Karlsruhe Institute of Technology

# The Factorization Method for Inverse Scattering Problems Part I

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# Outline of the Course

- Part I:
  - Introduction
  - The Direct Scattering Problem, Dirichlet Boundary Conditions
  - The Direct Scattering Problem, Inhomogeneous Medium
  - The Inverse Scattering Problem (Dirichlet Boundary Conditions)
- Part II:
  - A Factorization of the Far Field Operator
  - Range Identities
  - The Factorization Method
  - Some Numerical Simulations
- Part III:
  - The Inverse Scattering Problem for Inhomogeneous Media
  - An Interior Transmission Eigenvalue Problem (see also Lassi!)
  - An Electromagnetic Inverse Scattering Problem (perhaps)

# Introduction

Propagation of (acoustic) waves is modelled by scalar wave equation in $\mathbb{R}^d$ for $d = 2$ or 3:

$$\frac{\partial^2 U(x,t)}{\partial t^2} = c^2 \Delta_x U(x,t), \quad x \in \mathbb{R}^d, \; t \geq 0,$$

where $U(x,t)$ is potential at location $x$ and time $t$.

| | |
|---|---|
| Speed of sound: | $c$ |
| Velocity: | $\nabla_x U(x,t)$ |
| Pressure: | $-\partial U(x,t)/\partial t$ |

Special Case: Time harmonic (=periodic) waves, in complex form:

$$U(x,t) = u(x)\, e^{-i\omega t}$$

with circular frequency $\omega > 0$; that is $\lambda = 2\pi/\omega$ is wavelength.

Physical wave: $\operatorname{Re} U(x,t) = \operatorname{Re} u(x) \cos(\omega t) + \operatorname{Im} u(x) \sin(\omega t)$.

# Introduction

$u$ satisfies Helmholtz equation (reduced wave equation)

$$\Delta u(x) + k^2 u(x) = 0 \quad \text{in (part of) } \mathbb{R}^d ,$$

with wave number $k = \omega / c > 0$.

Examples:

(a) Plane wave of direction $\hat{\theta} \in S^{d-1}$ (= unit sphere in $\mathbb{R}^d$):

$$u(x) = e^{ik\,\hat{\theta}\cdot x} , \quad x \in \mathbb{R}^d .$$

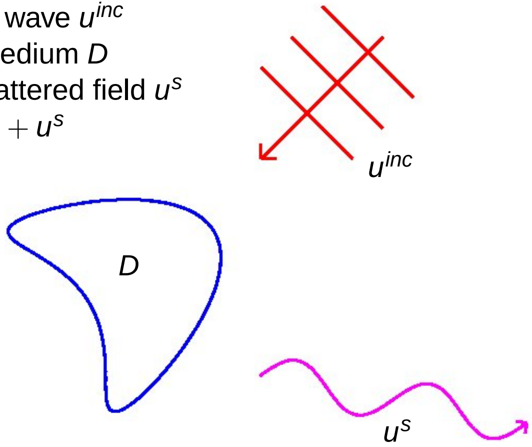
(b) Spherical wave with source point $y \in \mathbb{R}^d$:

$$u(x) = \Phi(x,y) := \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) , & d = 2, \\ \dfrac{\exp(ik|x-y|)}{4\pi|x-y|} , & d = 3, \end{cases} \qquad x \neq y .$$

# Introduction



For a general scattering problem
a given (“incident”) wave $u^{inc}$
is disturbed by a medium $D$
and produces a scattered field $u^s$
Total field: $u = u^{inc} + u^s$



# Introduction

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The direct scattering problem is to determine the scattered and total field when the wave number $k > 0$, the incident field $u^{inc}$, and the scattering medium $D$ is given.

In the inverse scattering problem the incident and the scattered fields are known ("measured"), and the medium $D$ has to be determined.

Literature on time harmonic (inverse) scattering theory:

- D. Colton, R. Kress: Inverse Acoustic and Electromagnetic Scattering Theory. 2nd edition, Springer, 1998.
- A. Kirsch: Introduction to the Mathematical Theory of Inverse Problems. Springer 1996, 2011.
- J.-C. Nédélec: Acoustic and Electromagnetic Equations. Springer, 2001.

