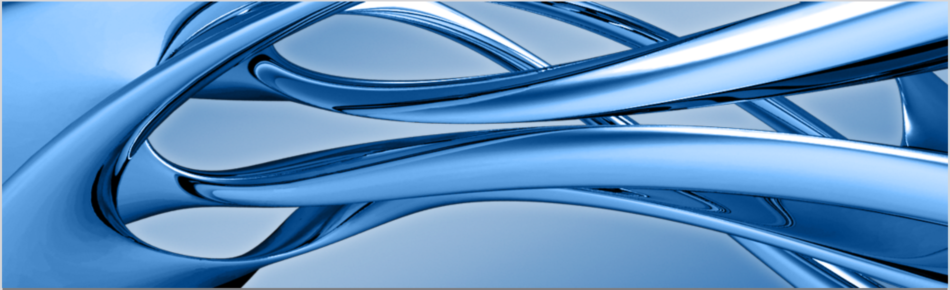
Karlsruhe Institute of Technology

# The Factorization Method for Inverse Scattering Problems Part II

Andreas Kirsch | Madrid 2011

Department of Mathematics



# Recall: Outline of the Course

- Part I:
  - Introduction
  - The Direct Scattering Problem, Dirichlet Boundary Conditions
  - The Direct Scattering Problem, Inhomogeneous Medium
  - The Inverse Scattering Problem (Dirichlet Boundary Conditions)
- Part II:
  - A Factorization of the Far Field Operator
  - Range Identities
  - The Factorization Method
  - Some Numerical Simulations
- Part III:
  - The Inverse Scattering Problem for Inhomogeneous Media
  - An Interior Transmission Eigenvalue Problem
  - An Electromagnetic Inverse Scattering Problem (perhaps)

# The Factorization Method

Recall: Incident plane wave $u^{inc}(x) = \exp(ik\hat{\theta} \cdot x)$ is scattered by obstacle $D \subset \mathbb{R}^d$ and produces scattered field $u^s = u^s(x, \hat{\theta})$ and total field $u = u^{inc} + u^s$ satisfying

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \setminus \overline{D}\,, \quad u = 0 \text{ on } \partial D\,,$$

and $u^s$ satisfies Sommerfeld's radiation condition (SRC).

$u^s$ has asyptotic behaviour

$$u^s(x, \hat{\theta}) \;=\; \gamma_d \, \frac{\exp(ik|x|)}{|x|^{(d-1)/2}} \left[ u^\infty(\hat{x}, \hat{\theta}) \;+\; \mathcal{O}(1/|x|) \right], \quad |x| \to \infty\,,$$

uniformly with respect to $\hat{x} := x/|x| \in S^{d-1}$ and $\hat{\theta} \in S^{d-1}$.

For the inverse problem we know $u^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^{d-1}$.

Define far field operator $F : L^2(S^{d-1}) \longrightarrow L^2(S^{d-1})$ by

$$(Fg)(\hat{x}) \;:=\; \int_{S^{d-1}} u^\infty(\hat{x}, \hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta})\,, \quad \hat{x} \in S^{d-1}\,.$$

# The Factorization Method

Recall: $(Fg)\big(\hat{x}\big) := \int_{S^{d-1}} u^\infty(\hat{x},\hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta}), \quad \hat{x} \in S^{d-1}.$

$$u^\infty(\hat{x},\hat{\theta}) \quad \longleftrightarrow \quad u^{inc}(x,\hat{\theta}) = \exp(ik\,\hat{\theta}\cdot x)$$

$$(Fg)(\hat{x}) \quad \longleftrightarrow \quad \int_{S^{d-1}} u^{inc}(x,\hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta})$$

- $F$ is compact.
- $F$ is one-to-one with dense range $\mathcal{R}(F)$ provided $k^2$ is not a Dirichlet-eigenvalue of $-\Delta$ in $D$.
- $F$ is normal; that is, $F^*F = F\,F^*$, and even: $\mathcal{S} := I + \frac{ik}{2\pi} F$ is unitary (=scattering matrix).

