Some Open Problems in Fourier Analysis

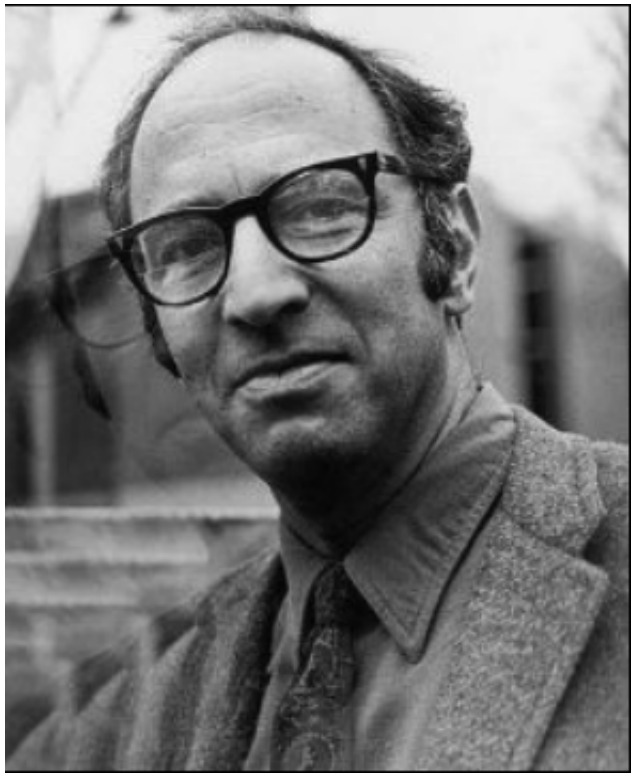
Stefan Steinerberger

El Escorial 2025



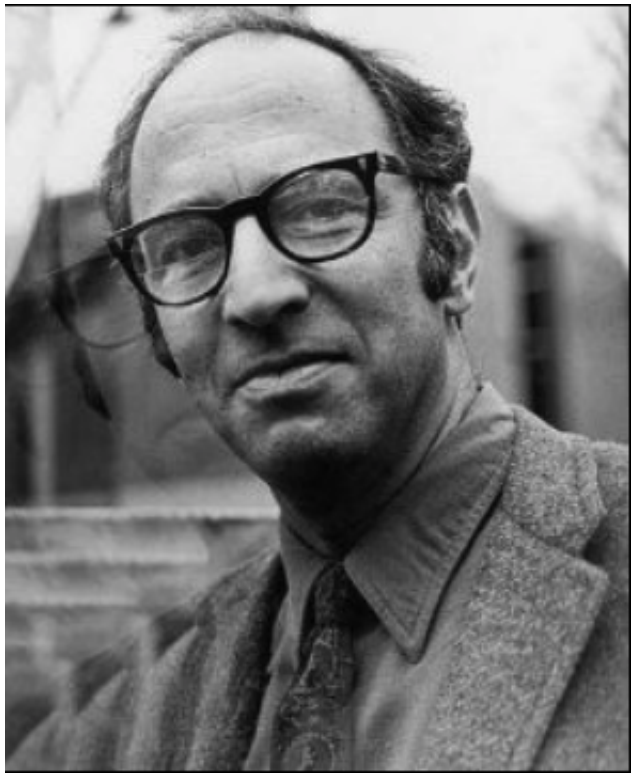
El Escorial 2012





Thomas Kuhn

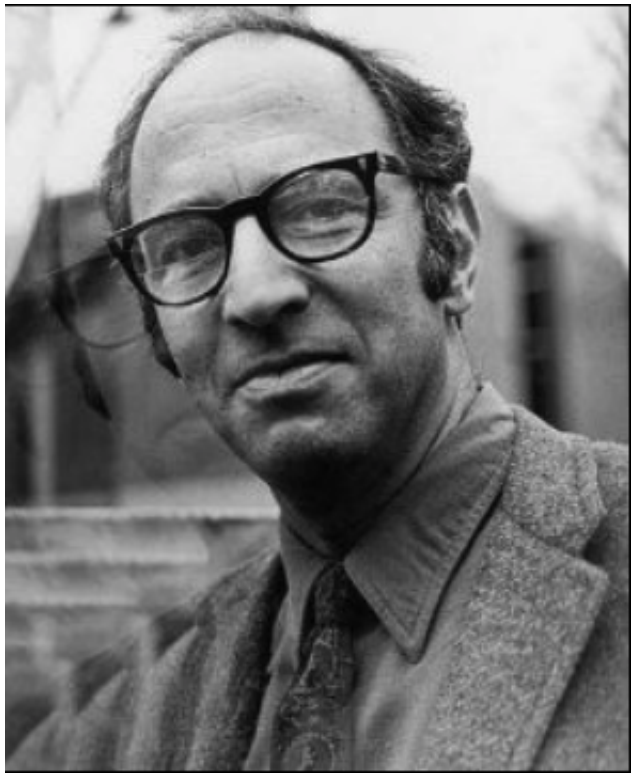
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The point of this talk is to mention some open problems and some partial results.



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The point of this talk is to mention some open problems and some partial results.

The hope is that these are not (all) insanely hard, that they are not very well known and *that progress is possible*.

The first Erdős problem

Well known but not in Fourier Analysis.

[1] *perhaps my first serious conjecture which goes back to 1931 or 32*

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Examples with $a_n \leq 0.23 \cdot 2^n$ exist (Conway, Guy '68, Bohman '98).

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Reformulation (Elkies, 1986)

For any $a_1, \ldots, a_n \in \mathbb{N}$

$$\int_0^1 \prod_{i=1}^{n} \cos(2\pi a_i x)^2 dx \geq \frac{1}{2^n}.$$

Does equality force $a_n \gtrsim 2^n$?

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Does equality force $a_n \gtrsim 2^n$? **Does it force arithmetic structure on the extremal set $a_1, a_2, \ldots, a_n$?**

Littlewood's Cosine Problem

Littlewood (1950s)

Given $A \subset \mathbb{N}$, consider $f : [0, 2\pi] \to \mathbb{R}$ given by

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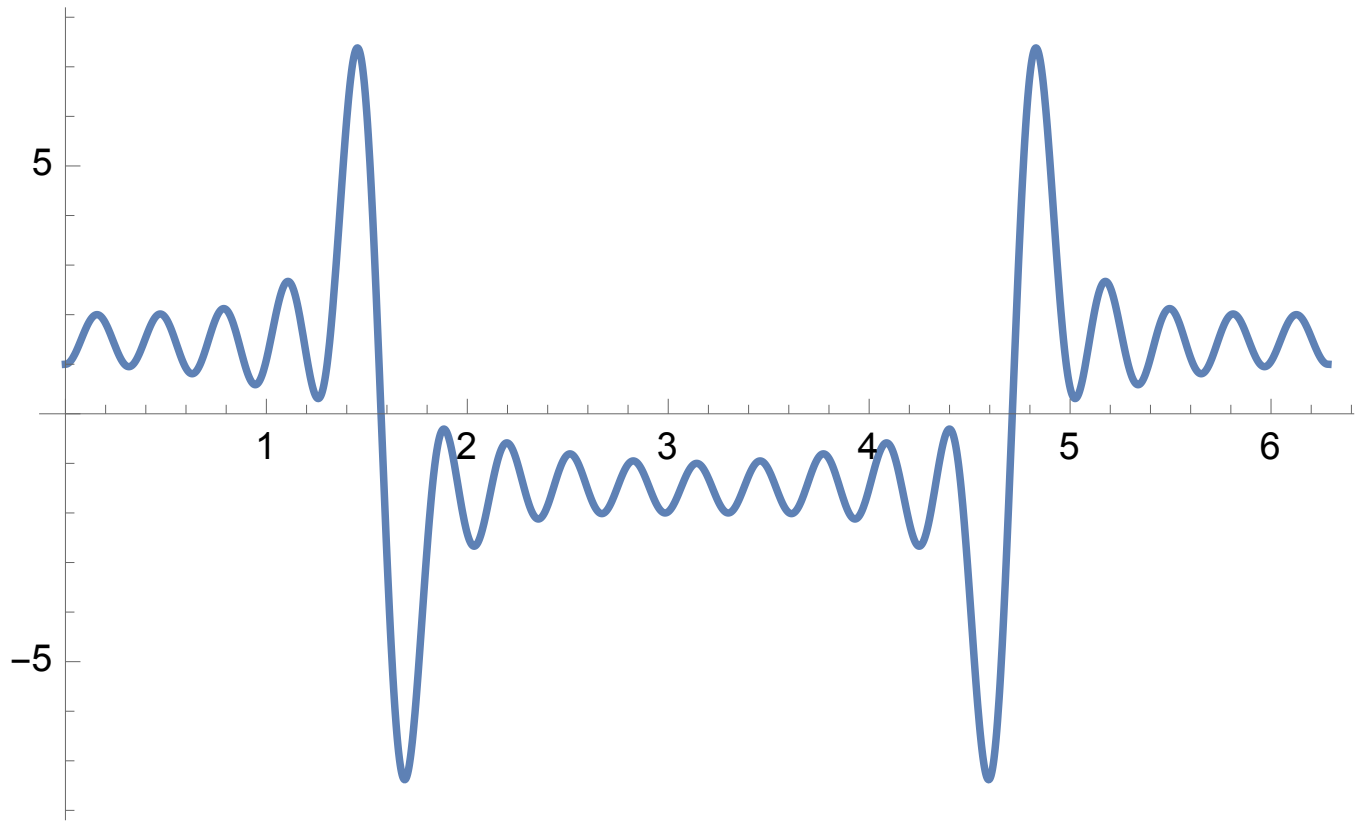
Warning (Sahasrabudhe), the function

$$f(x) = 2\cos(x) + \sum_{k=2}^{2n} \sin\left(\frac{k\pi}{2}\right) \cos(kx)$$

has all coefficients in $\{-1, 0, 1\}$ (and one 2) and only 2 roots.

