HILBERT SCHEMES OF POINTS, ENUMERATIVE GEOMETRY OF CURVES, AND TROPICAL GEOMETRY

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The Hilbert scheme of points $X[n]$ on a projective variety $X$ parametrizes 0-dimensional subschemes of length $n$ on $X$. It is a simple but important example of a moduli space, with connections to many other subjects, like moduli of sheaves, enumerative geometry of curves, Donaldson-Thomas invariants. For $C$ a smooth curve, $C[n]$ is just the symmetric power $C^{(n)}$ of $C$. For $C$ a singular curve, $C[n]$ is different from $C^{(n)}$: the topology of $C[n]$ contains nontrivial information about the singularities of $C$.

The information contained in the Hilbert schemes of points $C[n]$, can be best used by organizing them into generating functions. The simplest example is the generating function $\sum_{n \geq 0} e(C[n]) t^n$ for the topological Euler numbers. It turns out that this generating function contains information about the deformation theory of the singularities of $C$, it can be related to knot invariants, it has been used to give a mathematical definition of the Gopakumar-Vafa invariants from physics, and to count singular curves on surfaces.

Replacing the Euler number by finer invariants like the $\chi_y$-genus, one gets polynomial curve counting invariants. These can be related to the count of real algebraic curves and also to tropical geometry.

In these lectures we want to give an overview over this circle of ideas. On the way we want to introduce many concepts and techniques, e.g. Hilbert schemes of points, compactified Jacobians, Grothendieck groups of varieties, generating functions, $\chi_y$-genus, cobordism ring, equivariant localization, Severi degrees, moduli spaces of pairs, Welschinger invariants, tropical geometry.