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An elementary introduction to Multiple Zeta Values

by

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Abstract

One main open problem in transcendental number theory is to describe all algebraic relations among the values $\zeta(s)$ at the integers $s \geq 2$ of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}\cdot$$

L. Euler proved that the numbers $\zeta(2n)/\pi^{2n}$ are rational for $n \geq 1$. The expected answer to the above question is that the relations $\zeta(2n)/\pi^{2n} \in \mathbf{Q}$ generate the ideal of all algebraic relations. In other terms, the numbers

$$\zeta(3),\ \zeta(5),\ \dots,\ \zeta(2n+1),\ \dots$$

are expected to be algebraically independent over the field $\mathbf{Q}(\pi)$. So far, very few results are known in this direction.

This problem of algebraic independence can be seen as a special case of a problem of $\mathbf{Q}$–linear independence, by introducting the Multiple Zeta Values (MZV)

$$\zeta(s_1,\dots,s_k) = \sum_{n_1>n_2>\dots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}},$$

where $k \geq 1$ and $s_j \geq 1$ $(1 \leq j \leq k-1)$, $s_k \geq 2$ are again integers. Indeed, the product of two MZVs is a $\mathbf{Q}$–linear combination of MZVs, and often there are several such linear combinations giving the same value. Hence

these numbers satisfy many $\mathbf{Q}$–linear relations, giving rise to rich algebraic structures. The goal of these lectures is to introduce in a elementary way these algebraic structures, with the *shuffle* and *stuffle* products, as well as the *regularized double shuffle relations*.

One central theme will be a conjecture due to D. Zagier, which predicts the value of the dimension d_p of the $\mathbf{Q}$–space spanned by the $\zeta(s_1,\dots,s_k)$ for $s_1+\cdots+s_k=p$ and $p\geq 1$. The value of d_p given by Zagier is actually an upper bound for that dimension: this amounts to check that there are sufficiently many $\mathbf{Q}$–linear relations among these numbers, and it looks like a combinatoric result. But the proofs which are known so far (due to Goncharov, Terasoma and Brown) involve heavy machinery from algebraic geometry. The proof by F. Brown yields a generating set for this space, with the expected number of generators: this is the set of $\zeta(s_1,\dots,s_k)$ for $s_1+\cdots+s_k=p$ and $s_i\in\{2,3\}$ $(i=1,\dots,k)$. The proof of Brown's result requests a lemma due to D. Zagier, the proof of which we plan to discuss, together with its variant by Zhonghua Li.

No preliminary background is required; some of the results, which are easy to state, have a proof which requires deep tools far above the level of this course: in such cases, we will only discuss the statements, not the proofs.

Reference:

A course on multizeta values (51 pages).
http://www.math.jussieu.fr/∼miw/articles/pdf/MZV2011IMSc.pdf

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