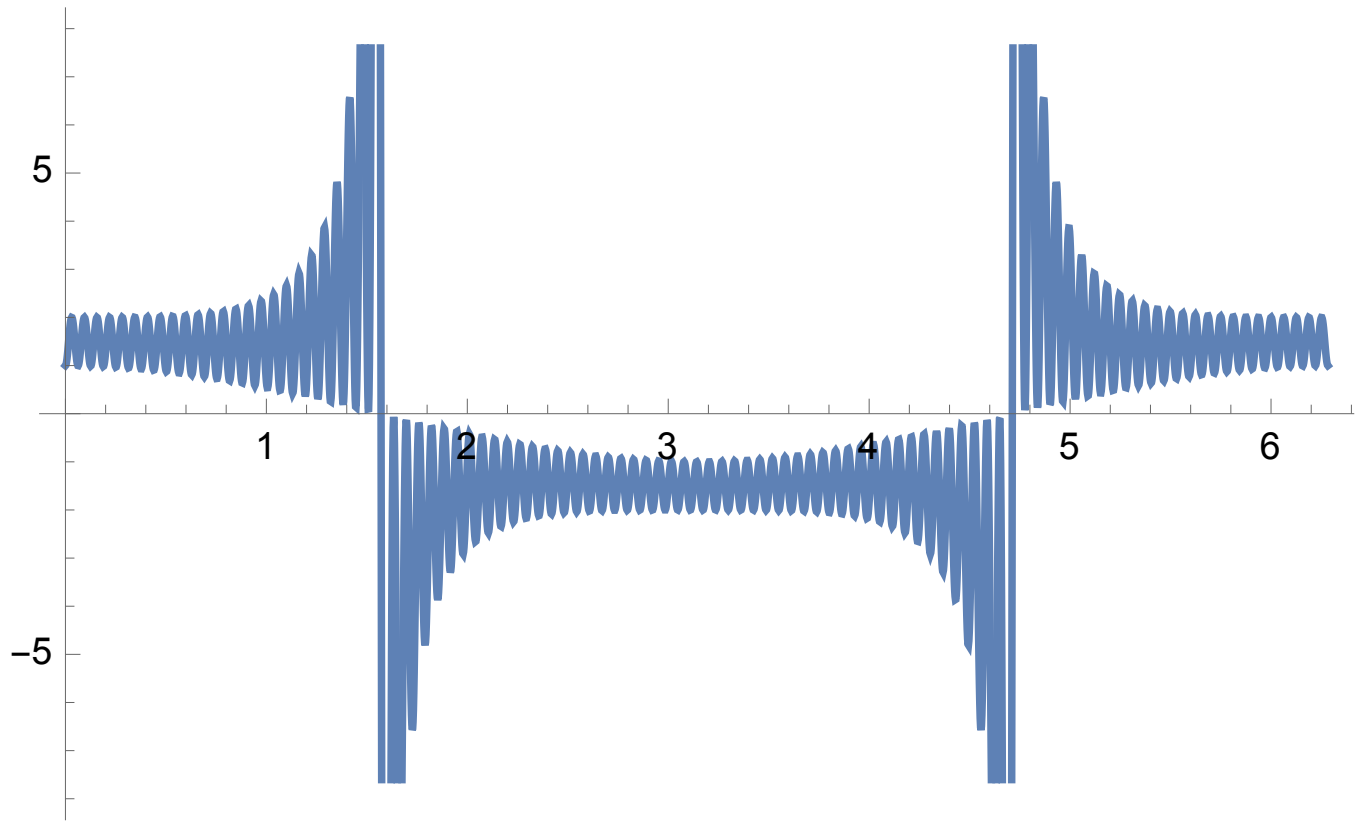
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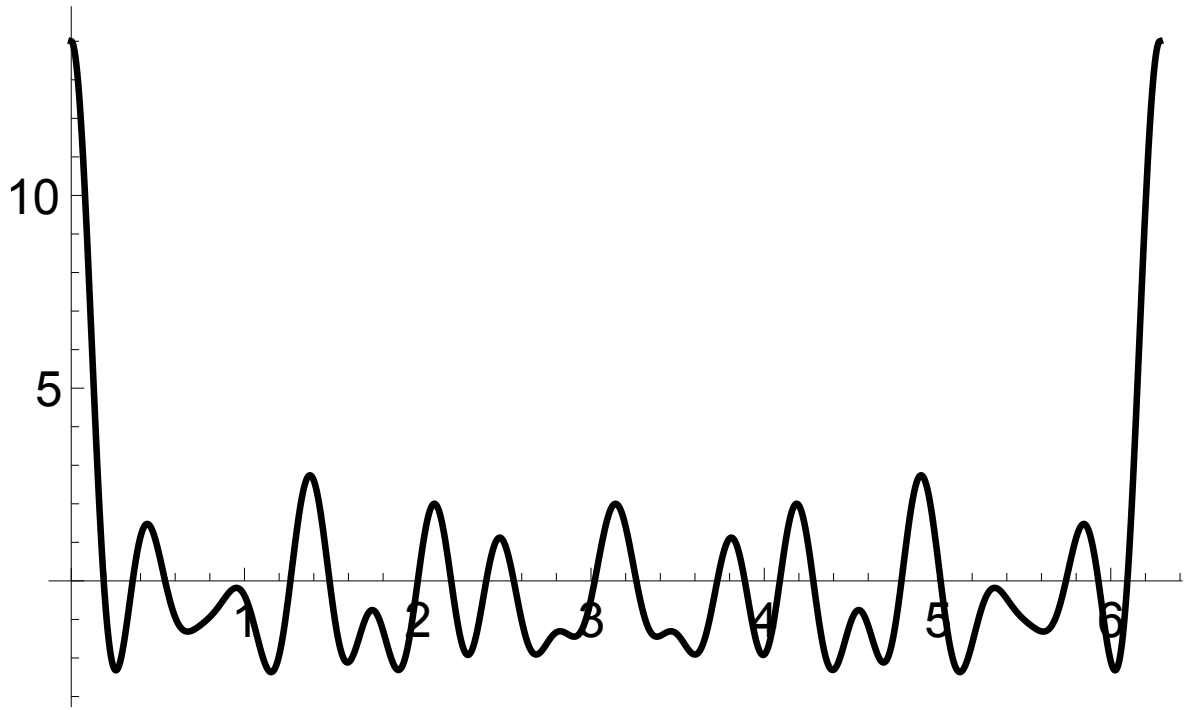
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Example. $A = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 18\}$



Bourgain, Ruzsa, Borwein: $\min \leq -c \cdot e^{c \cdot \sqrt{\log n}}$

Hardy-Littlewood Maximal Function

Let $f \in L^1(\mathbb{R})$. Everybody knows the **Hardy-Littlewood Maximal Function**

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What are the **maximal radii**? What can be said?

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Here is an amazing and somewhat confusing example

$$\frac{1}{2r}\int_{x-r}^{x+r}\sin(y)dy = \frac{\sin(r)}{r}\sin(x).$$

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There are only two maximal radii that show up!

Theorem (S, Studia Mathematica, 2014)

Suppose $f \in C^{\alpha}(\mathbb{R})$ is periodic and $\alpha > 1/2$. If the function A_x given by

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Weird and somewhat unexpected ingredient, fixed points of $\tan(x)$

$$\{x \in \mathbb{R}_{>0} : \tan(x) = x\} \quad \text{are linearly independent over } \mathbb{Q}.$$

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1. F. Temur, Level set estimates for the discrete frequency function, JFAA 2018
2. F. Temur, The frequency function and its connections to the Lebesgue points and the Hardy-Littlewood maximal function, Turkish Journal of Mathematics 2019

