# Microlocal analysis and inverse problems
# Lecture 1 : Introduction

David Dos Santos Ferreira

LAGA – Université de Paris 13

Tuesday May 10 – Instituto de Ciencias Matemáticas, Madrid

# Outline

# The Calderón problem

In a foundational paper of 1980, A. Calderón asked the following question: Is it possible to determine the electrical conductivity of a body by making current and voltage measurements at the boundary?

The mathematical formulation is as follows: let $\Omega \subset \mathbf{R}^n$ be a smooth bounded open set, the conductivity is modelled by a bounded measurable function $\gamma$ bounded from below by a positive constant $c$, If we consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma \operatorname{grad} u) = 0 \\ u|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega) \end{cases}$$

and define the associated Dirichlet-to-Neumann map

$$\Lambda_\gamma f = \gamma \partial_\nu u|_{\partial\Omega}$$

the question is whether $\Lambda_\gamma$ determine $\gamma$?

## Remarks

1. There are a few problems related to this question (identifiability, stability, reconstruction methods,...) but we will be concerned with identifiability, i.e. injectivity of the map $\gamma \to \Lambda_\gamma$.
2. The map $\gamma \to \Lambda_\gamma$ is nonlinear, which explains part of the difficulty of the problem (the other is that the problem is ill-posed).
3. Calderón dealt with the linearized problem near constant conductivities.
4. There are substancial differences between dimension $n = 2$ and higher dimensions.
5. The problem is solved in dimension $n = 2$ (Astala-Päivärinta), open in higher dimensions.
6. Partial data problems are of interests and tend to be more difficult (e.g. the linearized problem is already much more difficult).

# The Schrödinger equation

There is a classical argument to remove first order terms in elliptic equations. Use conjugation:

$$\begin{aligned}\operatorname{div}(\gamma \operatorname{grad} u) &= \gamma\Delta u + \operatorname{grad}\gamma \cdot \operatorname{grad} u \\ &= \sqrt{\gamma}(\Delta + q)v\end{aligned}$$

where $v = \sqrt{\gamma}u$ and

$$q = -\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}.$$

This requires the conductivity $\gamma$ to be smooth enough ($C^2, W^{2,\infty}$, etc.).

# Inverse problem on the Schrödinger equation

Just as for the conductivity equation, one can define a Dirichlet-to-Neumann map. Consider the Dirichlet problem

$$\begin{cases} (\Delta + q)u = 0 \\ u|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega) \end{cases}$$

and define the associated Dirichlet-to-Neumann map

$$\Lambda_q f = \partial_\nu u|_{\partial\Omega}.$$

This is well defined provided $0$ is not a Dirichlet eigenvalue of $\Delta + q$.
Inverse problem: Does $\Lambda_q$ determine $q$?

If one knows the conductivity at the boundary (boundary determination), then $\Lambda_q$ is known, so inverse problem on Schrödinger $\Rightarrow$ Calderón problem

## Some references (full data case)

1980 Calderón: linearized problem, introduction of harmonic exponentials

1984 Kohn-Vogelius: boundary determination, (piecewise) analytic case

1987 Sylvester-Uhlmann: Case $n \geq 3$, $C^2$ conductivities, use of complex geometrical optics with linear weights

1996 Nachman: Case $n = 2$, $W^{2,p}$ conductivities, $\bar{\partial}$-method

1997 Brown-Uhlmann: Case $n = 2$, $W^{1,p}$ conductivities, conductivity equation seen as a system

2003 Päivärinta-Panchenko-Uhlmann: Case $n \geq 3$, $W^{\frac{3}{2},\infty}$ conductivities,

2006 Astala-Päivärinta: Case $n = 2$, $L^\infty$ conductivities, use of quasiconformal geometry

# Some references (partial data case)

2002 Bukhgeim-Uhlmann: big subsets of the boundary, $n \geq 3$, global Carleman estimates with linear weights

2007 Kenig-Sjöstrand-Uhlmann: small subsets of the boundary, $n \geq 3$,global Carleman estimates with logarithmic weights, introduction of limiting Carleman weights.