# Direct Scattering Problem, Dirichlet BC

Simple model for scattering problem:

Helmholtz equation: $\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{D}$

Boundary condition: $u = 0 \quad \text{on } \partial D$

Sommerfeld's radiation condition (SRC):

$$\frac{\partial u^s(x)}{\partial r} - ik\, u^s(x) \;=\; \mathcal{O}\left(r^{-(d+1)/2}\right), \quad r = |x| \to \infty ,$$

uniformly with respect to $\hat{x} = x/|x| \in S^{d-1}$.

This is a classical boundary value problem in the (unbounded) exterior domain $\mathbb{R}^d \setminus \overline{D}$ for $u^s$ of the type:

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^d \setminus \overline{D}, \quad v = f \text{ on } \partial D, \quad v \text{ satisfies SRC}$$

# Direct Scattering Problem, Dirichlet BC

How to solve this exterior boundary value problem

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^d \setminus \overline{D}\,, \quad v = f \text{ on } \partial D\,, \quad v \text{ satisfies SRC ?}$$

First uniqueness: This is based on Lemma of Rellich:

For $k > 0$ (real valued) and $\Delta v + k^2 v = 0$ for $|x| > R_0$ it holds that

$$\lim_{R\to\infty} \int_{|x|=R} |v|^2 ds \; = 0 \quad \text{implies} \quad v = 0 \text{ for } |x| > R_0$$

Proof of uniqueness: Assume $f = 0$. Green's theorem yields

$$o(1) \;=\; \int\limits_{|x|=R} \left| \frac{\partial v}{\partial r} - ikv \right|^2 ds = \int\limits_{|x|=R} \left| \frac{\partial v}{\partial r} \right|^2 + k^2 |v|^2 ds$$

$$+\, 2k \,\mathrm{Im} \underbrace{\int\limits_{|x|=R} v \,\frac{\partial \overline{v}}{\partial r}\, ds}_{=0}$$

by Green's theorem in $B_R \setminus D$ and $v = 0$ on $\partial D$.
Thus $v = 0$ in $\mathbb{R}^d \setminus D$.

# Direct Scattering Problem, Dirichlet BC

Existence: At least two approaches. Let $f \in H^{1/2}(\partial D)$.

(A) Variational approach: Let $B$ be ball with radius $R$ such that $\overline{D} \subset B$. Solution of ext. bvp for $B$ is given by series ($d = 2$):

$$v(r,\phi) \;=\; \sum_{n\in\mathbb{Z}} \frac{f_n}{H_n^{(1)}(kR)}\, H_n^{(1)}(kr)\, e^{in\phi}\,,$$

with
$$f_n \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(R\phi)\, e^{-in\phi}\, d\phi\,.$$

This defines Dirichlet-Neumann operator $\Lambda : f \mapsto \partial v/\partial r|_{\partial B}$.
Green's formula in $B \setminus \overline{D}$:

$$\iint\limits_{B\setminus\overline{D}} \left[\nabla v \cdot \nabla\psi - k^2 v\psi\right] dx \;=\; \int\limits_{\partial B} \psi\, \frac{\partial v}{\partial \nu}\, ds \quad \text{for all } \psi \in X\,,$$

where
$$X \;=\; \left\{\psi \in H^1(B \setminus \overline{D}) : \psi = 0 \text{ on } \partial D\right\}.$$

# Direct Scattering Problem, Dirichlet BC

Substituting $\partial v/\partial \nu|_{\partial B} = \Lambda v$ yields: Determine $v \in H^1(B \setminus \overline{D})$ with $v = f$ on $\partial D$ and

$$\iint\limits_{B\setminus\overline{D}} \left[\nabla v \cdot \nabla \psi - k^2 v \psi\right] dx \;=\; \int\limits_{\partial B} \psi\, \Lambda v\, ds \quad \text{for all } \psi \in X\,.$$