Sums of two squares

Integers that can be written as the sum of two squares

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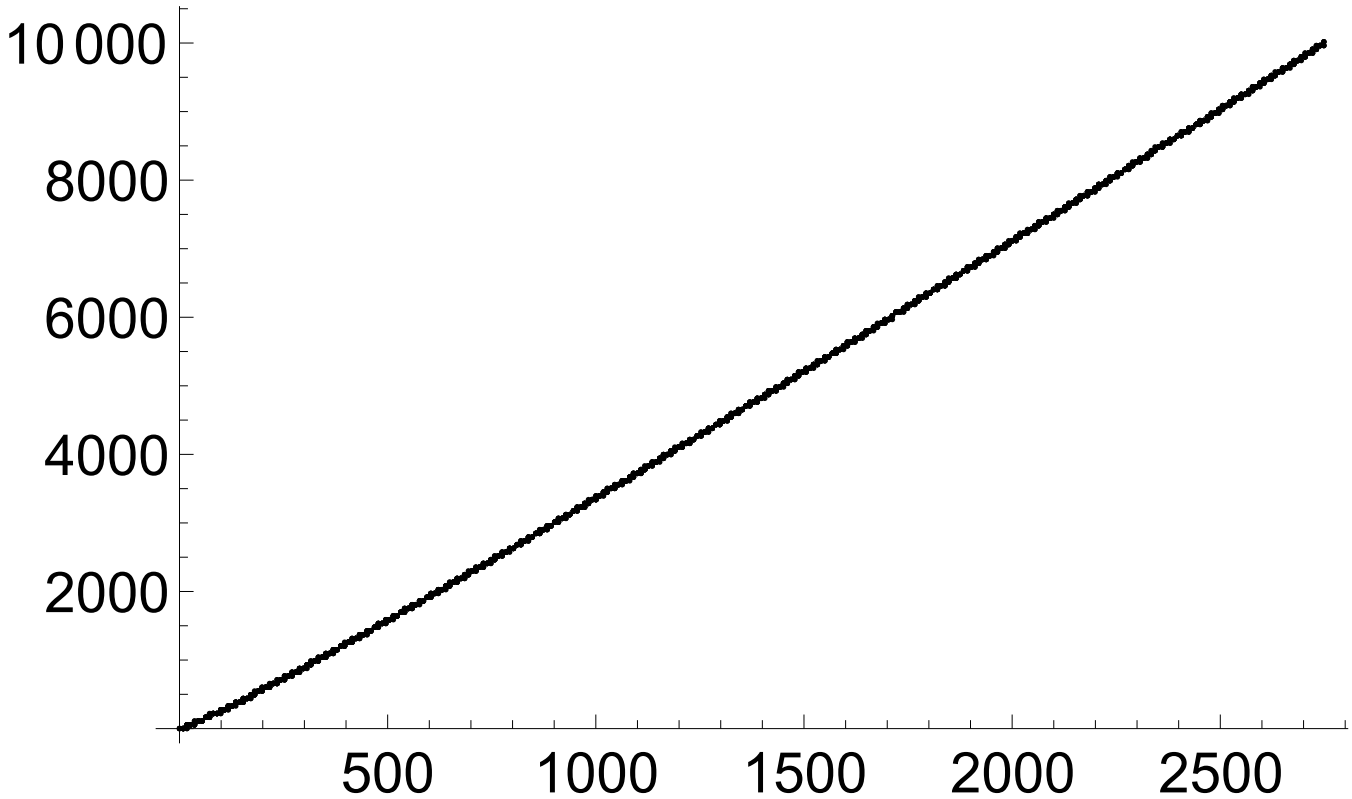
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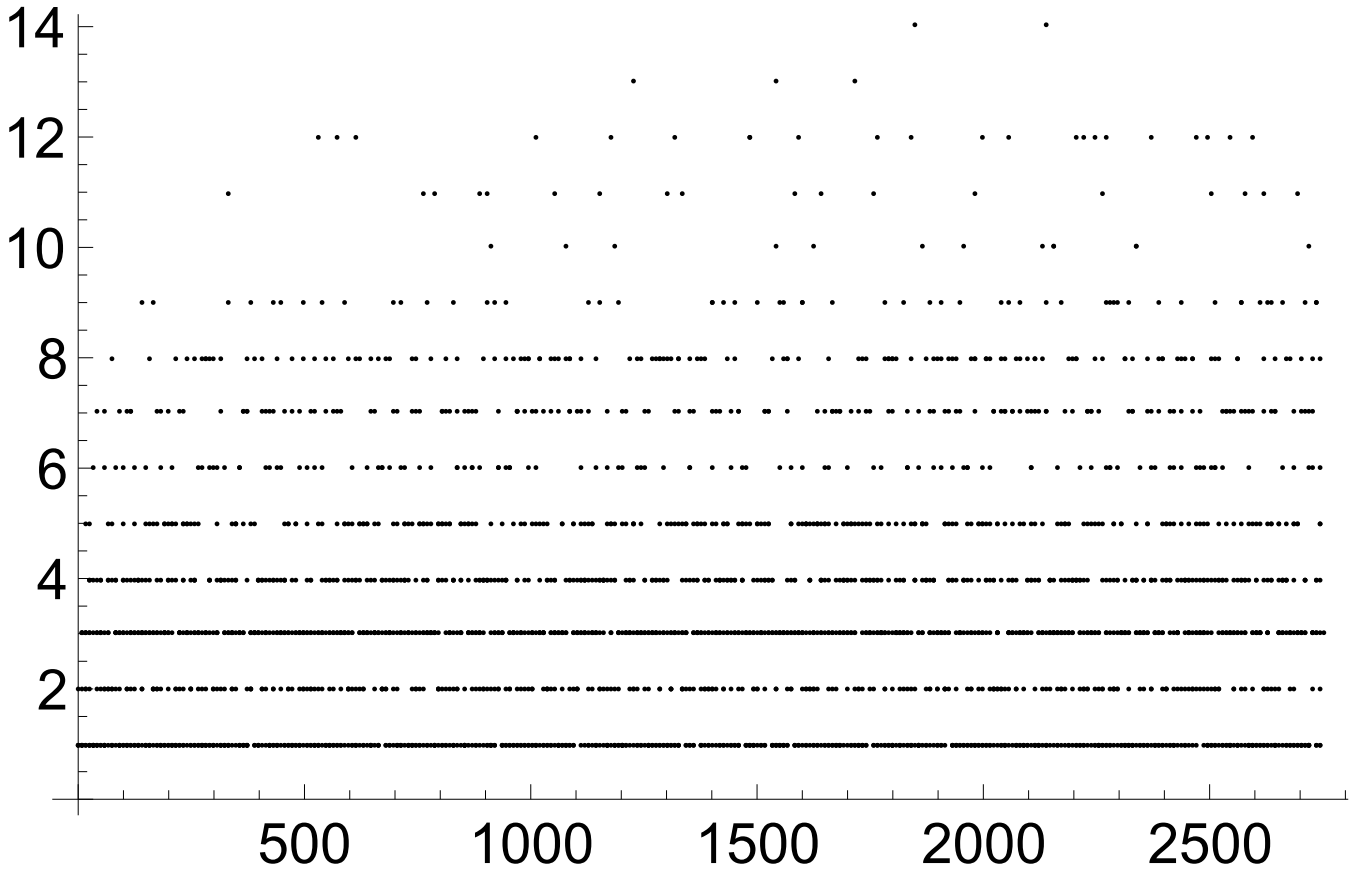
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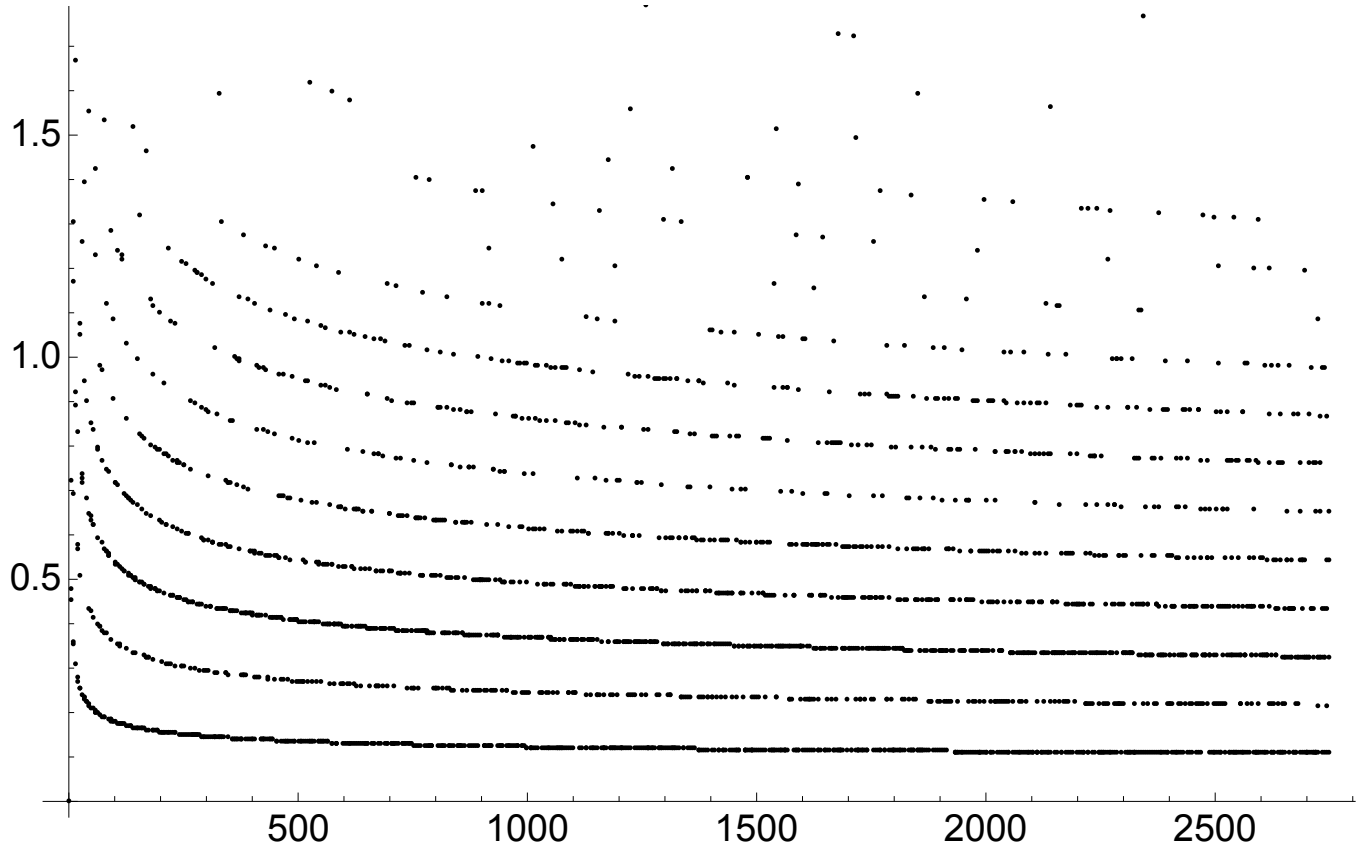
The sequence of sums of two squares (a_n)



The sequence of gaps $(a_{n+1} - a_n)$



The sequence of rescaled gaps $(a_{n+1} - a_n)/\log a_n$





Ram Bambah
(1925 – **26 May 2025**)

Theorem (Bambah-Chowla, '47)

Every interval

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$$n \leq \lfloor \sqrt{n} \rfloor^2 + \left\lceil \sqrt{n - \lfloor \sqrt{n} \rfloor^2} \right\rceil^2 < n + \sqrt{8}n^{1/4} + 1.$$



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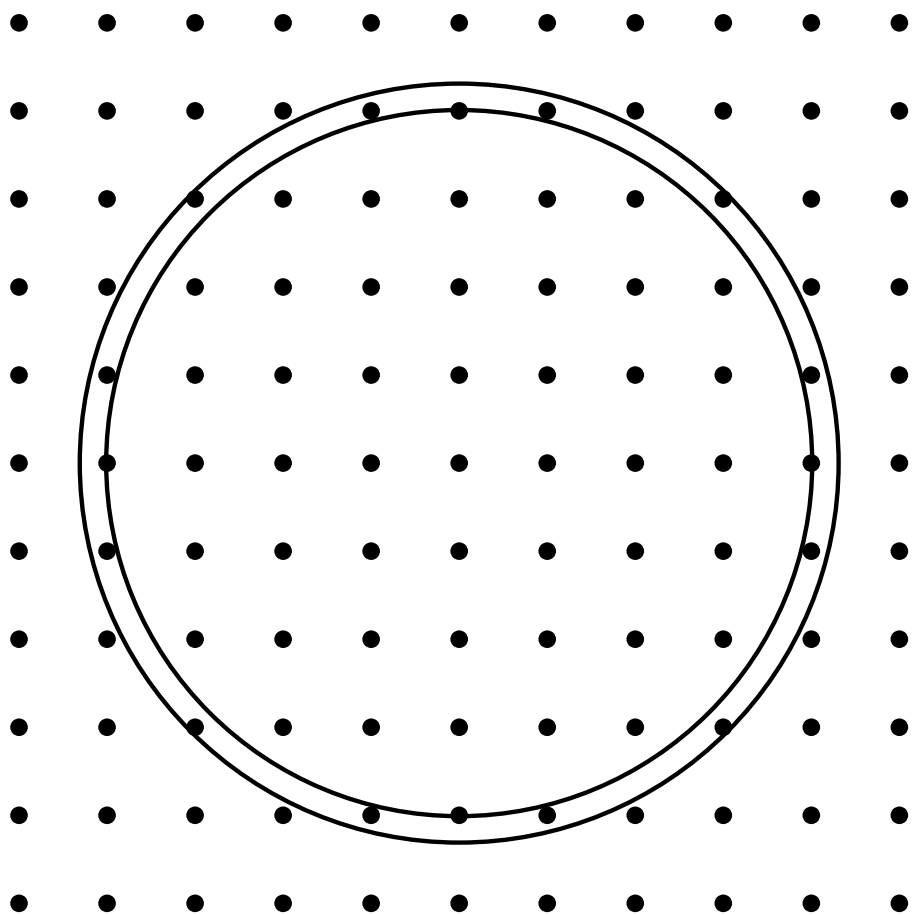
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Never been improved, not even the $\sqrt{8}$.



Every ‘wide’ annulus contains lattice points. Lattice points and circles, the Poisson Summation Formula....

Cosine Sign Correlation

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Motivation: sign correlation of different eigenfunction of Schrödinger operators, spooky action at a distance and so on – but also intrinsically interesting.

Cosine Sign Correlation

Lemma (Gonçalves, Oliveira e Silva, S, J. Spectral Theory '19)

For any integers $a < b$, we have

$$\frac{1}{2\pi} |\{x \in [0, 2\pi] : \text{sign}(\cos(ax)) = \text{sign}(\cos(bx))\}| \geq \frac{1}{3}$$

with equality if and only if $b = 3a$.

