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The Factorization Method for Inverse Scattering Problems Part III

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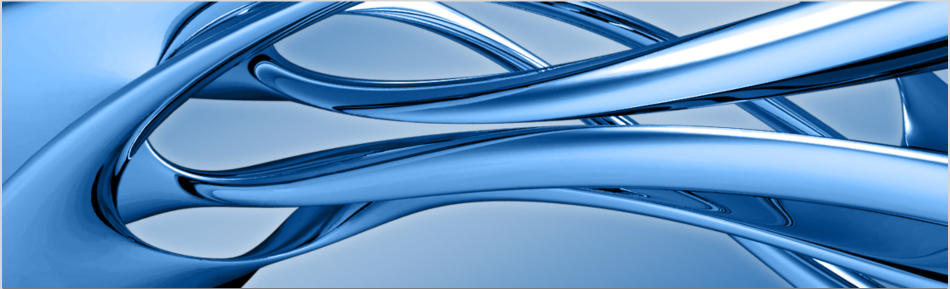

Recall: Outline of the Course

- Part I:
  - Introduction
  - The Direct Scattering Problem, Dirichlet Boundary Conditions
  - The Direct Scattering Problem, Inhomogeneous Medium
- Part II:
  - A Factorization of the Far Field Operator
  - Range Identities
  - The Factorization Method
  - Some Numerical Simulations
- Part III:
  - The Inverse Scattering Problem for Inhomogeneous Media
  - An Interior Transmission Eigenvalue Problem
  - An Electromagnetic Inverse Scattering Problem

Inhomogeneous Medium

Direct scattering problem: Given bounded domain D and $k > 0$ and $q \in L^\infty(D)$ with $q \geq \hat{q} > 0$ on D and $u^{inc}(x) = \exp(ik\hat{\theta} \cdot x)$, determine total field u and scattered field $u^s = u - u^{inc}$ such that

$$\operatorname{div}\big[(1+q)\nabla u\big] + k^2 u = 0 \text{ in } \mathbb{R}^d\,, \quad u^s \text{ satisfies SRC.}$$

Inverse scattering problem: Given $u^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^{d-1}$, determine q – or at least D!

Far field operator $F : L^2(S^{d-1}) \to L^2(S^{d-1})$ defined by

$$(Fg)(\hat{x}) \ := \ \int_{S^{d-1}} u^\infty \hat{x}, \hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta})\,, \quad \hat{x} \in S^{d-1}\,,$$

is again compact and normal and $I + \frac{ik}{2\pi} F$ is unitary.

Inhomogeneous Medium

Theorem: F has factorization $\boxed{F = -H^* T H}$ where $H : L^2(S^{d-1}) \to L^2(D)^d$ and $T : L^2(D)^d \to L^2(D)^d$ are defined as

$$(Hg)(x) \;=\; \sqrt{q(x)}\,\nabla \int_{S^{d-1}} g(\hat{\theta})\, e^{ikx\cdot\hat{\theta}} ds(\hat{\theta})\,, \quad x \in D\,,$$

$Tf = f - \sqrt{q}\,\nabla v|_D$ where v is the radiating solution of

$$\operatorname{div}\big[(1+q)\nabla v\big] + k^2 v = \operatorname{div}(\sqrt{q}\,f) \quad \text{in } \mathbb{R}^d\,.$$

Check assumption for range identity. Crucial: T coercive in general sense on $\overline{\mathcal{R}(H)}$ and F one-to-one.

Theorem: F is one-to-one if k^2 is not an interior transmission eigenvalue of

$$\operatorname{div}\big[(1+q)\nabla u\big] + k^2 u = 0 \text{ in } D\,, \quad \Delta w + k^2 w = 0 \text{ in } D\,,$$

$$u = w \text{ on } \partial D\,, \quad (1+q)\,\partial_\nu u = \partial_\nu w \text{ on } \partial D\,.$$

Inhomogeneous Medium

Theorem: T coercive in general sense on $\overline{\mathcal{R}(H)}$ provided k^2 is not an interior transmission eigenvalue.

Theorem: $z \in D$ if, and only if, $\phi_z \in \mathcal{R}(H^*)$.

