Microlocal analysis and inverse problems
Lecture 2 : Limiting Carleman weights

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Thursday May 12 – Instituto de Ciencias Matemáticas, Madrid

Outline

Construction of complex geometrical optics

Recall that the proof of the identifiability of the potential from the DN map was based on the construction of solutions to the Schrödinger equation by complex geometrical optics

$$e^{\frac{1}{h}(\varphi+i\psi)}\big(a(x)+\mathcal{O}(h)\big)$$

Euclidean case:

1. Sylvester-Uhlmann: linear phase: $\varphi + i\psi = -ix\cdot\zeta$
2. Kenig-Sjöstrand-Uhlmann: nonlinear phases:

$$\varphi + i\psi = \log|x| + id_{S^{n-1}}\left(\frac{x}{|x|}, N\right)$$

The question we will address is: when is this construction possible?

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Formal WKB expansion

Conjugated operator : $P_\varphi = \mathrm{e}^{\varphi/h} h^2 \Delta_g \mathrm{e}^{-\varphi/h}$ $\qquad P_\varphi^* = P_{-\varphi}$

Principal symbol : $p_\varphi = |\xi|^2 - |d\varphi|^2 + 2i\langle \xi, d\varphi \rangle$

We have

$$\Delta_g(\mathrm{e}^{\frac{1}{h}(\varphi+i\psi)}a) = \mathrm{e}^{\frac{1}{h}\varphi} P_\varphi(\mathrm{e}^{\frac{i}{h}\psi}a) = \mathrm{e}^{\frac{1}{h}(\varphi+i\psi)}\Big(h^0 p_\varphi(x, \mathrm{d}\psi) + 2h\Big[(\mathrm{grad}_g\varphi + i\mathrm{grad}_g\psi)a + \frac{1}{2}\Delta_g(\varphi+i\psi)a\Big] + h^2\Delta_g a\Big).$$

Eikonal equation: $p_\varphi(x, \mathrm{d}\psi) = 0$

Transport equation: $(\mathrm{grad}_g\varphi + i\mathrm{grad}_g\psi)a + \frac{1}{2}\Delta_g(\varphi+i\psi)a = 0$

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A few important remarks

1. Remember that in the Sylvester-Uhlmann approach, we were using CGOs u_1, u_2 with antagonizing exponential behaviour in order to have a product $u_1 u_2$ without exponential growth. This has to do with the fact that the important object is the product of CGOs.
   This means here that we want to be able ton construct CGOs of the form $e^{\frac{1}{h}(\pm\varphi+i\psi)}(a(x)+\mathcal{O}(h))$!
2. The construction of the construction terms necessitates the derivation of some weighted *a priori* estimates, namely Carleman estimates. So we need to know when these are true.
3. Already the solvability of the eikonal equation implies

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = 0.$$

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Hörmander's criterium of solvability

Our purpose is to determine for which type of weights φ, the Carleman estimate

$$\|u\| \leq Ch^s \|P_\varphi u\|, \quad s \in \mathbf{R}$$

might be true ?

Our claim is

Theorem

If the previous Carleman estimate is true then $\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} \geq 0$ *on* $p_\varphi^{-1}(0)$.

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Quasimodes

The proof presented here follows the original proof of Hörmander but uses the modern concept of positive Lagrangians introduced by Hörmander and extensively studied by Melrose and Sjöstrand.

We proceed by contradiction and suppose that there exists (x_0, ξ_0) such that

$$p_\varphi(x_0, \xi_0) = 0 \text{ and } \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\}(x_0, \xi_0) < 0.$$

To lighten the notation, we set $p = p_\varphi$. Note that $\{\operatorname{Re} p, \operatorname{Im} p\} = \frac{2}{i}\{\bar{p}, p\}$.We will construct a quasimode, i.e. a function $u(h) = e^{\frac{i}{h}w}a$ such that $\|u(h)\| \simeq 1$ and

$$P_\varphi(e^{\frac{i}{h}w}a) = \mathcal{O}(h^m), \quad \forall m$$

This will prevent any Carleman estimate (with polynomial constant in h) to be true.