Define $H : L^2(S^{d-1}) \longrightarrow H^{1/2}(\partial D)$ by

$$(Hg)(x) = \int_{S^{d-1}} u^{inc}(x,\hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta}) = \int_{S^{d-1}} e^{ik\,x\cdot\hat{\theta}}\, g(\hat{\theta})\, ds(\hat{\theta})\,,\ x \in \partial D\,.$$

# The Factorization Method

Define $G : H^{1/2}(\partial D) \longrightarrow L^2(S^{d-1})$ by $f \mapsto v^\infty$, where $v$ solves exterior boundary value problem:

$$\Delta v \;+\; k^2 v \;=\; 0 \quad \text{in } \mathbb{R}^d \setminus \overline{D}\,,$$

$$v \;=\; f \quad \text{on } \partial D\,, \quad \text{and} \quad \text{(SRC)}\,.$$

Since $u^s = -u^{inc}$ on $\partial D$ we conclude $\quad F = -GH$. $\quad$ Furthermore,

$$(H^*\varphi)(\hat{x}) \;=\; \int_{\partial D} e^{-ik\,\hat{x}\cdot y}\,\varphi(y)\,ds(y)\,, \quad \hat{x} \in S^{d-1}\,,$$

thus $H^* = GS$ with single layer boundary operator

$$(S\varphi)(x) \;=\; \int_{\partial D} \Phi(x,y)\,\varphi(y)\,ds(y)\,, \quad x \in \partial D\,.$$

Theorem: $$F \;=\; -\,G\,S^*G^*$$

# The Factorization Method

Recall factorization: $\boxed{F = -G\,S^* G^*}$

$$
\begin{array}{ccc}
L^2(S^{d-1}) & \xrightarrow{\quad F \quad} & L^2(S^{d-1}) \\
G^* \Big\downarrow & & \Big\uparrow G \\
H^{-1/2}(\partial D) & \xrightarrow{\quad -S^* \quad} & H^{+1/2}(\partial D)
\end{array}
$$

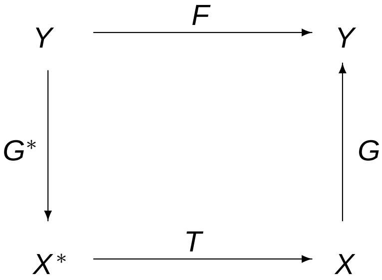
Theorem: Define $\phi_z \in L^2(S^{d-1})$ for $z \in \mathbb{R}^d$ by

$$\phi_z(\hat{x}) \;=\; e^{-ik\hat{x}\cdot z}\,, \quad \hat{x} \in S^{d-1}\,.$$

Then $z \in D \Longleftrightarrow \phi_z \in \mathcal{R}(G)$.

# Range Identity

Aim: Express range $\mathcal{R}(G)$ by known operator $F$ with use of $F = -GS^*G^*$. General situation $F = GTG^*$:




Theorem: If $T : X^* \to X$ is selfadjoint and coercive; that is,

$$\langle \psi, T\varphi \rangle = \overline{\langle \varphi, T\psi \rangle}\,, \qquad \langle \varphi, T\varphi \rangle \geq c\|\varphi\|^2 \quad \text{for all } \psi, \varphi \in X^*\,,$$

then
$$\mathcal{R}(G) \;=\; \mathcal{R}(F^{1/2})\,.$$

# Range Identity

Karlsruhe Institute of Technology

More general:

Theorem: Let $F = GTG^* : Y \to Y$ and $T : X^* \to X$ coercive in the generalized sense

$$|\langle \varphi, T\varphi \rangle| \;\geq\; c\|\varphi\|^2 \quad \text{for all } \varphi \in \mathcal{R}(G^*)\,.$$

Then: $\phi \in \mathcal{R}(G) \iff \inf\{|\langle \psi, F\psi \rangle_Y| : \psi \in Y,\ \langle \psi, \phi \rangle_Y = 1\} > 0$

Corollary: If $F = G_1 T_1 G_1^* = G_2 T_2 G_2^*$ and $T_1, T_2$ coercive in the generalized sense, then $\mathcal{R}(G_1) = \mathcal{R}(G_2)$.

Previous example where $T$ selfadjoint and coercive:
$F = GTG^* = F^{1/2}F^{1/2}$, thus $\mathcal{R}(G) = \mathcal{R}(F^{1/2})$.

Applications in impedance tomography: $F = \Lambda - \Lambda_0$ is relative Neumann-Dirichlet operator which is selfadjoint and coercive!