Application of range identity:

Theorem: Let k^2 be no interior transmission eigenvalue. Then:

$$\begin{aligned} z \in D \quad &\Longleftrightarrow \quad \phi_z \in \mathcal{R}\big((F^*F)^{1/4}\big) \\ &\Longleftrightarrow \quad \sum_{j\in\mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} < \infty \\ &\Longleftrightarrow \quad w(z) = \left[\sum_{j\in\mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} \right]^{-1} > 0\,. \end{aligned}$$

The Interior Transmission EVP

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$$\operatorname{div}\big[(1+q)\nabla u\big] + \lambda u = 0 \text{ in } D\,, \quad \Delta w + \lambda w = 0 \text{ in } D\,,$$

$$u = w \text{ on } \partial D\,, \quad (1+q)\partial_\nu u = \partial_\nu w \text{ on } \partial D\,.$$

Example: B ball and q constant:

$$\Delta u + \frac{\lambda}{1+q}\, u = 0 \text{ in } D\,, \quad \Delta w + \lambda w = 0 \text{ in } D\,,$$

$$u = w \text{ on } \partial D\,, \quad (1+q)\partial_\nu u = \partial_\nu w \text{ on } \partial D\,.$$

Ansatz ($d = 2$): $\quad u(r) = \alpha J_0\big(\sqrt{\lambda/(1+q)}\, r\big)\,, \quad w(r) = \beta J_0\big(\sqrt{\lambda}\, r\big)$

Determine α and β from transmission conditions! Asymptotics of Besselfunctions yields infinite sequence $\lambda_j \to \infty$ with vanishing determinant.

The Interior Transmission EVP

Now general case: $q \in L^\infty(D)$ real-valued with $q(x) \geq \hat{q} > 0$ on D.

$$\begin{aligned} \operatorname{div}\big[(1+q)\nabla u\big] \;+\; \lambda u &= 0 \quad \text{in } D\,, \\ \Delta w \;+\; \lambda w &= 0 \quad \text{in } D\,. \end{aligned}$$

Replace u by $v = w - u$. Then $v \in H_0^1(D)$ and

$$\operatorname{div}\big[(1+q)\nabla v\big] \;+ \lambda v = \operatorname{div}(q \nabla w) \quad \text{in } D\,.$$

Note that $\iint_D v dx = 0$. Define $T_\lambda : H^1_{0,\diamond}(D) \to H^1_\diamond(D)$ by $v \mapsto w$.
Then $T_\lambda = T_0 + \lambda T_1$.Find $v \in H^1_{0,\diamond}(D)$ such that $\Delta w + \lambda w = 0$ in D; that is

$$a_\lambda(v,\psi) \;=\; \iint_D \big[\nabla w \cdot \nabla \overline{\psi} - \lambda w \overline{\psi}\big]\, dx \;=\; 0\,, \quad \psi \in H^1_{0,\diamond}(D)\,,$$

where $w = T_\lambda v$.

The Interior Transmission EVP

λ transmission eigenvalue if there exists nontrivial $v \in H^1_{0,\diamond}(D)$ such that $a_\lambda(v,\psi) = 0$ for all $\psi \in H^1_{0,\diamond}(D)$.

$a_\lambda = a_0 + \lambda a_1 + \lambda^2 a_2$ with hermitean forms a_j and a_0 is coercive and a_2 positive. This leads to compact, self-adjoint operators $K_1, K_2 : H^1_{0,\diamond}(D) \to H^1_{0,\diamond}(D)$ by representation theorem of Riesz. Variational equation takes the form:

$$v \;+\; \lambda K_1 v \;+\; \lambda^2 K_2 v \;=\; 0\,.$$

K_2 positive implies existence of $K_2^{1/2}$. With $z = \lambda K_2^{1/2} v$:

$$\begin{pmatrix} v \\ z \end{pmatrix} \;+\; \lambda \begin{pmatrix} K_1 & K_2^{1/2} \\ -K_2^{1/2} & 0 \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus: spectrum is discrete!