2007 DSF-Kenig-Sjöstrand-Uhlmann: magnetic Schrödinger equation, $n \geq 3$, $L^\infty$ potential and $C^2$ magnetic potential, Radon transform and microlocal Holmgren approach

2008 Bukhgeim: full data case but Schrödinger equation with $L^\infty$ potentials, harmonic weights, stationary phase

2009 DSF-Kenig-Sjöstrand-Uhlmann: linearized problem, $n \geq 2$, Watermelon principle

2010 Imanuvilov-Uhlmann-Yamamoto: $n = 2$, local problem

2011 Guillarmou-Tzou: $n = 2$, Schrödinger equation on Riemann surfaces.

# The anisotropic Calderón problem

In some applications to medical imaging, it might be interesting to consider the case where the conductivity depends on the direction. This amounts to taking $\gamma$ to be a matrix. If we consider the Dirichlet problem

$$\begin{cases} \dfrac{\partial}{\partial x_j}\left(\gamma^{jk}\dfrac{\partial}{\partial x_k}\right)u = 0 \\ u|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega) \end{cases}$$

and define the associated Dirichlet-to-Neumann map

$$\Lambda_\gamma f = \gamma^{jk}\nu_j \frac{\partial u}{\partial x_k}\bigg|_{\partial\Omega}$$

the question is whether $\Lambda_\gamma$ determine $\gamma = (\gamma^{jk})$?

# Remarks

1. The answer is no, because as was observed by Tartar there is a gauge invariance
$$\Lambda_{\varphi_*\gamma} = \Lambda_\gamma$$
where $\varphi$ is a diffeomorphism which is the identity on the boundary and the pushforward is defined as
$$(\varphi_*\gamma)^{jk} = \frac{1}{\det \varphi'} \varphi'_{lj} \gamma^{lm} \varphi'_{mj}.$$

2. The inverse problem has to be reformulated modulo this gauge invariance.

3. In dimension $n = 2$, there are isothermal coordinates (as observed by Sylvester), which makes the problem not so different from the isotropic one.

# Main focus

Indeed the analogue of the Astala-Päivärinta was proved by Astala-Lassas-Päivärinta (2005).

Therefore, we are now concerned with the case $n \geq 3$, and mainly with smooth conductivities.

# Riemannian rigidity

In fact, one can give a more geometric flavour to the problem.

Let $(M, g)$ be a compact Riemannian manifold with boundary $\partial M$ of dimension $n \geq 3$ and $q$ a bounded measurable function. Consider the Dirichlet problem

$$\begin{cases} (\Delta_g + q)u = 0 \\ u|_{\partial M} = f \in H^{\frac{1}{2}}(\partial M) \end{cases}$$

and define the associated Dirichlet-to-Neumann map (under a natural spectral assumption)

$$\Lambda_{g,q} u = \partial_\nu u|_{\partial M}$$

where $\nu$ is a unit normal to the boundary.
If $q = 0$, we use $\Lambda_g = \Lambda_{g,0}$ as a short notation.

# Riemannian rigidity

As for the conductivity equation, there is a gauge invariance, that is by isometries which leave the boundary points unchanged:

$$\Lambda_{\varphi^* g} = \Lambda_g, \quad \varphi|_{\partial M} = \mathrm{Id}_{\partial M}$$

Inverse problem: Does the Dirichlet-to-Neumann map $\Lambda_{g,q}$ determine the potential $q$ and the metric $g$ modulo such isometries?

If $n \geq 3$ and $q = 0$ this is a generalization of the anisotropic conductivity problem and one passes from one to the other by

$$\gamma^{jk} = \sqrt{\det g}\, g^{jk}, \quad g^{jk} = (\det \gamma)^{-\frac{2}{n-2}} g^{jk}.$$

# Conformal metrics

As for the conductivity equation, there is a conformal gauge transformation

$$\Delta_{cg} u = c^{-1}(\Delta_g + q_c)(c^{\frac{n-2}{4}} u), \quad q_c = c^{\frac{n+2}{4}} \Delta_g (c^{\frac{n-2}{4}})$$

which translates at the boundary into

$$\Lambda_{cg,q} f = c^{-\frac{n+2}{4}} \Lambda_{g,q+q_c}(c^{\frac{n-2}{4}} u) + \frac{n-2}{4} c^{-\frac{1}{2}} \partial_\nu c f.$$

So if one knows $c$ at the boundary (boundary determination) then one can deduce one DN map from the other.

