An elementary introduction to
Multiple Zeta Values

by

Michel Waldschmidt

Abstract

One main open problem in transcendental number theory is to describe all algebraic relations among the values $\zeta(s)$ at the integers $s \geq 2$ of the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$ 

L. Euler proved that the numbers $\zeta(2n)/\pi^{2n}$ are rational for $n \geq 1$. The expected answer to the above question is that the relations $\zeta(2n)/\pi^{2n} \in \mathbb{Q}$ generate the ideal of all algebraic relations. In other terms, the numbers

$$\zeta(3), \zeta(5), \ldots, \zeta(2n+1), \ldots$$

are expected to be algebraically independent over the field $\mathbb{Q}(\pi)$. So far, very few results are known in this direction.

This problem of algebraic independence can be seen as a special case of a problem of $\mathbb{Q}$–linear independence, by introducing the Multiple Zeta Values (MZV)

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

where $k \geq 1$ and $s_j \geq 1$ ($1 \leq j \leq k-1$), $s_k \geq 2$ are again integers. Indeed, the product of two MZVs is a $\mathbb{Q}$–linear combination of MZVs, and often there are several such linear combinations giving the same value. Hence
these numbers satisfy many \( \mathbb{Q} \)-linear relations, giving rise to rich algebraic structures. The goal of these lectures is to introduce in a elementary way these algebraic structures, with the shuffle and stuffle products, as well as the regularized double shuffle relations.

One central theme will be a conjecture due to D. Zagier, which predicts the value of the dimension \( d_p \) of the \( \mathbb{Q} \)-space spanned by the \( \zeta(s_1, \ldots, s_k) \) for \( s_1 + \cdots + s_k = p \) and \( p \geq 1 \). The value of \( d_p \) given by Zagier is actually an upper bound for that dimension: this amounts to check that there are sufficiently many \( \mathbb{Q} \)-linear relations among these numbers, and it looks like a combinatoric result. But the proofs which are known so far (due to Goncharov, Terasoma and Brown) involve heavy machinery from algebraic geometry. The proof by F. Brown yields a generating set for this space, with the expected number of generators: this is the set of \( \zeta(s_1, \ldots, s_k) \) for \( s_1 + \cdots + s_k = p \) and \( s_i \in \{2, 3\} \) (\( i = 1, \ldots, k \)). The proof of Brown’s result requests a lemma due to D. Zagier, the proof of which we plan to discuss, together with its variant by Zhonghua Li.

No preliminary background is required; some of the results, which are easy to state, have a proof which requires deep tools far above the level of this course: in such cases, we will only discuss the statements, not the proofs.

Reference:
A course on multizeta values (51 pages).

Michel WALDSCHMIDT
Université Pierre et Marie Curie-Paris 6
Institut de Mathématiques de Jussieu IMJ UMR 7586
Théorie des Nombres Case Courrier 247
4 Place Jussieu
F–75252 Paris Cedex 05 France
e-mail: miw@math.jussieu.fr
URL: http://www.math.jussieu.fr/~miw/