Transformation to homogeneous boundary data: Choose $F \in H^1(B \setminus \partial D)$ with $F = f$ on $\partial D$ and $F = 0$ on $\partial B$ and make ansatz $v = F + w$. Then $w \in X$ has to solve

$$\iint\limits_{B\setminus\overline{D}} \left[\overline{\nabla w} \cdot \nabla \psi - k^2 \overline{w} \psi\right] dx - \int\limits_{\partial B} \psi\, \overline{\Lambda w}\, ds \;=\; \ell(\psi)\,, \quad \psi \in X$$

where $$\ell(\psi) \;=\; -\iint\limits_{B\setminus\overline{D}} \left[\overline{\nabla F} \cdot \nabla \psi - k^2 \overline{F} \psi\right] dx\,, \quad \psi \in X\,.$$

# Direct Scattering Problem, Dirichlet BC

$$\iint_{B\setminus\overline{D}} \left[\nabla\overline{w}\cdot\nabla\psi - k^2\overline{w}\psi\right] dx - \int_{\partial B} \psi\,\overline{\Lambda w}\,ds \;=\; \ell(\psi)\,, \quad \psi\in X\,.$$

Remark:
$$\Lambda f \;=\; \Lambda_k f \;=\; k\sum_{n\in\mathbb{Z}} f_n\,\frac{(H_n^{(1)})'(kR)}{H_n^{(1)}(kR)}\,e^{in\phi}$$

is bounded from $H^{1/2}(\partial B)$ into $H^{-1/2}(\partial B)$. Integral over $\partial B$ is dual form $\langle\Lambda_k w,\psi\rangle$. Difference $\Lambda_k - \Lambda_i$ compact, sesqui-linear forms:

$$a(w,\psi) \;=\; \iint_{B\setminus\overline{D}} \nabla\overline{w}\cdot\nabla\psi\,dx \;-\; \langle\Lambda_i w,\psi\rangle \quad \text{is coercive,}$$

$$b(w,\psi) \;=\; k^2\iint_{B\setminus\overline{D}} \overline{w}\psi\,dx \;-\; \langle(\Lambda_i-\Lambda_k)w,\psi\rangle \quad \text{is compact.}$$

Theorem of Lax-Milgram and Riesz-theory yield existence.

# Direct Scattering Problem, Dirichlet BC

(B) Boundary integral equation approach:

Recall fundamental solution

$$\Phi(x,y) := \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|)\,, & d = 2, \\ \dfrac{\exp(ik|x-y|)}{4\pi|x-y|}\,, & d = 3, \end{cases} \qquad x \neq y\,.$$

Make ansatz for $v$ as combination of double and single layer:

$$v(x) = \int_{\partial D} \left[ \frac{\partial}{\partial \nu(y)} \Phi(x,y) + i\,\Phi(x,y) \right] \varphi(y)\, ds(y)\,, \quad x \notin \overline{D}\,,$$

with density $\varphi \in H^{1/2}(\partial D)$. Then $v$ satisfies SRC and $\Delta v + k^2 v = 0$ outside of $\overline{D}$. Continuity properties of single and double layer with $H^{1/2}(\partial D)-$density yields boundary integral equation

# Direct Scattering Problem, Dirichlet BC

$$\varphi + D\varphi + i\,S\varphi = f \quad \text{in } H^{1/2}(\partial D)\,,$$

where

$$(D\varphi)(x) = \int_{\partial D} \varphi(y)\,\frac{\partial}{\partial\nu(y)}\Phi(x,y)\,ds(y)\,, \quad x \in \partial D\,,$$

$$(S\varphi)(x) = \int_{\partial D} \varphi(y)\,\Phi(x,y)\,ds(y)\,, \quad x \in \partial D\,,$$

are compact in $H^{1/2}(\partial D)$. Again, Riesz-theory yields existence.

Literature on boundary integral equation approach:

- D. Colton, R. Kress: Inverse Acoustic and Electromagnetic Scattering Theory, 2nd edition, Springer, 1998.
- W. McLean: Strongly Elliptic Systems and Boundary Integral Operators, Cambridge University Press, 2000

# Direct Scattering Problem, Dirichlet BC

Fundamental solution has asymptotic form

$$\Phi(x,y) \;=\; \gamma_d \,\frac{\exp(ik|x|)}{|x|^{(d-1)/2}} \left[e^{-ik\,\hat{x}\cdot y} \;+\; \mathcal{O}(1/|x|)\right], \quad |x| \to \infty\,,$$

uniformly with respect to $\hat{x} := x/|x| \in S^{d-1}$ and $y \in \partial D$, where $\gamma_2 = (1+i)/(4\sqrt{k\pi})$ and $\gamma_3 = 1/(4\pi)$. Ansatz yields

$$v(x) \;=\; \gamma_d \,\frac{\exp(ik|x|)}{|x|^{(d-1)/2}} \left[v^\infty(\hat{x}) \;+\; \mathcal{O}(1/|x|)\right], \quad |x| \to \infty\,,$$

uniformly with respect to $\hat{x} := x/|x| \in S^{d-1}$.