A more reasonable inverse problem: $\Lambda_{cg} = \Lambda_g \Rightarrow$ c=1.
Note that there is no isometry gauge invariance in this case.

# Some references $n \geq 3$

1989 Lee-Uhlmann: boundary determination, analytic metrics, no potential, determination of the metric

2001 Lassas-Uhlmann: improvement on topological assumptions

2009 Guillarmou-Sa Baretto: Einstein manifolds, no potential, determination of the metric, unique continuation argument

2009 DSF-Kenig-Salo-Uhlmann: fixed admissible geometries, determination of a smooth potential, CGOs

2011 DSF-Kenig-Salo: fixed admissible geometries, determination of an unbounded potential, CGOs

# Remarks

1. Analytic metrics case fairly well understood. The smooth case remains a challenging problem.
2. There are limitations in the method using CGO construction as the following lectures will show.
3. We will concentrate on the case of identifiability of the metric within a conformal class
$$\Lambda_{cg} = \Lambda_g \Rightarrow c = 1.$$
4. With boundary determination
$$\Lambda_{cg} = \Lambda_g \Rightarrow c|_{\partial M} = 1.$$
it is enough to solve the inverse problem on the Schrödinger equation with a fixed metric
$$\Lambda_{g,q_1} = \Lambda_{g,q_2} \Rightarrow q_1 = q_2.$$

# Outline

# An integration by parts

Let $u_1, u_2$ be solutions to the Schrödinger equations

$$\Delta u_1 + q_1 u_1 = 0 \tag{1}$$
$$\Delta u_2 + q_2 u_2 = 0 \tag{2}$$

then

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, \mathrm{d}x = \int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) u_1 u_2 \, \mathrm{d}\sigma.$$

Hence the inverse problem is implied by the following density property:

the set of products $u_1 u_2$ of solutions to the Schrödinger equations with respective potentials $q_1, q_2$ is dense in $L^1$.

# Complex geometrical optics with linear weights

Goal: Construct solutions to the Schrödinger equation

We start with harmonic exponentials (used by Calderón to deal with the linearized problem)

$$\mathrm{e}^{-ix\cdot\zeta}, \quad \zeta^2 = \zeta_1^2 + \cdots + \zeta_n^2 = 0.$$

Sylvester and Uhlmann constructed complex geometrical optics solutions by perturbation of the form

$$\mathrm{e}^{-ix\cdot\zeta}\big(1 + \mathcal{O}(|\operatorname{Im}\zeta|^{-1})\big).$$

It is convenient to introduce a small parameter $h$

$$\mathrm{e}^{-\frac{i}{h}x\cdot\zeta}\big(1 + \mathcal{O}(h)\big).$$

# Carleman estimates with linear weights

The correction term $\mathcal{O}(h)$ is constructed using solvability properties of $\Delta + q$ in $L^2$ weighted spaces (with exponential weights $\mathrm{e}^{-\frac{1}{h}\operatorname{Im}\zeta\cdot x}$).

The *a priori* estimates in those $L^2$ weighted spaces are provided by Carleman estimates with exponential with linear weights.

**Theorem**

*There exists a constant $C > 0$ such that for all $h \in (0,1]$, all $\omega \in S^n$ and all $u \in C_0^\infty(\mathbf{R}^n)$ the following estimate holds*

$$\|\mathrm{e}^{\frac{1}{h}\omega\cdot x}u\| + \|\mathrm{e}^{\frac{1}{h}\omega\cdot x}hDu\| \leq Ch\|\mathrm{e}^{\frac{1}{h}\omega\cdot x}(\Delta+q)u\|$$

# An important remark on the construction

In fact, there is some freedom in the CGO construction. Indeed

$$\mathrm{e}^{\frac{i}{h}x\cdot\zeta}h^2\Delta\mathrm{e}^{-\frac{i}{h}x\cdot\zeta} = -(hD+i\zeta)^2 = -h^2D^2 - 2i\zeta\cdot hD$$

hence if $a$ satisfies the Cauchy-Riemann equation

$$\zeta\cdot Da = 0$$

then $\mathrm{e}^{\frac{i}{h}x\cdot\zeta}a$ is still an approximate solution and the CGOs construction work as before.
For instance, we have solutions of the form

$$\mathrm{e}^{-\frac{i}{h}x\cdot\zeta}\big(\mathrm{e}^{-ix\cdot\xi}+\mathcal{O}(h)\big)$$

with $\xi\perp\zeta$.