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Quasimodes

The phase w will be complex-valued and $\mathrm{Im}\, w \simeq |x - x_0|^2$.
Note that this allows to make a local construction since if $\chi \in C_0^\infty$ is 1 near x_0

$$P_\varphi(\chi e^{\frac{i}{h}w}\alpha) = \chi P_\varphi(e^{\frac{i}{h}w}\alpha) + \underbrace{[P_\varphi, \chi]e^{\frac{i}{h}w}\alpha}_{=\mathcal{O}(h^\infty)}.$$

Similarly, in the complex WKB construction, one only needs to solve an approximate eikonal equation

$$p(x, \mathrm{d}w) = \mathcal{O}(|x - x_0|^{2m})$$

then $p(x, \mathrm{d}w)e^{\frac{i}{h}w} = \mathcal{O}(|x - x_0|^{2m})e^{-\frac{c}{h}|x-x_0|^2} = h^m$.
In particular, we can replace p by its Taylor expansion p_{2m-1} of order $2m - 1$ at x_0 and suppose p analytic!

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Construction of the phase

Lemma

There exist a neighbourhood ω of x_0 and an analytic function $w : \omega \to \mathbf{C}$ which satisfies the eikonal equation

$$p(x, \mathrm{d}w) = 0$$

with $\mathrm{d}w(x_0) = \xi$ and whose Hessian[a] at x_0 has imaginary definite positive part

$$\operatorname{Im} w''(x_0) > 0.$$

[a]Note that the Hessian of the imaginary part of w is invariantly defined since $\mathrm{d}w(x_0) = \xi \in \mathbf{R}^n$, i.e. x_0 is a critical point of $\operatorname{Im} w$.

The proof is similar to the construction of solutions to Hamilton-Jacobi equations but one has to keep track of the positivity of the imaginary part.

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Positive Lagrangian submanifolds

Definition

A complex Lagrangian linear subspace H in $\mathbf{C}^{2n}$ is positive if

$$\frac{1}{i}\sigma(Z,\overline{Z}) > 0, \quad \forall Z \in H \setminus 0.$$

Similarly a Lagrangian manifold Λ in $\mathbf{C}^{2n}$ is said to be positive at $\varrho_0 = (z_0, \zeta_0) \in \mathbf{C}^{2n}$ if its tangent space $T_{\varrho_0}\Lambda$ at ϱ_0 is positive.

If the Lagrangian manifold Λ is parametrized by a holomorphic function w

$$\Lambda = \Lambda_w = \big\{(z, \partial_z w(z)) : z \in U\big\}$$

then the positivity of the tangent space $\big\{(t, \partial_z^2 w(z_0)t) : t \in \mathbf{C}^n\big\}$ reads

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Proof of the Lemma

Possibly after a rotation

$$\operatorname{Re}\frac{\partial p}{\partial \eta}(x_0,\xi) = \big(0,\dots,0,\operatorname{Re}\partial p/\partial\eta_n(x_0,\xi)\big) \neq 0$$

thus by the Weierstrass preparation theorem,

$$p(x,\eta) = e(x,\eta)\big(\eta_n - a(x,\eta')\big), \quad \eta = (\eta',\eta_n)$$

Let $w_0 : U' \to \mathbf{C}$ be an analytic function such that

$$\frac{\partial w_0}{\partial z'}(x_0') = \xi', \quad \operatorname{Im}\frac{\partial^2 w_0}{\partial z'^2}(x_0') > 0.$$

We then consider the submanifold of $\mathbf{C}^{2n}$

$$\Lambda_0 = \Big\{\big(z', x_0^n, \partial w_0/\partial z', a(z', x_0^n, \partial w_0/\partial z')\big) : z' \in U'\Big\}$$

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This is an isotropic complex manifold because if we take the exterior derivative of the canonical one form restricted to Λ_0

$$(\zeta_j \mathrm{d}z^j)|_{\Lambda_0} = \partial_{z_\alpha} w_0 \, \mathrm{d}z^\alpha$$

we get

$$\sigma|_{\Lambda_0} = \partial^2_{z_\alpha z_\beta} w_0 \, \mathrm{d}z^\alpha \wedge dz^\beta = 0.$$

Besides it is also positive since if

$$W = \big(t', 0, \partial^2_{z'} w_0(x_0)t', \langle \partial_{z'} a(x_0, \xi'), t' \rangle\big) \in {}^{\mathbf{C}}T_{(x_0,\xi)}\Lambda_0$$

then one has

$$\frac{1}{i}\sigma(W, \overline{W}) = \operatorname{Im} \langle \partial^2_{z'} w_0(x_0)t', \overline{t'} \rangle > 0, \quad t' \neq 0.$$