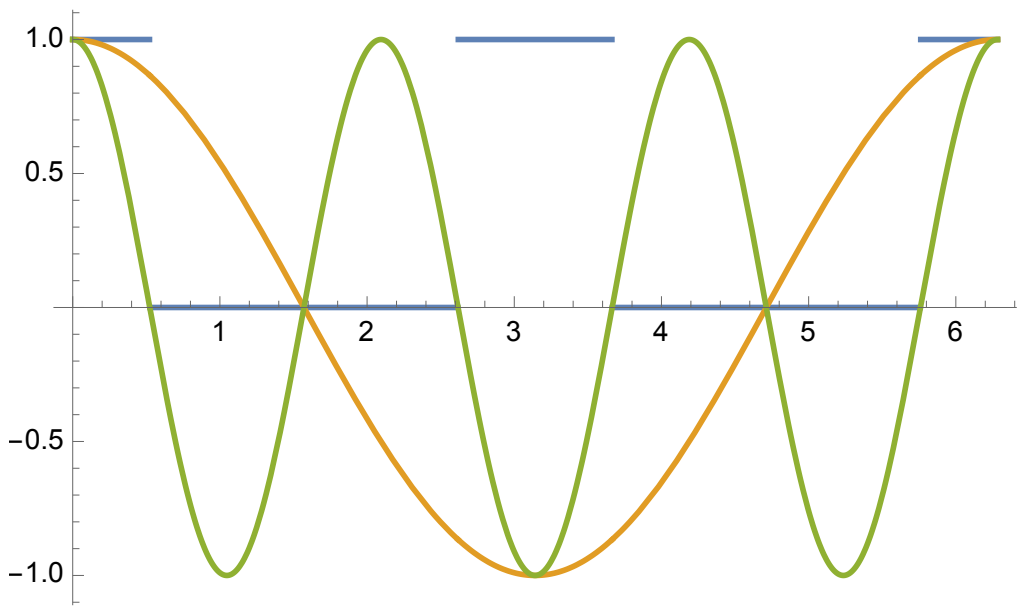
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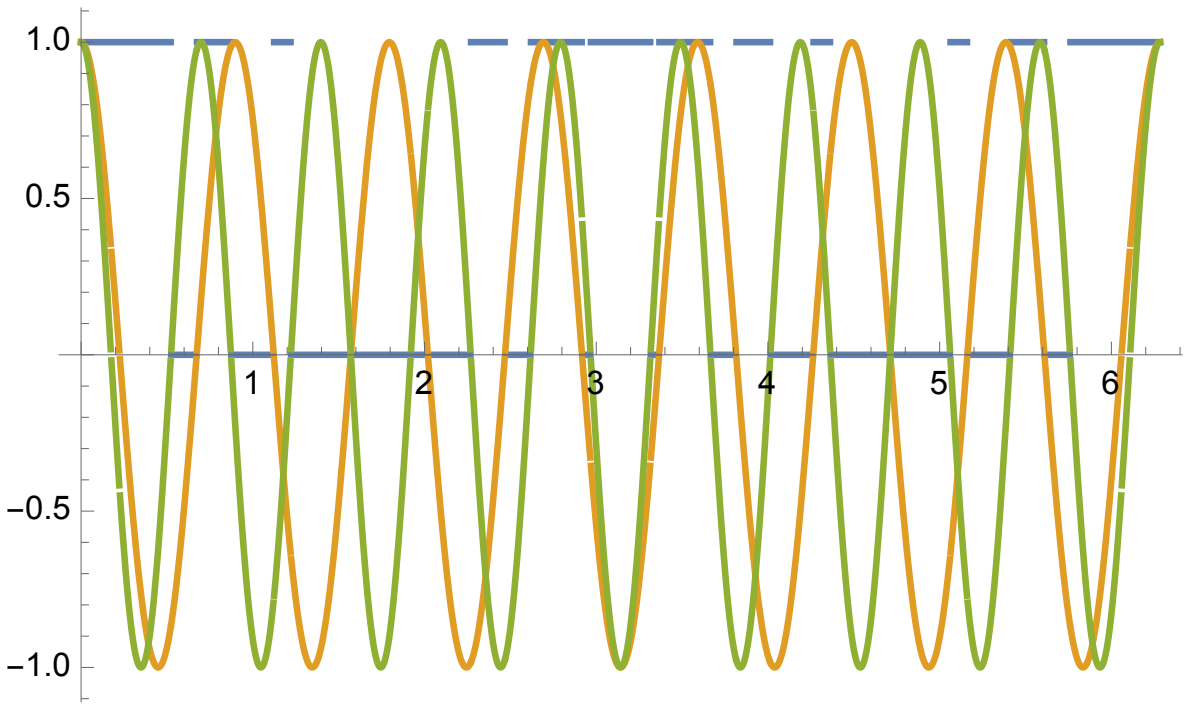
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Big Question

Given $a_1, a_2, \ldots, a_n \in \mathbb{N}$ and consider the functions

$$\cos(a_1 x), \cos(a_2 x), \ldots, \cos(a_n x) \quad \text{on } [0, 2\pi].$$

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Consider set $S \subseteq [0, 2\pi]$ where all n functions have the same sign. The set S is non-empty (because $\cos(0) = 1$). How small can it be (as a function of n)?

Let us denote the answer by $F(n)$. We know $F(2) = 1/3$.

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Theorem (Dou, Goh, Liu, Legate, Pettigrew, JFAA 2022)

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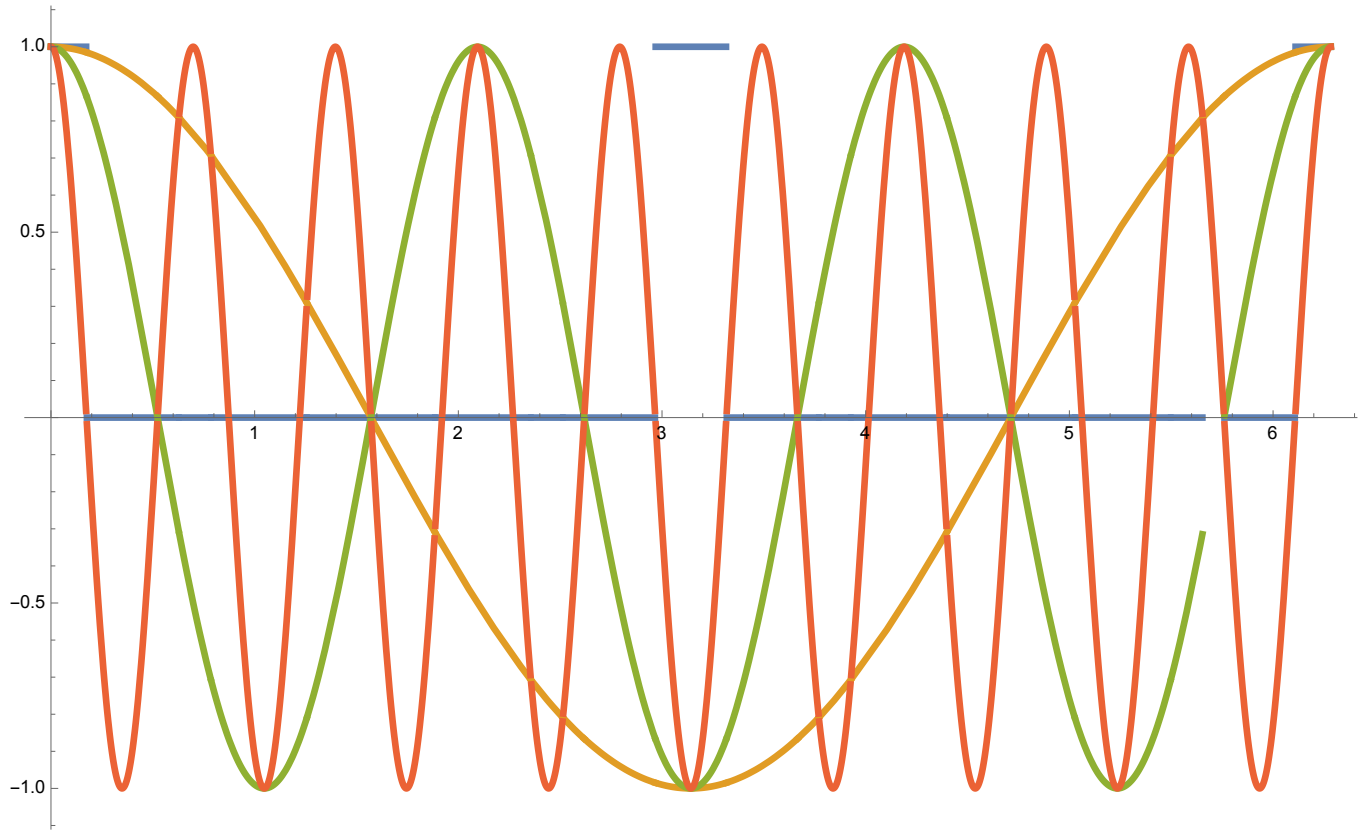
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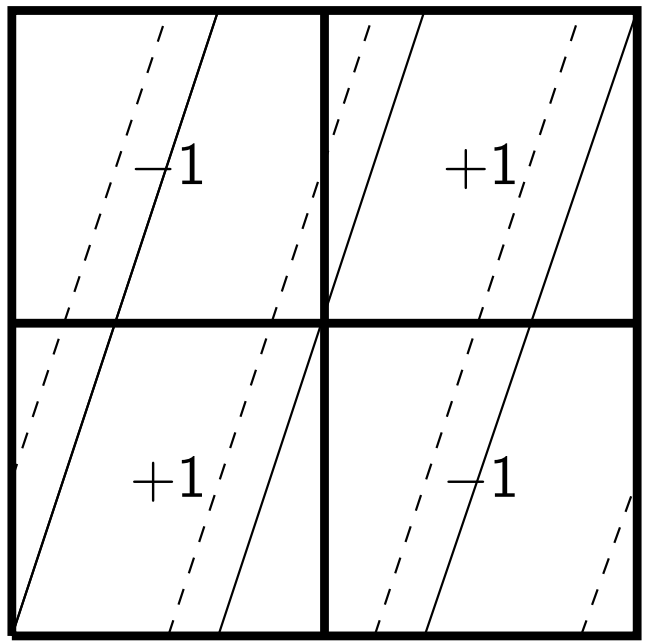
by taking $1, 3, 11, 35, 105$. **Is** $F(n) \geq C^{-n}$? **Probably?**

A Maximal Geodesic Problem

If we try to understand the behavior of signs of $\sin(ax)$ and $\sin(bx)$, then one way to do is to integrate a function attaining the values ± 1 and integrate that over a closed geodesic.