# Identifiability of the potential

Plugging our CGOs solutions in the integration by parts formula, we get

$$\int_\Omega \mathrm{e}^{-\frac{i}{h}(\zeta_1+\zeta_2)} \mathrm{e}^{-ix\cdot\xi}(q_1 - q_2)\, \mathrm{d}x = \mathcal{O}(h)$$

provided $\mathrm{Im}(\zeta_1 + \zeta_2) = 0$.
If we can choose $\zeta_1 = \zeta, \zeta_2 = -\zeta$ with

$$\zeta^2 = 0, \quad \zeta \perp \xi \quad |\mathrm{Im}\, \zeta| \geq 1,$$

then $1_\Omega \widehat{(q_1 - q_2)} = 0$ leading to $q_1 = q_2$.

This is only possible in dimension $n \geq 3$ !!

Note that there is some flexibility since by analiticity of the Fourier transform we don't need all frequencies $\xi \in \mathbf{R}^n$.

# Alternative endings

1. An amplitude of the form $a(x \cdot \xi), \xi \perp \zeta$ also satisfies the Cauchy-Riemann equation, and varying $a$ and $\xi$, and translating the phases, we get that the Radon transform of $q_1 - q_2$ vanishes $\mathcal{R}(1_\Omega(q_1 - q_2))(H) = 0$.
   Again, there is some flexibility, because one can use microlocal analytic theory to deal with the case where there is partial information on the hyperplanes $H$.
2. If $\zeta = e_1 + i\eta$, another possible amplitude is $\mathrm{e}^{i\lambda\zeta \cdot x} b(x \cdot \xi)$ and translating in $x$ and varying $\eta$ and $\xi$ one obtains information on the weighted X-ray transform

$$\int 1_\Omega \widehat{(q_1 - q_2)}(\lambda, x_0' + t\xi')\mathrm{e}^{-\lambda t}\,\mathrm{d}t = 0.$$

In all cases, one is in fact using the injectivity of some functional transform.

# Complex geometrical optics with logarithmic weights

Here we describe another construction by complex geometrical optics due to Kenig, Sjöstrand and Uhlmann.

Suppose $0 \notin \Omega$ and write the Laplace operator on $\mathbf{R}^n$ in polar coordinates

$$\Delta = \partial_r^2 + (n-1)r^{-1}\partial_r + r^{-2}\Delta_{S^{n-1}}$$

and make the change of variable $s = \log r$

$$\begin{aligned}\Delta &= \mathrm{e}^{-2s}\big(\partial_s^2 + (n-2)\partial_s + \Delta_{S^{n-1}}\big) \\ &= \mathrm{e}^{-\frac{n+2}{2}s}\left(\partial_s^2 + \Delta_{S^{n-1}} - \frac{(n-2)^2}{4}\right)\mathrm{e}^{\frac{n-2}{2}s}.\end{aligned}$$

Remark: This corresponds to seeing $\mathbf{R}^n$ as a warped product. By change of variables, it is conformal to a product.

# Complex geometrical optics with logarithmic weights

The Laplace-Beltrami operator on the sphere reads

$$\begin{aligned}\Delta_{S^{n-1}} &= \frac{1}{(\sin\theta)^{n-2}}\partial_\theta\big((\sin\theta)^{n-2}\partial_\theta\big) + \frac{1}{\sin^2\theta}\,\Delta_{S^{n-2}}\\ &= \partial_\theta^2 + (n-2)\cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\,\Delta_{S^{n-2}}.\end{aligned}$$