For special case $f(x) = -u^{inc}(x) = -\exp(ik\hat{\theta}\cdot x)$ the function $u^\infty = u^\infty(\hat{x},\hat{\theta})$ is called far field pattern or scattering amplitude.

# Direct Scattering Problem, Inhom. Medium

Medium penetrable with index of refraction $1+q$.

Direct scattering problem: Given bounded domain $D$ and $k>0$ and $q \in L^\infty(D)$ with $q \geq \hat{q} > 0$ on $D$ and $u^{inc}(x) = \exp(ik\hat{\theta}\cdot x)$, determine total field $u$ and scattered field $u^s = u - u^{inc}$ such that

$$\Delta u + k^2(1+q)u = 0 \text{ in } \mathbb{R}^d\,, \quad u^s \text{ satisfies SRC.}$$

Uniqueness again by Rellich's Lemma and unique continuation.
Existence by, e.g., Lippmann-Schwinger integral equation

$$u(x) \;=\; u^{inc}(x) \;+\; k^2 \iint_D q(y)\,u(y)\,\Phi_k(x,y)\,dy\,, \quad x\in D\,.$$

Again: $$u^s(x) \;=\; \gamma_d\,\frac{\exp(ik|x|)}{|x|^{(d-1)/2}}\left[u^\infty(\hat{x}) \;+\; \mathcal{O}(1/|x|)\right], \quad |x|\to\infty\,,$$

uniformly with respect to $\hat{x} := x/|x| \in S^{d-1}$ with far field pattern $u^\infty = u^\infty(\hat{x},\hat{\theta})$.

# Inverse Scattering Problem (Dirich. BC)

Recall: Incident plane wave $u^{inc}(x) = \exp(ik\hat{\theta}\cdot x)$ is scattered by obstacle $D \subset \mathbb{R}^d$ and produces scattered field $u^s = u^s(x,\hat{\theta})$ and total field $u = u^{inc} + u^s$ satisfying

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \setminus \overline{D}\,, \quad u = 0 \text{ on } \partial D\,,$$

and $u^s$ satisfies Sommerfeld's radiation condition (SRC).

$u^s$ has asyptotic behaviour

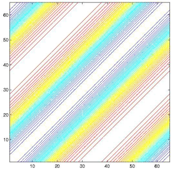
$$u^s(x,\hat{\theta}) \;=\; \gamma_d\,\frac{\exp(ik|x|)}{|x|^{(d-1)/2}}\left[u^\infty(\hat{x},\hat{\theta}) \;+\; \mathcal{O}(1/|x|)\right], \quad |x|\to\infty\,,$$

uniformly with respect to $\hat{x} := x/|x| \in S^{d-1}$ and $\hat{\theta} \in S^{d-1}$.

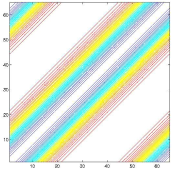
The inverse scattering problem is to determine the shape of $D$ from the knowledge of the far field pattern $u^\infty(\hat{x},\hat{\theta})$ for all $\hat{x},\hat{\theta} \in S^{d-1}$.

# The Inverse Scattering Problem

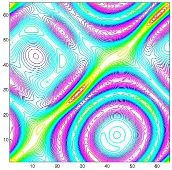
Which domain $D \subset \mathbb{R}^2$ corresponds to the following far fields $u^\infty(\phi, \theta)$, $\phi, \theta \in [0, 2\pi]$?



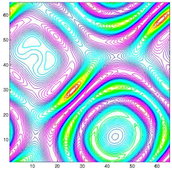
$\operatorname{Re} u^\infty$



$\operatorname{Im} u^\infty$



$\operatorname{Re} u^\infty$



$\operatorname{Im} u^\infty$

Left example simple:

Theorem of Karp: If $u^\infty(\hat{x}, \hat{\theta}) = \psi(\hat{x} \cdot \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^{d-1}$, then $D$ is a ball in $\mathbb{R}^{d-1}$.