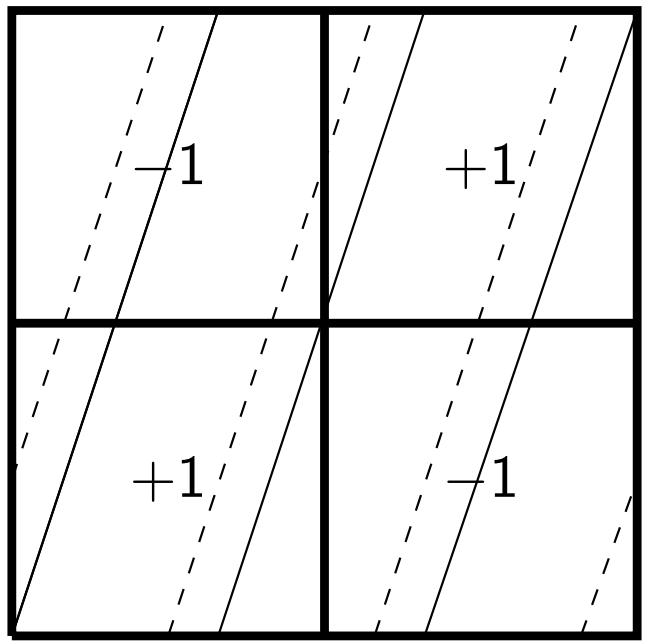
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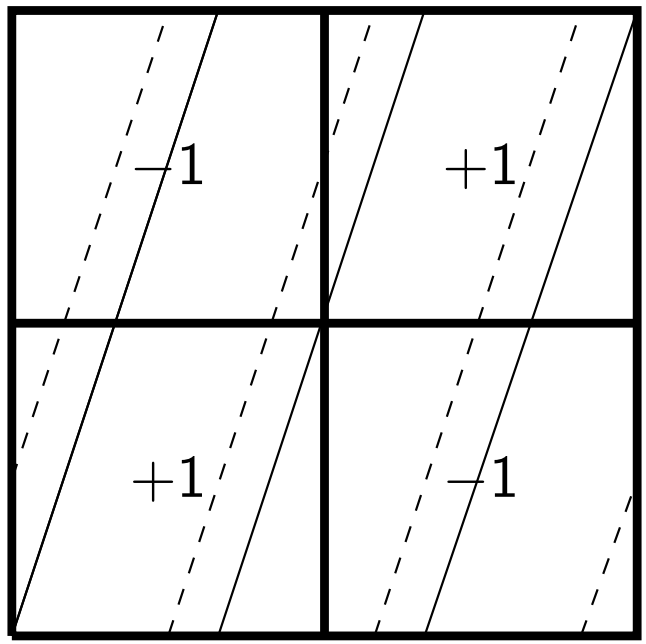
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Simple Idea. If the geodesic is very long, then it's probably also fairly uniformly distributed and the average value of the function is probably pretty close to the mean value. Extremal geodesics corresponding to extreme values **have to be short**.

Theorem (S, Bull. Aust. Math. Soc, 2018)

Let $f : \mathbb{T}^2 \to \mathbb{R}$ be at least $s \geq 2$ times differentiable and have mean value 0. Then

$$\sup_{\gamma \text{ closed geodesic}} \frac{1}{|\gamma|} \left| \int_\gamma f \ d\mathcal{H}^1 \right|,$$

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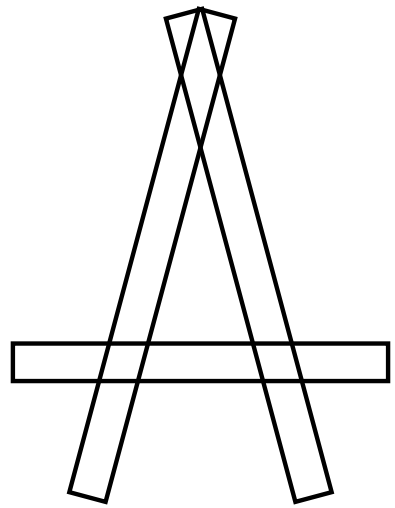
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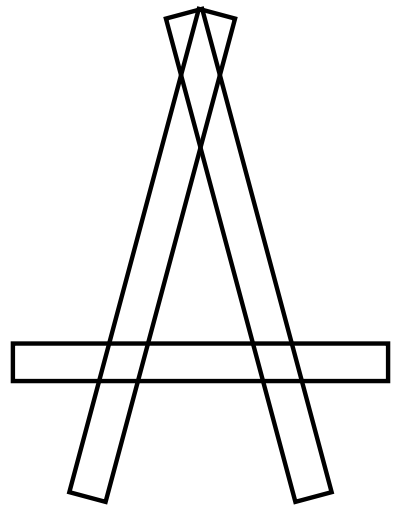
- **Open problem.** Higher dimensions. Groups?

Kakeya on the Sphere



Here's a cheap Kakeya-type statement in $\mathbb{R}^2$.

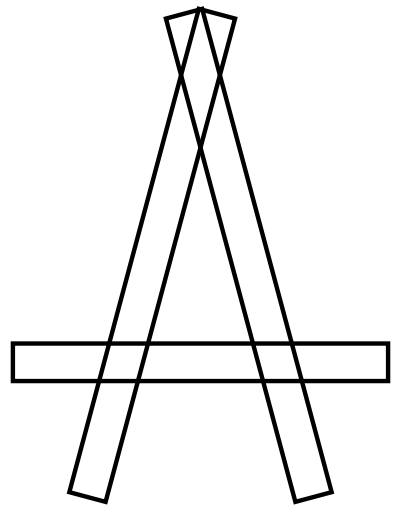
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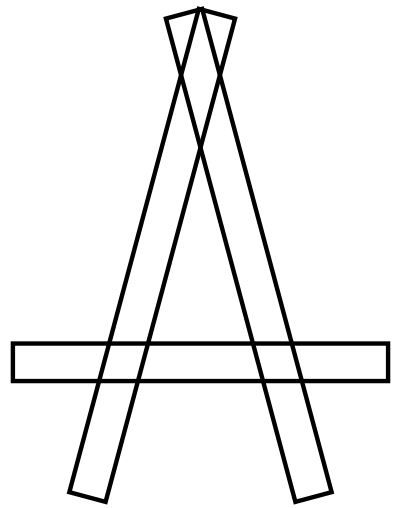


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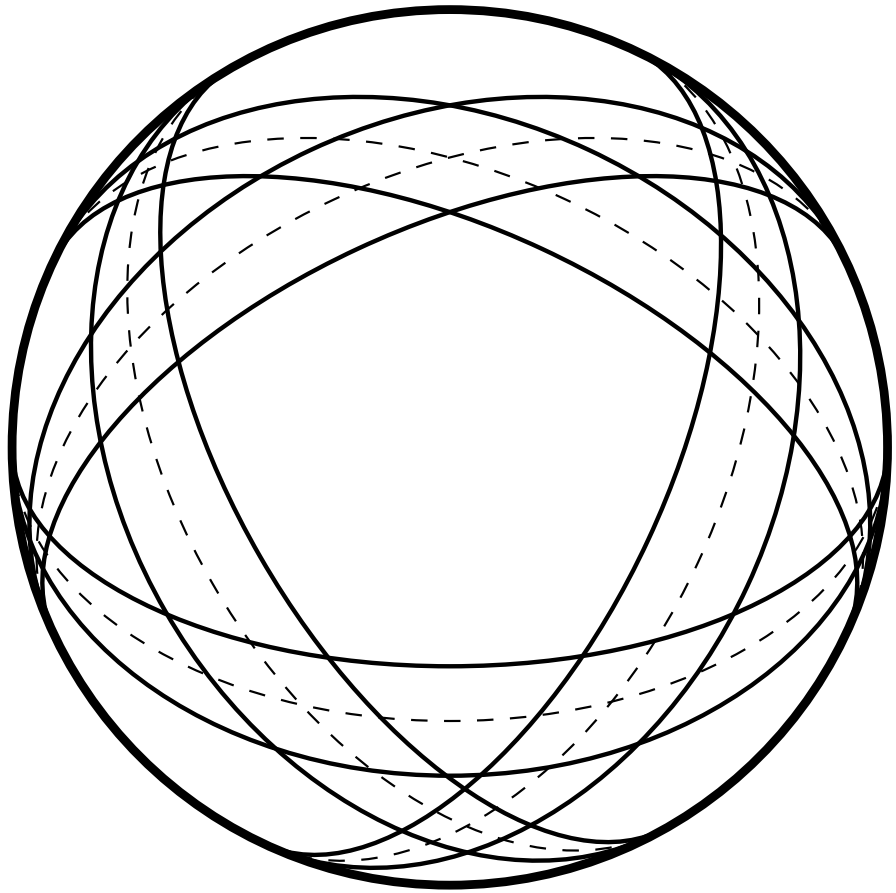


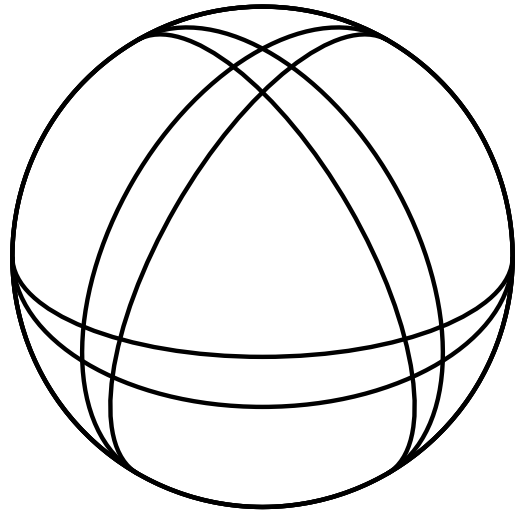
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Trivial bound is $\gtrsim 1$. What happens on $\mathbb{S}^2$?

Kakeya on the Sphere



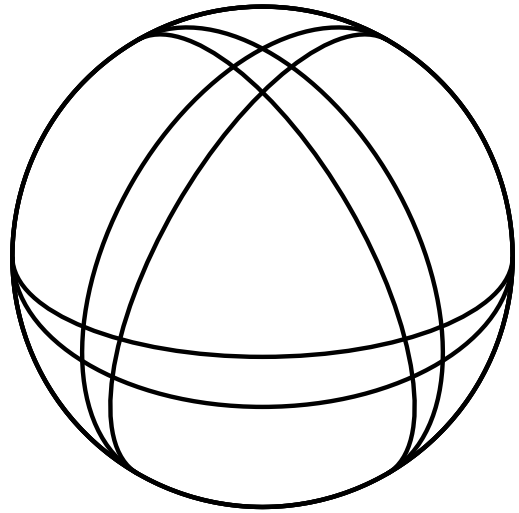


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and *this is best possible*[2].