We may rewrite the Laplace-Beltrami operator on the sphere as the conjugated operator

$$\Delta_{S^{n-1}} = (\sin\theta)^{-\frac{n-2}{2}}\left(\partial_\theta^2 + \frac{1}{\sin^2\theta}\,\widehat{\Delta}_{S^{n-2}} + \frac{(n-2)^2}{4}\right)(\sin\theta)^{\frac{n-2}{2}}$$

with

$$\widehat{\Delta}_{S^{n-2}} = \Delta_{S^{n-2}} - \frac{(n-2)(n-4)}{4}.$$

# Complex geometrical optics with logarithmic weights

We get

$$\Delta = \mathrm{e}^{-\frac{n+2}{2}s}(\sin\theta)^{-\frac{n-2}{2}}\left(\partial_s^2 + \partial_\theta^2 + \frac{1}{\sin^2\theta}\,\widehat{\Delta}_{S^{n-2}}\right)\mathrm{e}^{\frac{n-2}{2}s}(\sin\theta)^{\frac{n-2}{2}}.$$

Note that the Riemannian distance to the north pole $N = (0, \dots, 0, 1)$ is given by

$$d_{S^{n-1}}(y, N) = \theta.$$

An approximate solution to the Schrödinger equation looks like

$$u_{\pm}^{\mathrm{app}} = \mathrm{e}^{\pm\frac{1}{h}(s+i\theta)}\mathrm{e}^{-\frac{n-2}{2}s}(\sin\theta)^{-\frac{n-2}{2}} = \mathrm{e}^{\pm\frac{1}{h}\left(\log|x| + i d_{S^{n-1}}\left(\frac{x}{|x|}, N\right)\right)}a(x).$$

since $h^2(\Delta + q)u^{\mathrm{app}} = \mathrm{e}^{\pm\frac{1}{h}\log|x|}\mathcal{O}(h^2)$.

# Remarks on the construction

1. Note that $a = \mathrm{e}^{-\frac{n-2}{2}s}(\sin\theta)^{-\frac{n-2}{2}}$ and that

$$\mathrm{d}x = \mathrm{e}^{ns}(\sin\theta)^{n-2}\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\sigma_{S^{n-2}}.$$

Thus if one uses two approximate solutions $u_{+}^{\mathrm{app}}$, $u_{-}^{\mathrm{app}}$ then

$$\int_{\Omega}(q_1 - q_2)u_{+}^{\mathrm{app}}u_{-}^{\mathrm{app}}\mathrm{d}x = \int_{S^{n-2}}\iint \mathrm{e}^{2s}(q_1 - q_2)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\sigma_{S^{n-2}}.$$

2. This construction can be modified in the following way: any multiplication by a holomorphic function in $z = \varphi + i\psi$ and a smooth function in $\omega \in S^{n-2}$ yields a similar approximate solution

$$u_{\pm}^{\mathrm{app}} = \mathrm{e}^{\pm\frac{1}{h}(s+i\theta)}\mathrm{e}^{-\frac{n-2}{2}s}(\sin\theta)^{-\frac{n-2}{2}}f(s+i\theta)b(\omega).$$

with $f \in \mathrm{Hol}(\mathbf{C})$ and $b \in C^{\infty}(S^{n-2})$.

# Carleman estimates with logarithmic weights

Once again the correction term $\mathcal{O}(h)$ is constructed using solvability properties of $\Delta + q$ in $L^2$ weighted spaces (with weights $|x|^{\pm\frac{1}{h}}$).

The *a priori* estimates in those $L^2$ weighted spaces are provided by Carleman estimates with exponential with logarithmic phase.

**Theorem**

*There exists a constant $C > 0$ such that for all $h \in (0, 1]$, and all $u \in C_0^\infty(\Omega)$ the following estimate holds*

$$\||x|^{\pm\frac{1}{h}} u\| + \||x|^{\pm\frac{1}{h}} hDu\| \leq Ch\||x|^{\pm\frac{1}{h}}(\Delta + q)u\|$$

# Concluding remarks

The method to obtain the identifiability of the potential following Sylvester and Uhlmann is the following

1. Use integration by parts to relate the information on the boundary to the inside.
2. Construct an approximate solution to the Schrödinger equation using complex geometrical optics.
3. Construct a correction term (with corresponding estimates) using Carleman weights.
4. Use the injectivity of a certain functional transform (Fourier, Radon or X-ray transforms in the Euclidean case).

This is our roadmap.