Multiple Zeta Values, Multiple Polylogarithms and Quantum Field Theory

ICMAT, Campus de Cantoblanco, Madrid (Spain)
October 7-11, 2013
www.icmat.es/congresos/MZV-School/

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We are happy to invite you to participate in the international school *Multiple Zeta Values, Multiple Polylogarithms and Quantum Field Theory*, hosted by ICMAT (www.icmat.es), in Madrid, from October 7-11, 2013.
The subject of this school is the theory of **multiple zeta values** (MZVs) and their generalizations, i.e., multiple polylogarithms (MPLs). It recently attracted much attention, both in pure mathematics and theoretical physics. MZVs are special values of multiple zeta functions. The latter may be considered as a vast generalization of the Riemann zeta function. A systematic study of these numbers only started in the early 1990s with the work of M. Hoffman and D. Zagier, although the prehistory can be traced back to Euler in the 18th century.
Research on MZVs and MPLs comprises several mathematical areas, including algebra (Hopf and Lie algebras), combinatorics (double shuffle relations), algebraic geometry (Grothendieck's motives), Lie group theory (Kashiwara-Vergne conjecture), and, of course, number theory. Moreover, they have deep connections with modern high-energy physics (Feynman diagrams in perturbative quantum field theory).

This series of lectures consists of 3 mini-courses (6hrs each) combined with discussion and problem sessions. It aims at introducing in pedagogical lectures, accessible to advanced master and early PhD students in mathematics, the fascinating theory of MZVs and MPLs. The lecturers are going to present the basic structures and their generalizations as well as its ramifications into other areas of modern mathematics.

Lecturers:

• **Herbert Gangl** (Durham University, UK) -- 6hrs
An introduction to Abstract Multiple Polylogarithms and their Symbology
http://www.maths.dur.ac.uk/~dma0hg/

• **Leila Schneps** (CNRS, Inst. Jussieu, Paris, France) -- 6hrs
Multiple zeta values and their surprising connections to other mathematical problems
http://www.math.jussieu.fr/~leila/

• **Michel Waldschmidt** (Univ. Paris VI, Inst. Jussieu, Paris, France) -- 6hrs
An elementary introduction to Multiple Zeta Values
http://www.math.jussieu.fr/~miw/index2.html

The school is supported by

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Schedule

Arrival: Sunday, Oct. 6, 2013

Departure: Saturday, Oct. 12, 2013

Monday	Tuesday	Wednesday	Thursday	Friday
10:00 – 11:00 Waldschmidt 1	10:00 – 11:00 Waldschmidt 3	10:00 – 11:00 Waldschmidt 5	10:00 – 11:00 Waldschmidt 6	10:00 – 11:00 Schneps 5
Break	**Break**	**Break**	**Break**	**Break**
11:30 – 12:30 Waldschmidt 2	11:30 – 12:30 Schneps 1	11:30 – 12:30 Gangl 3	11:30 – 12:30 Schneps 3	11:30 – 12:30 Schneps 6
12:30 – 13:30 *Discussion*	12:30 – 13:30 *Discussion*	12:30 – 13:30 *Discussion*	12:30 – 13:30 *Discussion*	12:30 – 13:30 *Discussion*
Lunch 13:30 – 15:00	**Lunch** 13:30 – 15:00	**Lunch**	**Lunch** 13:30 – 15:00	**Lunch** 13:30 – 15:00
15:00 – 16:00 Gangl 1	15:00 – 16:00 Waldschmidt 4	**Free**	15:00 – 16:00 Gangl 4	15:00 – 16:00 Gangl 5
16:15 – 17:15 Gangl 2	16:15 – 17:15 Schneps 2		16:15 – 17:15 Schneps 4	16:15 – 17:15 Gangl 6

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An introduction to
Multiple Polylogarithms and their Symbology

by

Herbert Gangl

Abstract

The classical polylogarithm function, defined inside the unit disk by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \qquad s \geq 1,\ |z| < 1,$$

occurs in many different mathematical and physical contexts. On the mathematics side, they are e.g. particularly important in Number Theory where they play the role of so-called higher regulator functions, generalising the logarithm function as the classical Dirichlet regulator map (which is crucial when studying the units in a number ring). Apart from their definition as a sum, they can be analytically continued via an integral, and the two ways of expressing them assures a very rich algebraic structure.