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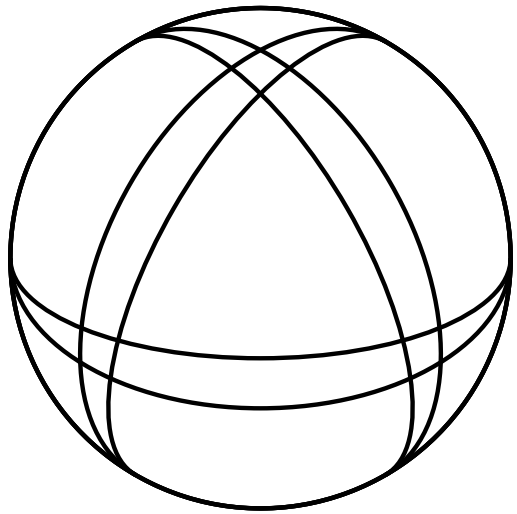
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and *this is best possible*[2]. Nontrivial sharp bound

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} |C_i \cap C_j|^2 \gtrsim \frac{\log n}{n^2}.$$

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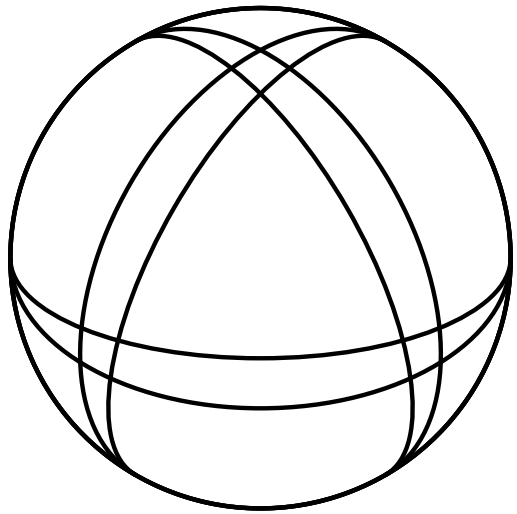


Question

We have (from Hölder and $p = 1$) that for all $p \geq 1$,

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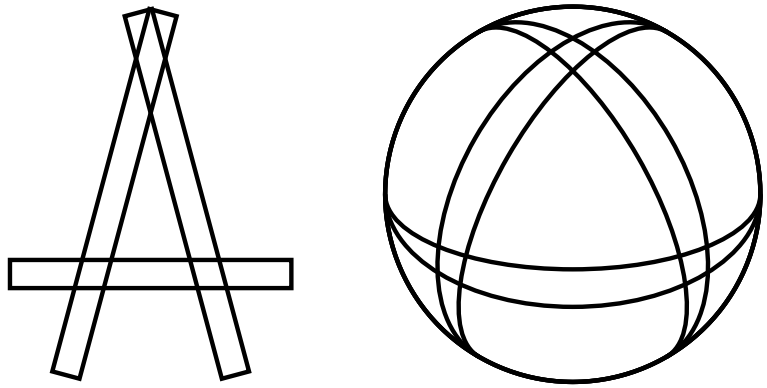
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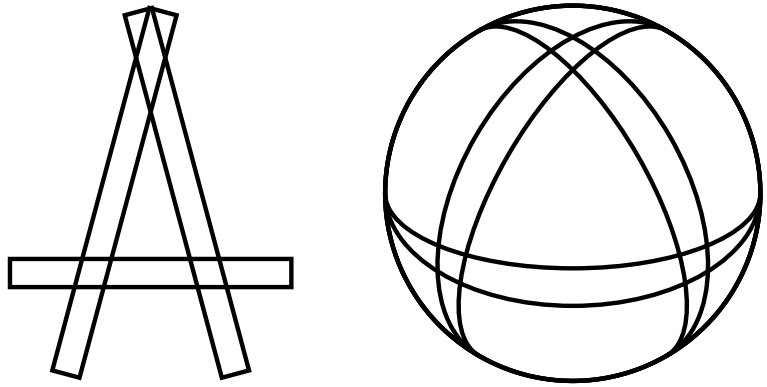
Sharp for $1 \leq p \leq 2$. *Probably not when* $p > 2$ **?**

Speaking of curvature?

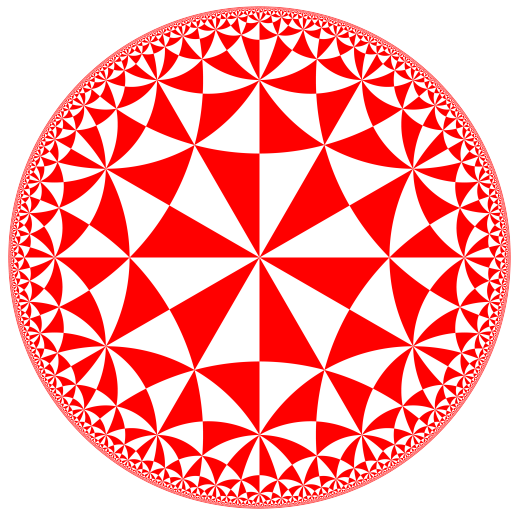


The curvature of $\mathbb{S}^2$ increases transversality, which decreases the area of intersection.

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?

The Motzkin-Schmidt Problem

Jean was very generous with problems. In December 2013, there was a very nice conference in Jerusalem that I especially remember for two things. We had the snow-storm of the century and that was the last time I met Jean in person. He told me there the following problem, which to the best of my knowledge, is still open. Is it possible to find n points in the unit square such that the $1/n$-neighborhood of any line contains no more than C of them for some absolute constant C? The motivation for this problem comes from a possible construction of spherical harmonics as a combination of Gaussian beams, which would have L^∞ norm bounded by a constant independently of the degree.

(Peter Varju, Remembering Jean Bourgain, AMS Notices 2021)

The Motzkin-Schmidt Problem

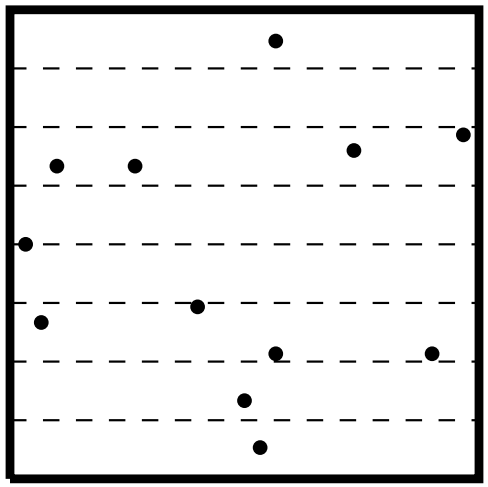
A Trivial Statement

Given n points in $[0,1]^2$, there exists exists a line $\ell \in \mathbb{R}^2$ whose $3/n-$neighborhood contains at least 3 points.

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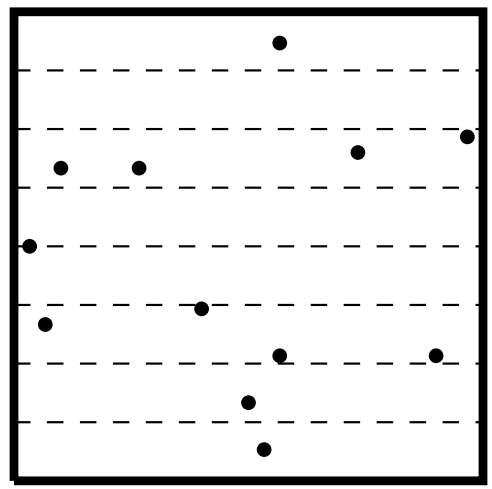
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The Motzkin-Schmidt Problem

Is this true with a $o(1/n)$ neighborhood?

A quick one: number of critical points

Trivial

Let $f : \mathbb{T} \to \mathbb{R}$ be smooth. Then

$$(\#\text{number of critical points}) \cdot \|f\|_{L^\infty(\mathbb{T})} \gtrsim \|f'\|_{L^1(\mathbb{T})}.$$

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Open problem. Is there a one-parameter family? Interpolation? Higher dimensions? Or it's just an accident?

Spooky Action at a Distance

When doing Fourier Analysis, there is no real difference between, say, the function $\sin(x)$ and the function $-\sin(x)$ because the sign cancels

$$f = \sum_{k \in \mathbb{N}} \langle f, \phi_k \rangle \, \phi_k.$$

[3] S, Quantum entanglement and the growth of Laplacian eigenfunctions. Communications in Partial Differential Equations, 48 (2023), 511-541.

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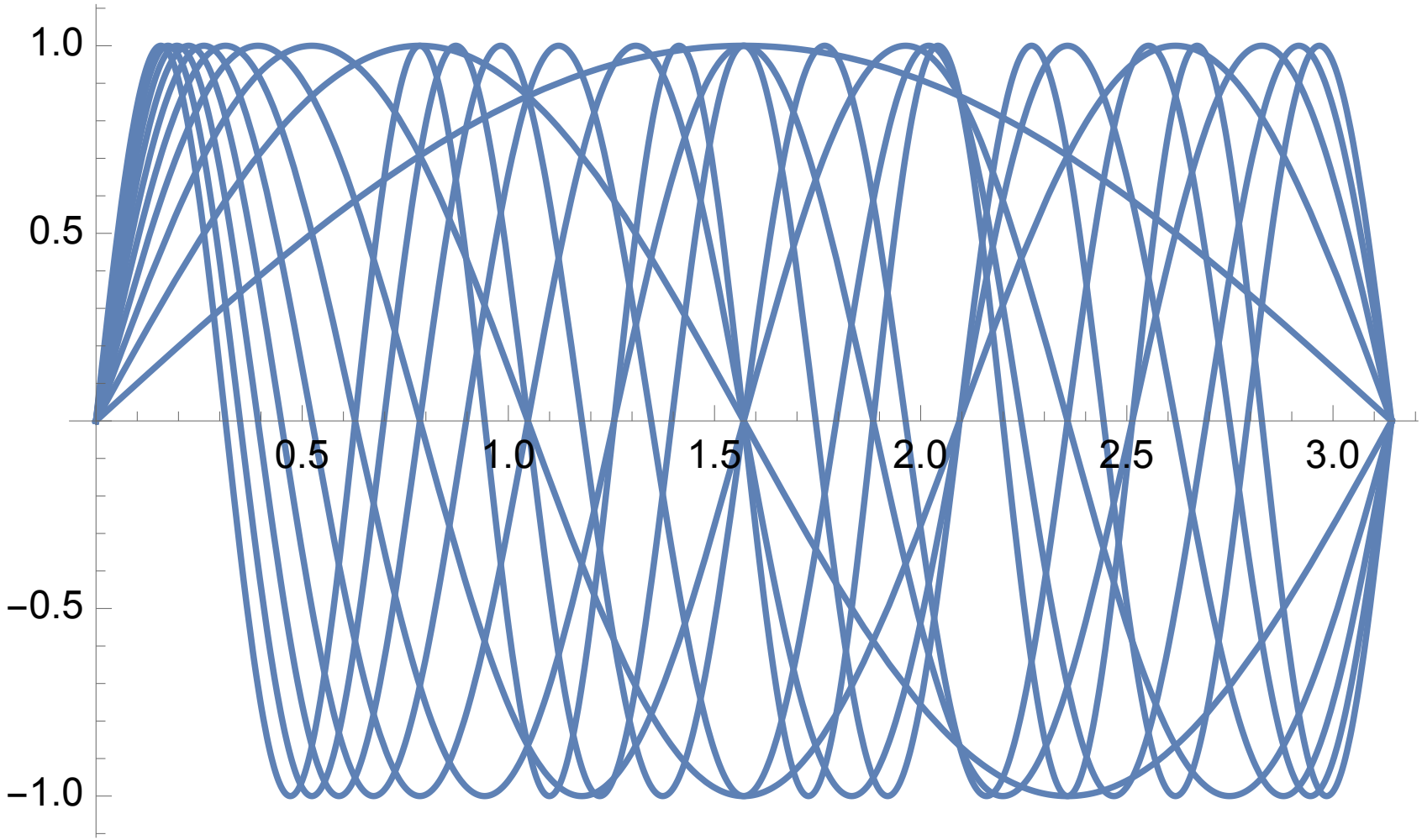
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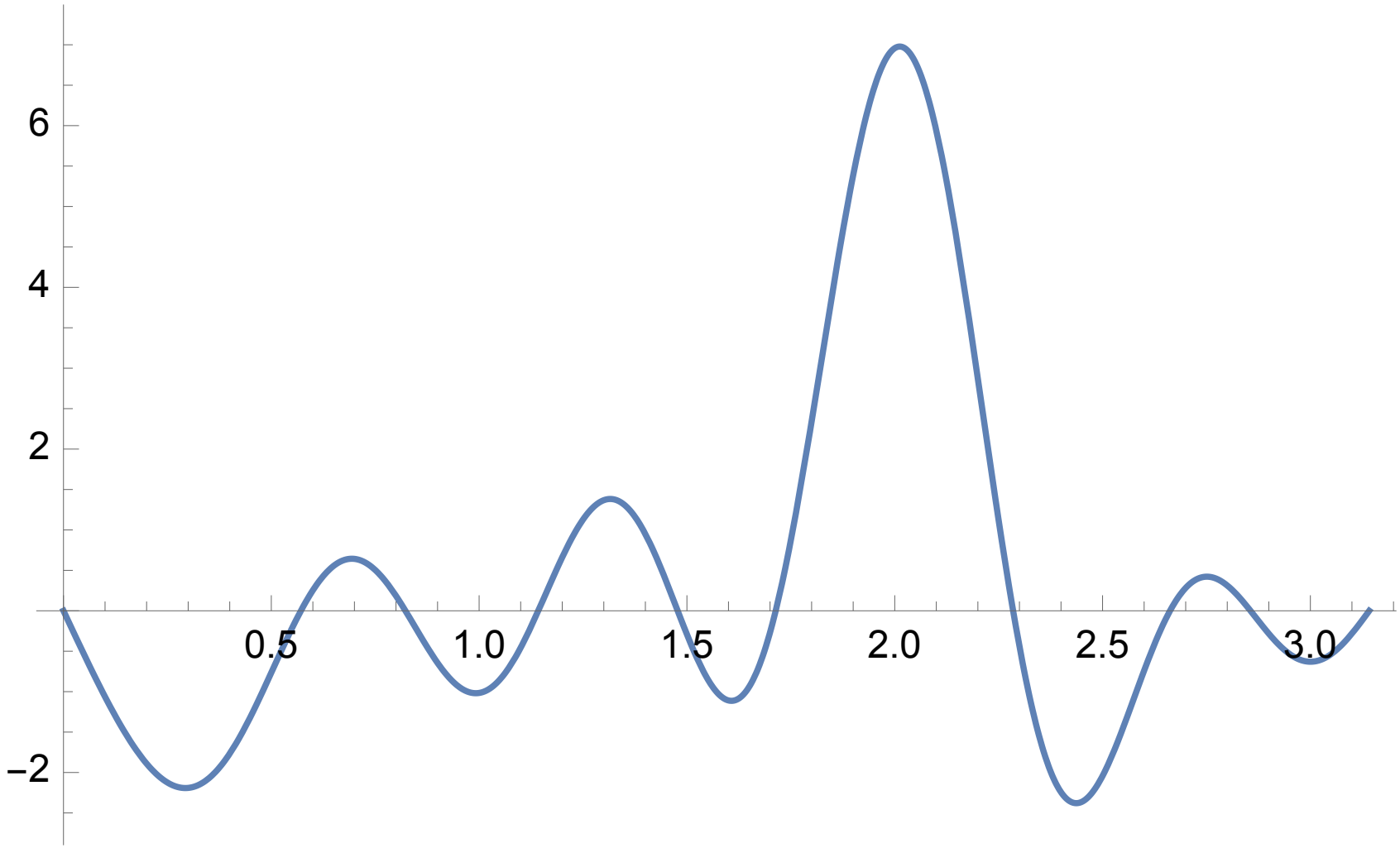
Motivated by some question on Laplacian eigenfunctions[3], we can flip all the signs so that the functions are all positive in x