Maxwell's Equations, Introduction

Electric field $E : \mathbb{R}^3 \to \mathbb{C}^3$ and magnetic field $H : \mathbb{R}^3 \to \mathbb{C}^3$ satisfy

$$\begin{aligned} \operatorname{curl} E &= i\omega\mu H \quad \text{in } \mathbb{R}^3, \\ \operatorname{curl} H &= -i\omega\varepsilon E + \sigma E \quad \text{in } \mathbb{R}^3, \end{aligned}$$

and tangential components of E and H are continuous at interfaces.

Parameters:

dielectricity $\varepsilon = \varepsilon_0 \varepsilon_r$, permeability $\mu = \mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}$,
conductivity σ, speed of light $c_0 = 1/\sqrt{\varepsilon_0 \mu_0} \approx 3 \cdot 10^8 \frac{m}{s}$
wave number: $k = \omega / c_0 = \omega \sqrt{\varepsilon_0 \mu_0}$

Assumption: $\varepsilon_r \in L^\infty(D)$ positive, $\sigma = 0$. Then $E, H \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ where

$$F \in H_{loc}(\operatorname{curl}, \mathbb{R}^3) \Longleftrightarrow F|_B, \operatorname{curl} F|_B \in L^2(B)$$

for all balls $B \subset \mathbb{R}^3$.

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Maxwell's Equations, Introduction

Solution has to be understood in the variational sense:

$$\iint\limits_{\mathbb{R}^3} [\operatorname{curl} E \cdot \operatorname{curl} \psi - k^2 \varepsilon_r E \cdot \psi] dx = 0$$

$$\iint\limits_{\mathbb{R}^3} [\varepsilon_r^{-1} \operatorname{curl} H \cdot \operatorname{curl} \psi - k^2 H \cdot \psi] dx = 0$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support.

(Additional assumption: $\varepsilon_r^{-1} \in L^\infty(D)$).

Scattering problem: Given E^{inc}, H^{inc}, determine total fields E, H and scattered fields $E^s = E - E^{inc}$ and $H^s = H - H^{inc}$ such that E, H solve system and E^s, H^s satisfy a radiation condition (Silver-Müller radiation condition).

Maxwell's Equations, Introduction

Every radiating solution v of

$$\mathrm{curl}^2 v - k^2 v \;=\; 0 \quad \text{outside some ball}$$

has asymptotic behaviour of the form

$$v(x) \;=\; \frac{\exp(ik|x|)}{4\pi|x|}\, v^\infty(\hat{x}) \;+\; \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty\,.$$

uniformly w.r.t. $\hat{x} = x/|x|$ where $v^\infty : S^3 \to \mathbb{C}^3$ is far field pattern.
Plane incident field of direction $\hat{\theta} \in S^2$ and polarization $p \in \mathbb{C}^3$ with $p \cdot \theta = 0$:

$$H^{inc}(x,\hat{\theta};p) = p\, e^{ik\hat{\theta}\cdot x} \quad \text{and} \quad E^{inc} = -\frac{1}{i\omega\varepsilon_0}\, \mathrm{curl}\, H^{inc}$$

corresponds to far field pattern $\boxed{H^\infty = H^\infty(\hat{x},\hat{\theta};p).}$

Maxwell's Equations, Introduction

Inverse scattering problem: Given $H^\infty(\hat{x},\hat{\theta};p)$ for all $\hat{x},\hat{\theta}\in S^2$ and $p\in\mathbb{C}^3$ with $p\cdot\theta=0$, find ε_r or (only) the support $\overline{D}$ of ε_r-1!

Uniqueness: D. Colton, L. Päivärinta 1992: ε_r is uniquely determined by $H^\infty\big(\hat{x},\hat{\theta};p\big)$ for all $\hat{x},\hat{\theta},p$.

Standard reference: D. Colton and R. Kress: Inverse Acoustic and Electromagnetic Scattering Theory, 2nd edition, Springer, 1998.

Also possible: Anisotropic case; that is, ε_r is matrix-valued and $\varepsilon_r(x)$ symmetric for a.a. x and $\varepsilon_r(x),\varepsilon_r^{-1}(x)$ are uniformly positive definite on D.