# The Inverse Scattering Problem

First uniqueness:

Theorem (Schiffer, before 1967)

The far field patterns $u^\infty(\hat{x},\hat{\theta})$ determine $D$ uniquely; that is, if $D_j \leftrightarrow u_j^\infty(\hat{x},\hat{\theta})$ for $j = 1, 2$, then:

$$u_1^\infty(\hat{x},\hat{\theta}) = u_2^\infty(\hat{x},\hat{\theta}) \text{ for all } \hat{x},\hat{\theta} \in S^{d-1} \implies D_1 = D_2\,.$$

Second stability:

Theorem (Isakov 1991, 1993)

Let $D_1, D_2$ be star-shaped with respect to the origin; that is, $D_j = \{r\hat{x} : r < d_j(\hat{x}),\ \hat{x} \in S^{d-1}\}$. Assume $\frac{1}{c_0} \leq \|d_j\|_{2+\tau} \leq c_0$ and

$$\left|u_1^\infty(\hat{x},\hat{\theta}) - u_2^\infty(\hat{x},\hat{\theta})\right| \leq \varepsilon \text{ for all } \hat{x} \in S^{d-1} \text{ and some } \hat{\theta} \in S^{d-1}\,.$$

Then

$$\|d_1 - d_2\|_\infty \leq c\,\ln(-\ln\varepsilon)^{-1/c}$$

where $c$ depends only on $c_0$.

# The Inverse Scattering Problem

Third reconstruction techniques:

(A) Iterative methods. Define mapping (for fixed incident field) $\mathcal{F} : D \mapsto u^\infty$. Apply iterative method to solve $\mathcal{F}(D) = f$ for $D$ where $f$ is given (measured) far field pattern.

After parametrization (for, e.g., star-shaped obstacles) this leads to $\mathcal{F} : C^2(S^{d-1}, \mathbb{R}_{>0}) \to C(S^{d-1}, \mathbb{C})$. Possible members of this group: Newton-type methods, gradient-type methods, second order methods. Derivatives are computed via, e.g., domain derivatives.

Advantages: Very general, accurate, incorporation of a priori information possible.

Disadvantages: “Expensive”, only local convergence, a priori information necessary (number of components, type of boundary condition).

# The Inverse Scattering Problem

(B) Methods based on analytic continuation. First step: Given (measured) far field pattern $f$ on $D^{d-1}$ determine (approximation of) scattered field $u^s$ with $u^\infty = f$. Second step: Determine surface Γ such that$u^s + u^{inc} = 0$ on Γ. Usually, both steps are combined into one functional to be minimized. Members of this group: *Dual space method* by Colton/Monk, *continuation method* by Kirsch/Kress, *point source method* by Potthast, *contrast source inversion method* by Kleinman/van den Berg.

Advantages: Avoids computation of direct problems, quite general, incorporation of a priori information possible.

Disadvantages: Only local convergence since methods are based on minimization of non-quadratic functionals, a priori information necessary (number of components, type of boundary condition).

# The Inverse Scattering Problem

(C) Sampling Methods. Choose set of *sampling objects*, e.g. points $z \in \mathbb{R}^d$, and construct binary criterium which uses only the data $u^\infty$ to decide whether or not $z$ belongs to $D$. Members of this group: *Linear sampling method* by Colton/Kirsch, *Factorization method* by Kirsch (both use points $z \in \mathbb{R}^d$ as sampling objects), *Probe method* by Ikehata (curves), *No-response-test* by Luke/Potthast (domains), *Singular sources method* by Potthast (points, in combination with point source method)

We discuss only Factorization method.

Advantages: Fast, avoids computation of direct problems, no a priori information on type of boundary condition or number of components necessary, mathematically elegant and rigorous, gives characteristic function explicitely.

Disadvantages: Needs $u^\infty(\hat{x}, \hat{\theta})$ for many (in theory: all) $\hat{x}, \hat{\theta}$, no incorporation of a-priory information possible, very sensitive to noise