On the physics side, they typically occur when evaluating Feynman integrals, e.g. as coefficients in perturbative expansions where one develops possibly divergent expressions with respect to a small parameter.

It turns out that it is profitable to study a larger class of functions, the *multiple* polylogarithms (note that by specialising all the arguments z_j to 1 one retrieves the *multiple zeta values* treated in M. Waldschmidt's course)

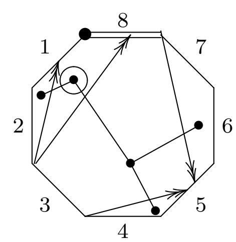
$$Li_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{z_1^{n_1}}{n_1^{s_1}} \cdots \frac{z_k^{n_k}}{n_k^{s_k}}, \qquad k \geq 1,\ s_j \geq 1$$

for $|z| < 1$ (they also called hyperlogarithms, or Goncharov polylogarithms, short "MPL's") and these suffice to express many more Feynman integral coefficients. Goncharov, Spradlin, Vergu and Volovich convincingly showed that

understanding their symmetry properties also helps for handling those Feynman integrals better, in particular they drastically simplified a certain breakthrough integral formula in particle physics (obtained originally by del Duca, Duhr and Smirnov). Moreover, it is useful to clarify the relationships—so-called functional equations—between different polylogarithmic expressions and, potentially even more rewarding, relations between different arguments for a *fixed* polylogarithm, like $Li_2(z) + Li_2(1-z) \equiv 0$ (modulo products of logarithms). A crucial tool in approaching such relations as well as the symmetries mentioned above is the notion of a *symbol* attached to a given MPL, due to Bloch (Li_2), to Zagier (Li_n), and to Goncharov for the general case.

The goal of these lectures is to introduce, in an elementary way, MPL's, to give a number of their basic properties and to explain in detail how to arrive—in different ways—at their associated symbols, using combinatorial notions like trees and polygons.

For an illustration of the type of objects that occur we depict a (decorated rooted oriented) polygon with a dissection into 5 parts, together with its inscribed dual tree.



Furthermore, we want to highlight and study functional equations—including some new ones—and indicate their role in a more conceptual framework (e.g. Zagier's Polylogarithm Conjecture).

References:

From polygons and symbols to polylogarithmic functions (Duhr, G., Rhodes)
The remarkable dilogarithm, by D. Zagier

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Multiple zeta values and their surprising connections to other mathematical problems

by

Leila Schneps

Abstract:

The multiple zeta values $\zeta(s_1, \ldots, s_k)$ form an algebra $\mathcal{Z}$ of real numbers over the rationals. They satisfy a double family of algebraic relations called the *double shuffle* relations. If we replace the real multiple zeta values by formal symbols $Z(s_1, \ldots, s_k)$ subject only to the double shuffle relations, we obtain a formal multizeta algebra $\mathcal{FZ}$ that surjects onto the $\mathcal{Z}$. It is a fundamental conjecture about multizeta values that the surjection is actually an isomorphism, i.e. that all algebraic relations between multiple zeta values lie in the ideal generated by the double shuffle relations.

Due to a construction by Goncharov, it is known that there is a third algebra, the *motivic multizeta algebra* $\mathcal{MZ}$, that fits between the other two, so that we have two surjections $\mathcal{FZ} \to \mathcal{MZ} \to \mathcal{Z}$.