Then uniqueness only up to "change of coordinates".

Our goal: Determine only support D of contrast ε_r-1 !

Maxwell's Equations, Far Field Operator

Far field operator $F : L^2_t(S^2) \to L^2_t(S^2)$, defined as

$$(Fp)(\hat{x}) := \int_{S^2} H^\infty\big(\hat{x}, \hat{\theta}; p(\hat{\theta})\big)\, ds(\hat{\theta}), \quad \hat{x} \in S^2,$$

satisfies

$$F - F^* = \frac{ik}{8\pi^2} F^* F.$$

Scattering operator $\mathcal{S} := I + \frac{ik}{8\pi^2} F$ is unitary and F is normal.

Maxwell's Equations, Factorization

Recall: $\operatorname{curl}\left[\varepsilon_r^{-1}\operatorname{curl} H\right] - k^2 H = 0\,;$ that is,

$$\operatorname{curl}\left[\varepsilon_r^{-1}\operatorname{curl} H^s\right] - k^2 H^s \;=\; \operatorname{curl}\left[q\,\operatorname{curl} H^{inc}\right] \quad \text{in } \mathbb{R}^3\,,$$

where $q = 1 - \varepsilon_r^{-1}$ denotes contrast.

Assumption: $q(x) \geq \hat{q} > 0$ on D. Define: $\mathcal{H}\colon L^2_t(S^2) \to L^2(D, \mathbb{C}^3)$ and $\mathcal{T}\colon L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ by

$$(\mathcal{H}p)(x) = \sqrt{q(x)}\,\operatorname{curl}\int_{S^2} p(\theta)\,e^{ikx\cdot\theta}\,ds(\theta)\,,\; x \in D\,, \text{ and}$$

$\mathcal{T}f = \left[f + \sqrt{q}\,\operatorname{curl} v\right]$ and $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ solves

$$\operatorname{curl}\left[\varepsilon_r^{-1}\operatorname{curl} v\right] - k^2 v \;=\; \operatorname{curl}\left[\sqrt{q}\,f\right] \quad \text{in } \mathbb{R}^3\,.$$

Theorem: The following factorization holds: $\boxed{F = \mathcal{H}^*\mathcal{T}\mathcal{H}}$

Factorization Method, cont.

Theorem: For $z \in \mathbb{R}^3$ and fixed $p \in \mathbb{C}^3$ define $\phi_z \in L^2_t(S^2)$ by

$$\phi_z(\hat{x}) \;=\; -ik\,(\hat{x} \times p)\,e^{-ik\hat{x}\cdot z}\,, \quad \hat{x} \in S^2\,.$$

Then: $$z \in D \quad \Longleftrightarrow \quad \phi_z \in \mathcal{R}(\mathcal{H}^*)$$

Theorem: Assume that k^2 is not an interior transmission eigenvalue (see below). Then

$$\mathcal{R}(\mathcal{H}^*) \;=\; \mathcal{R}\big((F^*F)^{1/4}\big)\,.$$

Corollary:

$$z \in D \quad \Leftrightarrow \; \phi_z \in \mathcal{R}\big((F^*F)^{1/4}\big) \;\Leftrightarrow\; \sum_j \frac{\big|\langle \phi_z, \psi_j \rangle_{L^2}\big|^2}{|\lambda_j|} \;<\; \infty$$

where $\{\lambda_j, \psi_j\}$ spectral system of (normal!) operator F.

Interior Transmission Eigenvalue

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Definition: $\lambda > 0$ is called interior transmission eigenvalue if there exists $u, w \in H(\text{curl}, D)$ with

$$\begin{aligned} \text{curl}\big[(1+q)\,\text{curl}\,u\big] \;-\; \lambda u &= 0 \quad \text{in } D\,, \\ \text{curl}\,\text{curl}\,w \;-\; \lambda w &= 0 \quad \text{in } D\,, \end{aligned}$$

$$\nu \times u = \nu \times w \text{ on } \partial D\,, \quad (1+q)\,\nu \times \text{curl}\,u = \nu \times \text{curl}\,w \text{ on } \partial D\,.$$