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An introduction to
Multiple Polylogarithms and their Symbology

by

Herbert Gangl

Abstract

The classical polylogarithm function, defined inside the unit disk by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}\,, \qquad s \geq 1,\ |z| < 1\,,$$

occurs in many different mathematical and physical contexts. On the mathematics side, they are e.g. particularly important in Number Theory where they play the role of so-called higher regulator functions, generalising the logarithm function as the classical Dirichlet regulator map (which is crucial when studying the units in a number ring). Apart from their definition as a sum, they can be analytically continued via an integral, and the two ways of expressing them assures a very rich algebraic structure.

On the physics side, they typically occur when evaluating Feynman integrals, e.g. as coefficients in perturbative expansions where one develops possibly divergent expressions with respect to a small parameter.

It turns out that it is profitable to study a larger class of functions, the *multiple* polylogarithms (note that by specialising all the arguments z_j to 1 one retrieves the *multiple zeta values* treated in M. Waldschmidt's course)

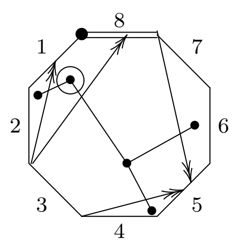
$$Li_{s_1,\dots,s_k}(z_1,\dots,z_k) = \sum_{n_1>n_2>\dots>n_k\geq 1} \frac{z_1^{n_1}}{n_1^{s_1}} \cdots \frac{z_k^{n_k}}{n_k^{s_k}}\,, \qquad k \geq 1,\ s_j \geq 1$$

for $|z| < 1$ (they also called hyperlogarithms, or Goncharov polylogarithms, short "MPL's") and these suffice to express many more Feynman integral coefficients. Goncharov, Spradlin, Vergu and Volovich convincingly showed that

understanding their symmetry properties also helps for handling those Feynman integrals better, in particular they drastically simplified a certain breakthrough integral formula in particle physics (obtained originally by del Duca, Duhr and Smirnov). Moreover, it is useful to clarify the relationships—so-called functional equations—between different polylogarithmic expressions and, potentially even more rewarding, relations between different arguments for a *fixed* polylogarithm, like $Li_2(z) + Li_2(1-z) \equiv 0$ (modulo products of logarithms). A crucial tool in approaching such relations as well as the symmetries mentioned above is the notion of a *symbol* attached to a given MPL, due to Bloch (Li_2), to Zagier (Li_n), and to Goncharov for the general case.

The goal of these lectures is to introduce, in an elementary way, MPL's, to give a number of their basic properties and to explain in detail how to arrive—in different ways—at their associated symbols, using combinatorial notions like trees and polygons.

For an illustration of the type of objects that occur we depict a (decorated rooted oriented) polygon with a dissection into 5 parts, together with its inscribed dual tree.



Furthermore, we want to highlight and study functional equations—including some new ones—and indicate their role in a more conceptual framework (e.g. Zagier's Polylogarithm Conjecture).

References:

From polygons and symbols to polylogarithmic functions (Duhr, G., Rhodes)
The remarkable dilogarithm, by D. Zagier

Herbert GANGL
Department of Mathematical Sciences, South Road
Durham DH1 3LE, United Kingdom
e-mail: herbert.gangl@durham.ac.uk
URL: http://www.maths.dur.ac.uk/~dma0hg/myres.html/