Then consider the union of complex bicharacteristic curves starting at Λ_0

$$\Lambda = \big\{ \exp(sH_p)(z_0, \zeta_0) : s \in \mathbf{C},\ |s| < \varepsilon,\ (z_0, \zeta_0) \in \Lambda_0 \big\}.$$

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Then consider the union of complex bicharacteristic curves starting at Λ_0

$$\Lambda = \big\{ \exp(sH_p)(z_0, \zeta_0) : s \in \mathbf{C}, \, |s| < \varepsilon, \, (z_0, \zeta_0) \in \Lambda_0 \big\}.$$

Proof of the Lemma

This is a complex Lagrangian manifold since the flow $\exp(sH_p)$ is simplectly invariant, Λ_0 is isotropic and

$$^{\mathbf{C}}T_{(x_0,\xi)}\Lambda = {}^{\mathbf{C}}T_{(x_0,\xi)}\Lambda_0 \oplus \mathbf{C}H_{p_{2m-1}}(x_0,\xi)$$

In particular, Λ is also positive since if one considers the tangent vector $Z = W + sH_p(x_0,\xi)$ then

$$\frac{1}{i}\sigma(Z,\overline{Z}) = \frac{1}{i}\sigma(W,\overline{W}) + \frac{|s|^2}{i}\sigma\big(H_p(x_0,\xi),\overline{H_{p_{2m-1}}(x_0,\xi)}\big) - 2\,\mathrm{Im}\,\big(s\mathrm{d}p(x_0,\xi)(\overline{W})\big).$$

Because of the positivity of Λ_0 and the assumption, this quantity is positive when $Z \neq 0$ provided

$$\frac{1}{i}\sigma\big(H_p(x_0,\xi),\overline{H_p(x_0,\xi)}\big) \times |\mathrm{d}p(\overline{W})|^2 \leq \frac{1}{i}\sigma(W,\overline{W})$$

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Since $\Lambda_0 \subset p^{-1}(0)$ we have $\mathrm{d}p(x_0,\xi)(W) = 0$, and the second factor in the left-hand side term can be replaced by

$$|\operatorname{Re}\mathrm{d}p(x_0,\xi)(W)|^2$$

which can be explicitly computed

$$\left|\langle \operatorname{Re}\partial_{x'}p(x_0,\xi),t'\rangle + (\operatorname{Re}\partial_{\xi_n}p(x_0,\xi))\langle \partial_{z'}a(x_0,\xi),t'\rangle\right|^2$$

and bounded by $C|t'|^2$ with a constant uniform with respect to w_0. The positivity condition is satisfied if

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Proof of the Lemma (end)

From the positivity of Λ, one deduces that $\Lambda \ni (z, \zeta) \to z$ is a local diffeomorphism, and since Λ is a complex Lagrangian submanifold of $\mathbf{C}^{2n}$, it is locally a graph, i.e. there exists a holomorphic function w which parametrizes Λ

$$\Lambda \cap \Gamma_0 = \big\{(z, \partial_z w(z)) : z \in U\big\}$$

From $\Lambda \subset p_{2m-1}^{-1}(0)$, we get that the restriction of w to a real neighbourhood of x_0 is a solution to the eikonal equation and $(x_0, \mathrm{d}w(x_0)) = (x_0, \xi)$. The positivity of Λ implies

$$\frac{1}{i}\sigma(Z, \overline{Z}) = \mathrm{Im}\langle \partial_z^2 w(x_0)t, t\rangle = \langle \mathrm{Im}\, w''(x_0)t, t\rangle > 0$$

hence $\mathrm{Im}\, w''(x_0) > 0$ as announced. □

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End of the construction

The end of the construction is relatively straightforward. In the WKB expansion, one has to solve a series of transport equations. Once again, it suffices to solve them approximatelvely therefore we can suppose that the coefficients of the transport equations are analytic and use the Cauchy-Kowalevskaia theorem to solve them locally. Besides one can suppose that $a \simeq h^{-\frac{n}{2}}$.

