\right\|_{L^{\frac{2n}{n+2}}(M)}$$

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Resolvent estimates

Theorem

Let (M, g) *be a compact Riemannian manifold (without boundary) of dimension* $n \geq 3$*, and let* δ *be a positive number. There exists a constant* $C > 0$ *such that for all* $u \in C^\infty(M)$ *and all*

$$z \in \mathbf{C} : \operatorname{Re} z + |z| \geq \delta$$

the following resolvent estimate holds

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C\|(\Delta_g - z)u\|_{L^{\frac{2n}{n+2}}(M)}.$$

Hadamard's parametrix

Fundamental solution F_0 of the flat Laplacian $-\Delta + z$ on $\mathbf{R}^n$:

$$F_0(|x|, z) = (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{\mathrm{e}^{ix\cdot\xi}}{|\xi|^2 + z} \, \mathrm{d}\xi,$$

For a radial function

$$\begin{aligned} \Delta_g f(r) &= f''(r)|\mathrm{d}r|_g^2 + f'(r)\Delta_g r \\ &= f''(r) + \frac{n-1}{r} f'(r) + \frac{\partial_r J}{J} f'(r). \end{aligned}$$

with $\mathrm{d}V_g = r^{n-1} J(r,\theta)\, \mathrm{d}r \wedge \mathrm{d}\theta$. Hence

$$(-\Delta_g + z) F_0 = \delta_0 - \underbrace{2\frac{\partial_r J}{2J} \partial_r F_0}_{\text{error term}}$$

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Hadamard's parametrix

We now take $r = d_g(x, y)$.

The Hadamard parametrix looks like

$$T_{\mathrm{Had}}(z)u = \int_M \chi(x,y) F_0\big(d_g(x,y), z\big) u(y)\, \mathrm{d}V_g(y)$$

with χ a localizing function and one has

$$(-\Delta_g + z) T_{\mathrm{Had}}(z)u = \chi(x,x)u + S(z)u$$

where $S(z)$ is an "error term".

In fact, the construction has to be slightly refined.

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Hadamard's parametrix

Theorem

The Hadamard parametrix is a bounded operator

$$|z|^{\frac{s}{2}}\, T_{\mathrm{Had}}(z) : L^p(M) \to L^q(M)$$

with a norm uniform with respect to the spectral parameter $z \in \mathbf{C}$, $|z| \geq 1$ *when* $|s| \leq 2$, $q \geq p \geq 2$

$$\frac{1}{p} - \frac{1}{q} + \frac{s}{n} = \frac{2}{n},$$

and

$$\min\left(\left|\frac{1}{p} - \frac{1}{2}\right|, \left|\frac{1}{q} - \frac{1}{2}\right|\right) > \frac{1}{2n}, \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{n-1}.$$

Hadamard's parametrix

Theorem

The Hadamard parametrix is a bounded operator

$$T_{\mathrm{Had}}(z) : L^p(M) \to L^q(M)$$

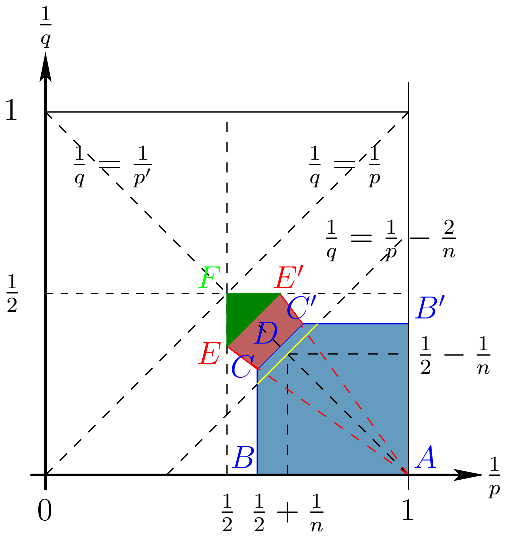
when $1 \leq p \leq 2 \leq q$ *and*

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n-1}, \quad \frac{n-1}{n+1}\frac{1}{p'} \leq \frac{1}{q} \leq \frac{n+1}{n-1}\frac{1}{p'},$$

with a norm bounded by

$$\|T_{\mathrm{Had}}(z)\|_{\mathcal{L}(L^p,L^q)} \leq C|z|^{\frac{n-1}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}.$$

Hadamard's parametrix: admissible exponents



Decay of Hadamard's parametrix

The combination of the two Theorems gives the following bound on the parametrix

$$\|T_{\mathrm{Had}}(z)u\|_{\mathcal{L}(L^p,L^q)} \leq C|z|^{-\sigma}$$

where the order σ is a piecewise linear function of $\delta = 1/p - 1/q$

$$\sigma = \begin{cases} -\dfrac{n-1}{4}\delta + \dfrac{1}{2} & \text{when } \delta \geq \dfrac{2}{n+1} \\ -\dfrac{n}{2}\delta + 1 & \text{when } \dfrac{2}{n+1} < 2 \leq 1 \end{cases}.$$

Decay of the parametrix: $\|T_{\mathrm{Had}}(z)u\|_{\mathcal{L}(L^p,L^q)} = \mathcal{O}(|z|^{-\sigma})$