In view of the main conjecture, it is natural to study the structure of $\mathcal{FZ}$. Although there are also many conjectures concerning $\mathcal{FZ}$, they are purely algebraic and combinatorial in nature, and therefore should be easier to prove; indeed, there has already been significant progress. In order to study $\mathcal{FZ}$ (resp. $\mathcal{MZ}$), we use the known fact that it is a Hopf algebra, so that its quotient modulo products is thus a Lie coalgebra, and we study the dual graded Lie algebra, the *double shuffle Lie algebra* **ds** (resp. the *motivic double shuffle Lie algebra* **mds**). The graded Lie algebra **ds** is easy to define, and the structure conjectures concerning $\mathcal{FZ}$ can then be rephrased for **ds**. We will explain the main structure conjectures and recent progress in the

direction of proving them. In particular, a major theorem due to Brown and Zagier shows that **mds** is a free Lie algebra with one generator in each odd weight, and the surjection $\mathcal{FZ} \to \mathcal{MZ}$ yields an injection $\mathbf{mds} \to \mathbf{ds}$ in the dual, thus giving lower bounds for the dimensions of the graded parts of **ds**.

What emerges in particular from the study of **ds** is that it is closely connected to several other Lie algebras that arise from very different problems, and have definitions that look very different. We will give the definitions and origins of the Grothendieck-Teichmüller Lie algebra and the Kashiwara-Vergne Lie algebra, and discuss the maps (conjectural isomorphisms) that exist between all three Lie algebras.

Reference:

An introduction to Grothendieck-Teichmüller theory (Lectures 2A,2B,4A,4B) http://www.math.jussieu.fr/̃leila/mit.html

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An elementary introduction to Multiple Zeta Values

by

Michel Waldschmidt

Abstract

One main open problem in transcendental number theory is to describe all algebraic relations among the values $\zeta(s)$ at the integers $s \geq 2$ of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}\cdot$$

L. Euler proved that the numbers $\zeta(2n)/\pi^{2n}$ are rational for $n \geq 1$. The expected answer to the above question is that the relations $\zeta(2n)/\pi^{2n} \in \mathbf{Q}$ generate the ideal of all algebraic relations. In other terms, the numbers

$$\zeta(3),\ \zeta(5),\ \ldots,\ \zeta(2n+1),\ \ldots$$

are expected to be algebraically independent over the field $\mathbf{Q}(\pi)$. So far, very few results are known in this direction.

This problem of algebraic independence can be seen as a special case of a problem of $\mathbf{Q}$–linear independence, by introducting the Multiple Zeta Values (MZV)

$$\zeta(s_1,\ldots,s_k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}},$$

where $k \geq 1$ and $s_j \geq 1$ $(1 \leq j \leq k-1)$, $s_k \geq 2$ are again integers. Indeed, the product of two MZVs is a $\mathbf{Q}$–linear combination of MZVs, and often there are several such linear combinations giving the same value. Hence

these numbers satisfy many **Q**–linear relations, giving rise to rich algebraic structures. The goal of these lectures is to introduce in a elementary way these algebraic structures, with the *shuffle* and *stuffle* products, as well as the *regularized double shuffle relations*.

One central theme will be a conjecture due to D. Zagier, which predicts the value of the dimension d_p of the **Q**–space spanned by the $\zeta(s_1, \dots, s_k)$ for $s_1 + \cdots + s_k = p$ and $p \geq 1$. The value of d_p given by Zagier is actually an upper bound for that dimension: this amounts to check that there are sufficiently many **Q**–linear relations among these numbers, and it looks like a combinatoric result. But the proofs which are known so far (due to Goncharov, Terasoma and Brown) involve heavy machinery from algebraic geometry. The proof by F. Brown yields a generating set for this space, with the expected number of generators: this is the set of $\zeta(s_1, \dots, s_k)$ for $s_1 + \cdots + s_k = p$ and $s_i \in \{2, 3\}$ $(i = 1, \dots, k)$. The proof of Brown's result requests a lemma due to D. Zagier, the proof of which we plan to discuss, together with its variant by Zhonghua Li.

No preliminary background is required; some of the results, which are easy to state, have a proof which requires deep tools far above the level of this course: in such cases, we will only discuss the statements, not the proofs.

Reference:

A course on multizeta values (51 pages).
http://www.math.jussieu.fr/∼miw/articles/pdf/MZV2011IMSc.pdf

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