We have

$$\|e^{\frac{i}{h}w}a\|^2 \simeq h^{-n} \int e^{-2c|x-x_0|^2}\, dx \simeq 1$$

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$$\|P_\varphi(e^{\frac{i}{h}w}a)\| = \mathcal{O}(h^m)$$

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Outline

Limiting Carleman weights

We want to be able to have Carleman estimates for $P_{\pm\varphi}$ by Hörmander's criterium this means

$$\frac{1}{i}\{\overline{p_{\pm\varphi}}, p_{\pm\varphi}\} = \pm\frac{1}{i}\{\overline{p_{\varphi}}, p_{\varphi}\} \geq 0$$

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A limiting Carleman weight on an open Riemannian manifold is smooth real-valued function without critical points such that

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Geometrical characterization

Question: Do all manifolds admitt LCW? Are there many LCW?

Lemma

Being a LCW is a conformally invariant property.

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$$\frac{1}{2i}\{\overline{p_\varphi}, p_\varphi\} = \nabla^2\varphi(\xi^\sharp, \xi^\sharp) + \nabla^2\varphi(\mathrm{grad}\varphi, \mathrm{grad}\varphi)$$

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If (M, g) is an open manifold having a limiting Carleman weight, then some conformal multiple of the metric g admits a parallel unit vector field. For simply connected manifolds, the converse is also true.

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Warped products

An open manifold which is conformally imbedded in the warped product of the Euclidean line $\mathbf{R}$ and a Riemannian manifold (M_0, g_0) of dimension $n-1$ admits limiting Carleman weights. Indeed the metric on $\mathbf{R} \times_{\mathrm{e}^{2\psi}} M_0$ is conformal to

$$\mathrm{e}^{-2\psi(t)}\mathrm{d}t^2 + g_0$$

and one can make the change of variable

$$s = \int_0^t \mathrm{e}^{-\psi(t')}\,\mathrm{d}t'$$

to reduce the metric to

$$\mathrm{d}s^2 + g_0$$

where $\varphi(s) = s$ is a natural limiting Carleman weight.

Outline

Level sets of LCW

The following result is well known.

Lemma

The only connected totally umbilical hypersurfaces in the Euclidean space of dimension $n \geq 3$ are parts of either hyperplanes or hyperspheres.

Let Σ be a connected totally umbilical hypersurface in an open subset $\Omega \subset \mathbf{R}^n$, let ν denote its unit exterior normal, and let λ be the common value of the principal curvatures. First, let us prove that λ is constant along Σ.

The second fundamental form reads $\ell(X,Y) = \langle \nabla_X \nu, Y \rangle = \lambda \langle X, Y \rangle$, therefore we have

$$\nabla_X \nu = \lambda X$$

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Therefore we deduce

$$\nabla_X \nabla_Y \nu - \nabla_Y \nabla_X \nu = (L_X \lambda) Y - (L_Y \lambda) X + \underbrace{\lambda [X, Y]}_{= \nabla_{[X,Y]} \nu}.$$

Since the Euclidean space is flat,

$$\nabla_X \nabla_Y \nu - \nabla_Y \nabla_X \nu - \nabla_{[X,Y]} \nu = R(X, Y) \nu = 0$$

therefore $(L_X \lambda) Y - (L_Y \lambda) X = 0$ and λ is constant along Σ. Consider $V = \sum_{j=1}^{n} x_j \partial_{x_j}$ we have

$$\nabla_X (\nu - \lambda V) = \lambda X - \lambda \nabla_X V = 0.$$

This means that $\nu - \lambda x$ is constant along the hypersurface. If $\lambda = 0$, the normal is constant and Σ is part of a hyperplane, and if $\lambda \neq 0$ then $\alpha = \lambda^{-1}(\lambda x - \nu)$ is constant, and Σ is a part of the hypersphere $|x - \alpha| = 1/|\lambda|$.

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(Local) determination of LCW

Theorem

Let Ω be an open subset of $\mathbf{R}^n$, $n \geq 3$, and let e be the Euclidean metric. The limiting Carleman weights in (Ω, e) are locally of the form

$$\varphi(x) = a\varphi_0(x - x_0) + b$$

where $a \in \mathbf{R} \setminus \{0\}$ and φ_0 is one of the following functions:

$$\langle x, \xi \rangle, \quad \arg\langle x, \omega_1 + i\omega_2 \rangle,$$
$$\log|x|, \quad \frac{\langle x, \xi \rangle}{|x|^2}, \quad \arg\left(e^{i\theta}(x + i\xi)^2\right), \quad \log\frac{|x+\xi|^2}{|x-\xi|^2}$$

with ω_1, ω_2 orthogonal unit vectors, $\theta \in [0, 2\pi)$ and $\xi \in \mathbf{R}^n \setminus \{0\}$.