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Multiple zeta values and their surprising connections to other mathematical problems

by
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Abstract:

The multiple zeta values $\zeta(s_1, \ldots, s_k)$ form an algebra $\mathcal{Z}$ of real numbers over the rationals. They satisfy a double family of algebraic relations called the *double shuffle* relations. If we replace the real multiple zeta values by formal symbols $Z(s_1, \ldots, s_k)$ subject only to the double shuffle relations, we obtain a formal multizeta algebra $\mathcal{FZ}$ that surjects onto the $\mathcal{Z}$. It is a fundamental conjecture about multizeta values that the surjection is actually an isomorphism, i.e. that all algebraic relations between multiple zeta values lie in the ideal generated by the double shuffle relations.

Due to a construction by Goncharov, it is known that there is a third algebra, the *motivic multizeta algebra* $\mathcal{MZ}$, that fits between the other two, so that we have two surjections $\mathcal{FZ} \to \mathcal{MZ} \to \mathcal{Z}$.

In view of the main conjecture, it is natural to study the structure of $\mathcal{FZ}$. Although there are also many conjectures concerning $\mathcal{FZ}$, they are purely algebraic and combinatorial in nature, and therefore should be easier to prove; indeed, there has already been significant progress. In order to study $\mathcal{FZ}$ (resp. $\mathcal{MZ}$), we use the known fact that it is a Hopf algebra, so that its quotient modulo products is thus a Lie coalgebra, and we study the dual graded Lie algebra, the *double shuffle Lie algebra* **ds** (resp. the *motivic double shuffle Lie algebra* **mds**). The graded Lie algebra **ds** is easy to define, and the structure conjectures concerning $\mathcal{FZ}$ can then be rephrased for **ds**. We will explain the main structure conjectures and recent progress in the

direction of proving them. In particular, a major theorem due to Brown and Zagier shows that **mds** is a free Lie algebra with one generator in each odd weight, and the surjection $\mathcal{FZ} \to \mathcal{MZ}$ yields an injection $\mathbf{mds} \to \mathbf{ds}$ in the dual, thus giving lower bounds for the dimensions of the graded parts of **ds**.

What emerges in particular from the study of **ds** is that it is closely connected to several other Lie algebras that arise from very different problems, and have definitions that look very different. We will give the definitions and origins of the Grothendieck-Teichmüller Lie algebra and the Kashiwara-Vergne Lie algebra, and discuss the maps (conjectural isomorphisms) that exist between all three Lie algebras.

Reference:

An introduction to Grothendieck-Teichmüller theory (Lectures 2A,2B,4A,4B) http://www.math.jussieu.fr/˜leila/mit.html

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