$$f(x,y) = \sum_{k=1}^{n} \text{sign}\left(\phi_k(x)\right) \phi_k(y).$$

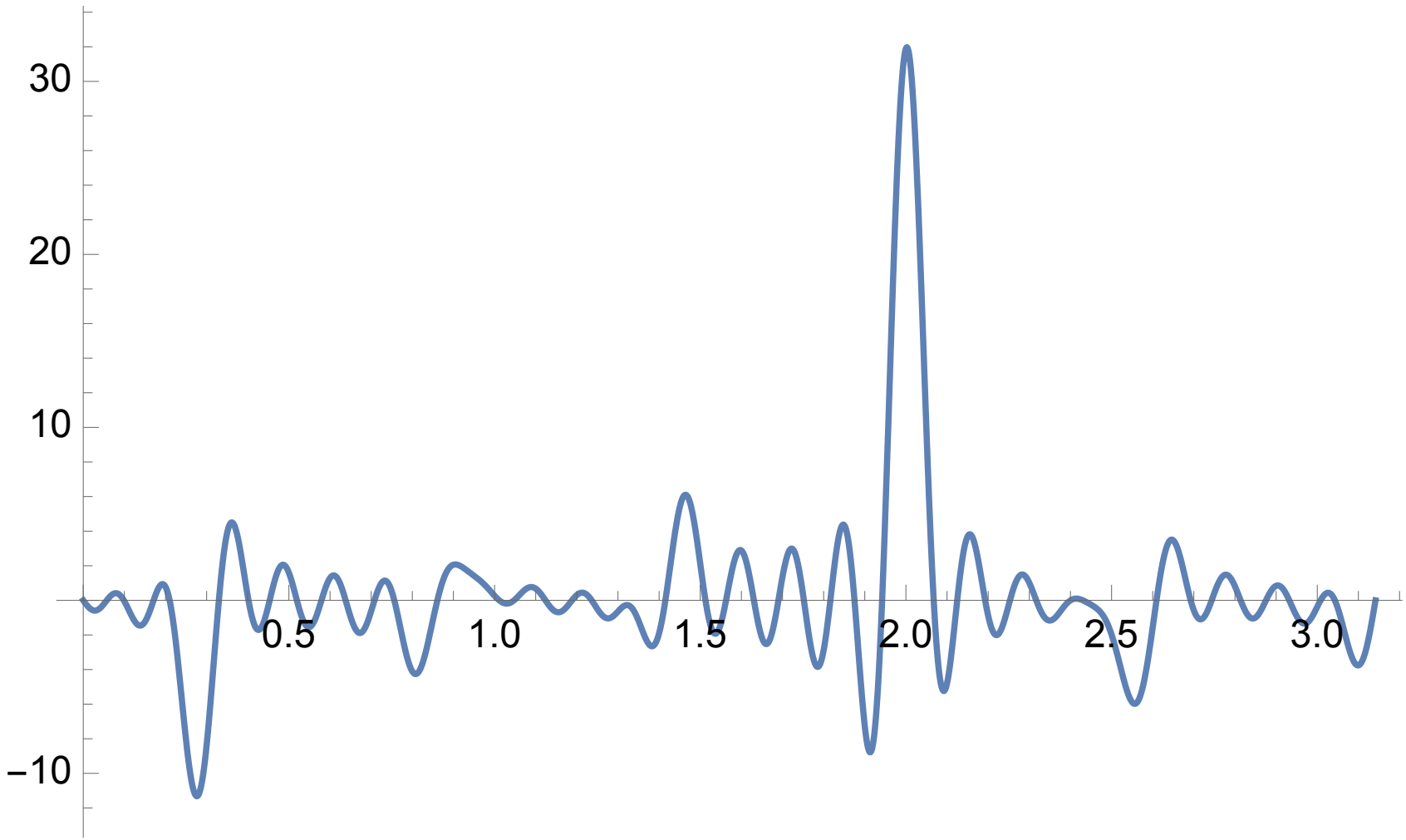
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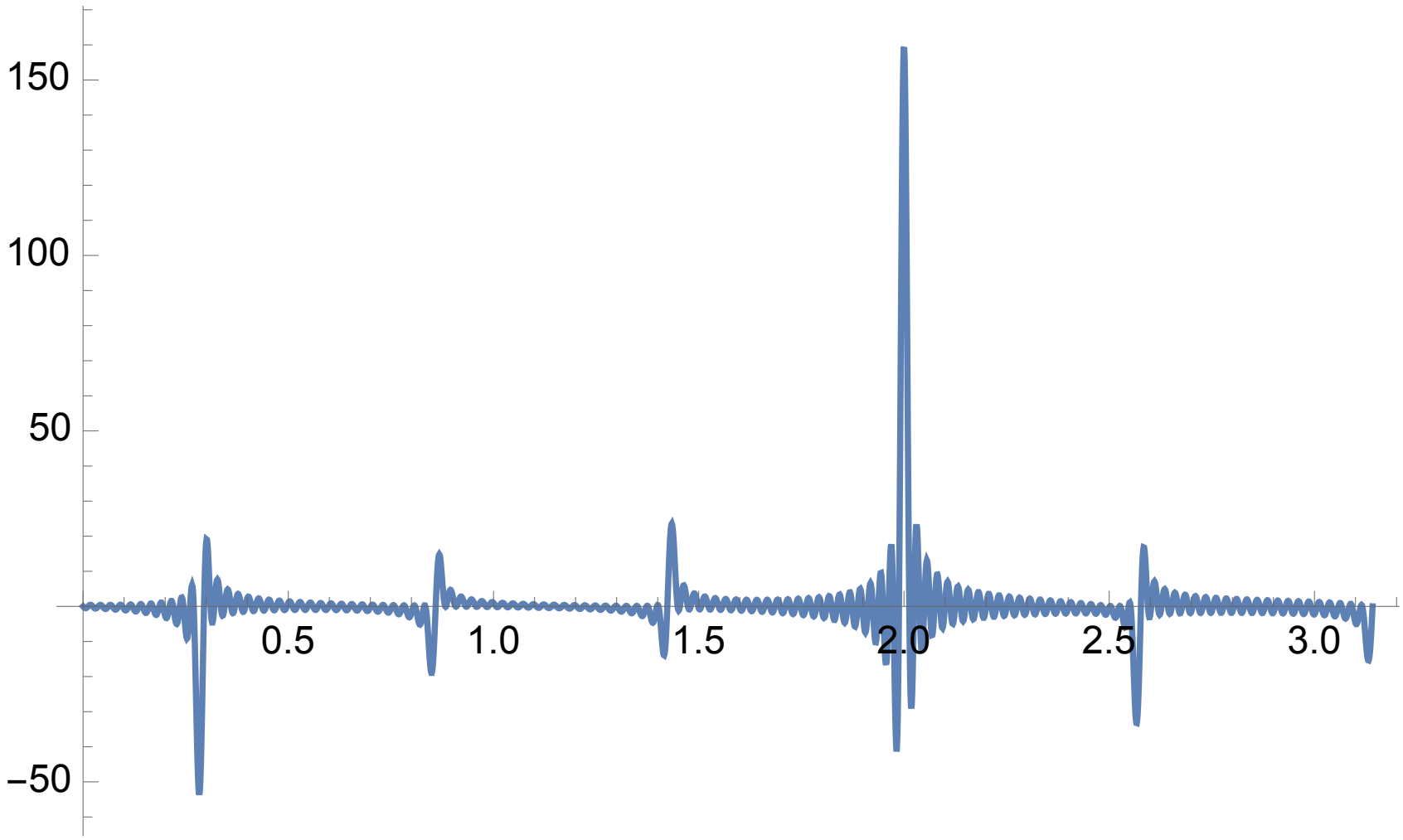
The functions $\sin(kx)$ on $[0, \pi]$.