Application of theorem in scattering theory:

# Application to Scattering Problem

Lemma: Let $k^2$ be no Dirichlet eigenvalue of $-\Delta$ in $D$. Then:

(a) $S$ is an isomorphism from $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$.

(b) $\mathrm{Im}\langle\varphi, S\varphi\rangle < 0$ for all $\varphi \in H^{-1/2}(\partial D)$ with $\varphi \neq 0$.

(c) Let $S_i$ be the operator corresponding to $k = i$. Then $S_i$ is selfadjoint and coercive.

(d) $S - S_i$ is compact from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$.

Corollary: There exists $c > 0$ with

$$|\langle\varphi, S\varphi\rangle| \;\geq\; c\|\varphi\|^2_{H^{-1/2}(\partial D)} \quad \text{for all } \varphi \in H^{-1/2}(\partial D)\,.$$

Theorem: Let $k^2$ be no Dirichlet eigenvalue of $-\Delta$ in $D$. Then

$$\begin{aligned} z \in D \quad &\Leftrightarrow \quad \phi_z \in \mathcal{R}(G) \quad \text{where } \phi_z(\hat{x}) = \exp(-ikz\cdot\hat{x}) \\ &\Leftrightarrow \quad \inf\left\{|\langle\psi, F\psi\rangle_{L^2}| : \psi \in L^2(S^{d-1}),\ \langle\psi,\phi_z\rangle_{L^2} = 1\right\} > 0 \end{aligned}$$

Characterization by range?

# Range Identity, Cont.

Theorem: Let $F = GTG^* : Y \to Y$ be one-to-one and such that $I + ir\,F$ is unitary for some $r > 0$. Furthermore, let $T : X^* \to X$ be compact perturbation of a selfadjoint and coercive operator and $\mathrm{Im}\langle\varphi, T\varphi\rangle \neq 0$ for all $\varphi \neq 0$. Then $\boxed{\mathcal{R}(G) = \mathcal{R}\big((F^*F)^{1/4}\big).}$

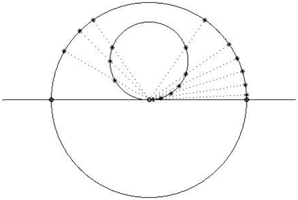
Idea of proof: $I + irF$ unitary implies $F$ normal and thus $F\psi_j = \lambda_j \psi_j$. Then $F = |F|^{1/2} W |F|^{1/2}$ where

$$|F|^{1/2}\psi = \sum_j \sqrt{|\lambda_j|}\,\langle\psi, \psi_j\rangle_Y\,\psi_j\,,$$

$$W\psi = \sum_j \frac{\lambda_j}{|\lambda_j|}\,\langle\psi, \psi_j\rangle_Y\,\psi_j\,.$$

$$|\langle W\psi, \psi\rangle| = \left|\sum_j \frac{\lambda_j}{|\lambda_j|}\,\big|\langle\psi, \psi_j\rangle_Y\big|^2\right|$$

$$\geq c\|\psi\|_Y^2$$



# Range Identity

Scattering media/impedance tomography with absorption:

Theorem: Let $F = GTG^* : Y \to Y$ such that
(a) $G : X \to Y$ compact with dense range,
(b) $\operatorname{Re} T = \frac{1}{2}(T + T^*) : X^* \to X$ compact perturbation of a selfadjoint and coercive operator,
(c) $\operatorname{Im} T = \frac{1}{2i}(T - T^*)$ compact and strictly positive on $\overline{\mathcal{R}(G^*)}$; that is, $\operatorname{Im}\langle\varphi, T\varphi\rangle < 0$ for all $\varphi \in \overline{\mathcal{R}(G^*)}$, $\varphi \neq 0$.

Then $\boxed{\mathcal{R}(G) = \mathcal{R}\big(F_\#^{1/2}\big)}$ where $F_\# = |\operatorname{Re} F| + \operatorname{Im} F$.

Idea of proof: Factorization $F = GTG^*$ yields

$$\operatorname{Re} F = G(\operatorname{Re} T)G^* \quad \text{and} \quad \operatorname{Im} F = G(\operatorname{Im} T)G^* .$$

Complicated: There exists $T_1$ such that $|\operatorname{Re} F| = GT_1G^*$, thus

$$F_\# = |\operatorname{Re} F| + \operatorname{Im} F = G\big[T_1 + \operatorname{Im} T\big]G^*$$

and $T_1 + \operatorname{Im} T$ is compact perturbation of coercive operator.

# Application to Scattering Problem

Let $k^2$ be no Dirichlet-eigenvalue of $-\Delta$ in $D$.

Recall: $F = -GS^*G^*$ and $F$ is one-to-one and $I + \frac{ik}{2\pi}F$ is unitary and $S^* = S_i + (S^* - S_i) : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ where $S_i$ is selfadjoint and coercive and $\mathrm{Im}\langle\varphi, S^*\varphi\rangle > 0$ for all $\varphi \neq 0$. Thus

$$\boxed{\mathcal{R}(G) = \mathcal{R}\big((F^*F)^{1/4}\big).}$$

Combination of previous theorems:

**Theorem:** Let $k^2$ be no Dirichlet-eigenvalue of $-\Delta$ in $D$. Then:

$$z \in D \iff \phi_z \in \mathcal{R}\big((F^*F)^{1/4}\big)$$

Let $\{\lambda_j : j \in \mathbb{N}\} \subset \mathbb{C}$ be eigenvalues of $F$ with normalized eigenfunctions $\psi_j \in L^2(S^{d-1})$ for $j \in \mathbb{N}$. Then:

$$z \in D \iff \sum_{j \in \mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} < \infty .$$

# Numerical Simulations

Recall:

$$z \in D \iff \sum_{j\in\mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} < \infty$$

$$\iff w(z) = \left[ \sum_{j\in\mathbb{N}} \frac{|\langle \phi_z, \psi_j \rangle_{L^2}|^2}{|\lambda_j|} \right]^{-1} > 0\,.$$

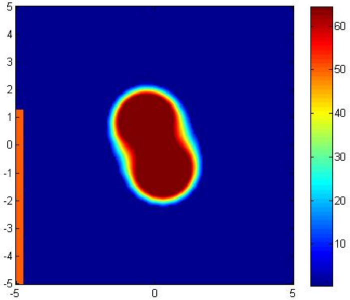
Therefore, sign($w$) is the characteristic function of $D$!

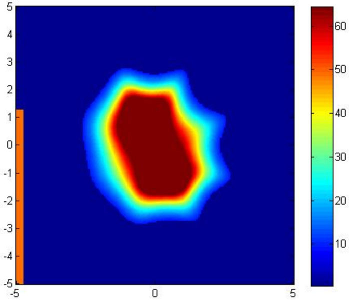
The following examples show plots of

$$w_N(z) = \left[ \sum_{j=1}^{N} \frac{|\langle \phi_z, \psi_j \rangle|^2}{|\lambda_j|} \right]^{-1}, \quad z \in \mathbb{R}^2 :$$

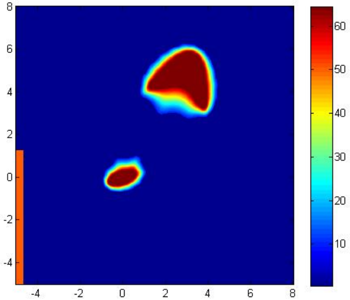
for $N = 32$ or $N = 36$, respectively.

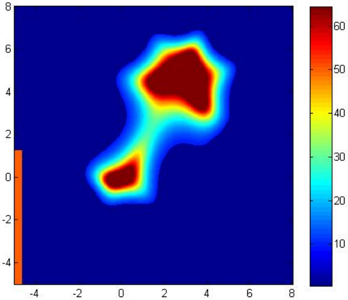
# Numerical Simulations



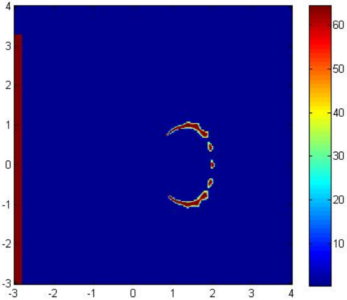



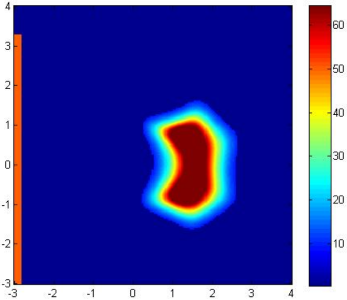
# Numerical Simulations



# Numerical Simulations



# Numerical Simulations