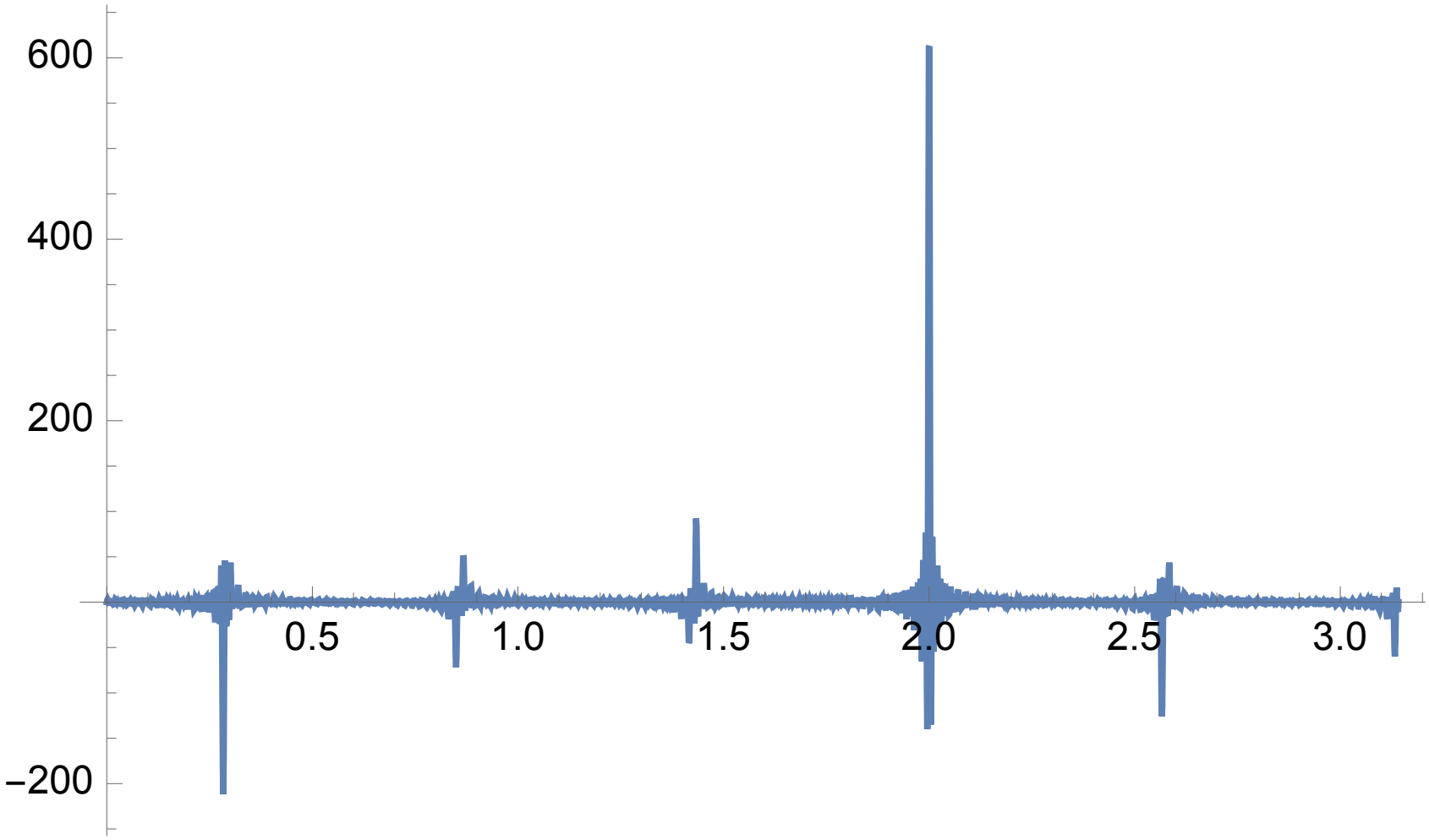
Summed sines (sign so that positive in $x = 2$), 10 terms.



Summed sines (sign so that positive in $x = 2$), 50 terms.



Summed sines (sign so that positive in $x = 2$), 250 terms.



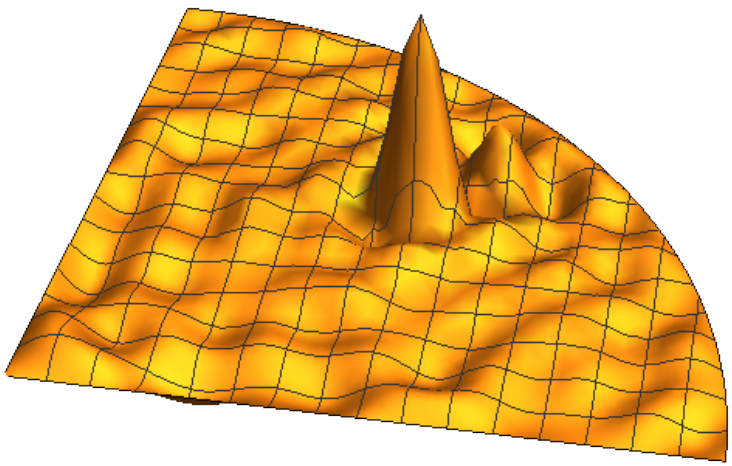
Summed sines (sign so that positive in $x = 2$), 1000 terms.

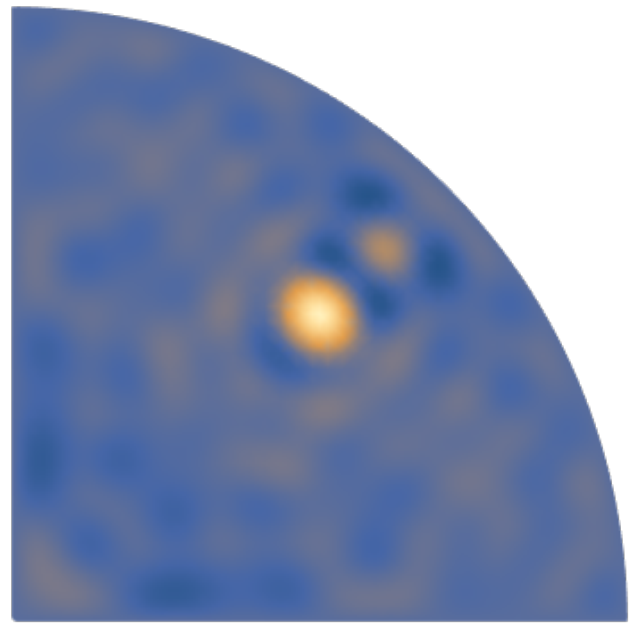
Spooky: General Orthogonal Functions

Eigenfunctions of Laplacian on $1/4-$disk with Dirichlet boundary conditions are an orthonormal family of basis functions in L^2.

Spooky: General Orthogonal Functions

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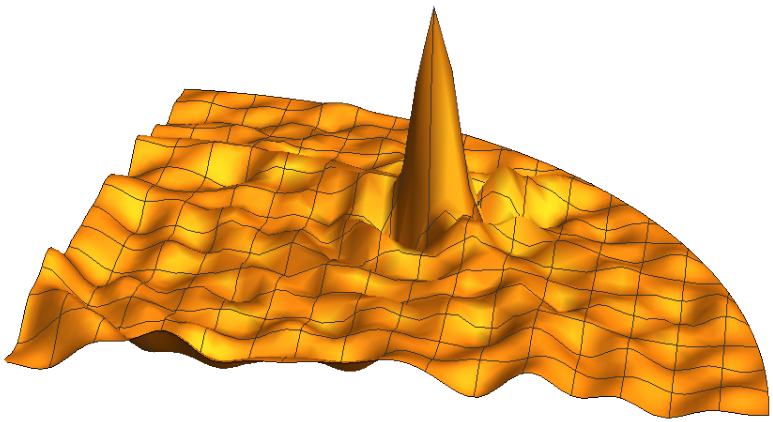
Bessel Function Identities?

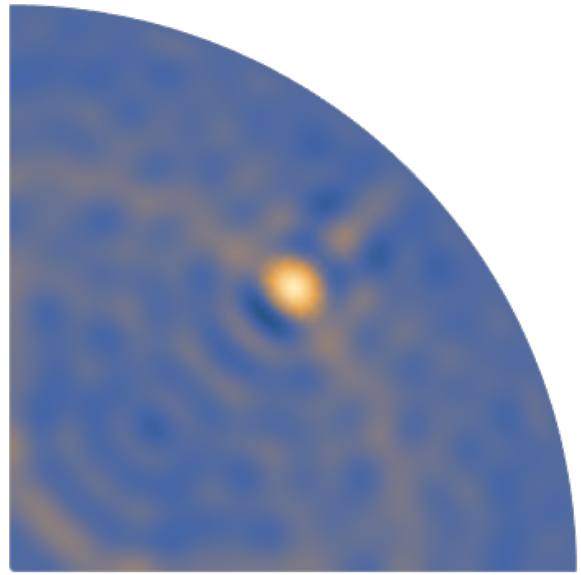
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Bessel Function Identities? Other examples?

Opaque Square (Mazurkiewicz 1916, Bagemihl 1959)

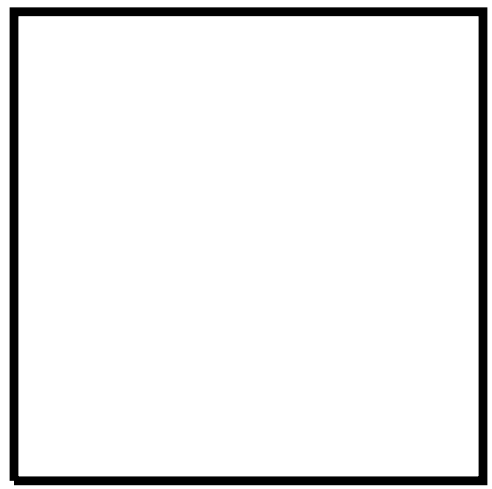
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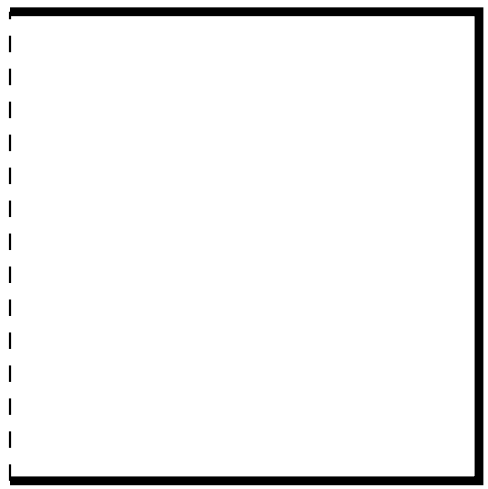
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Length $= 4$

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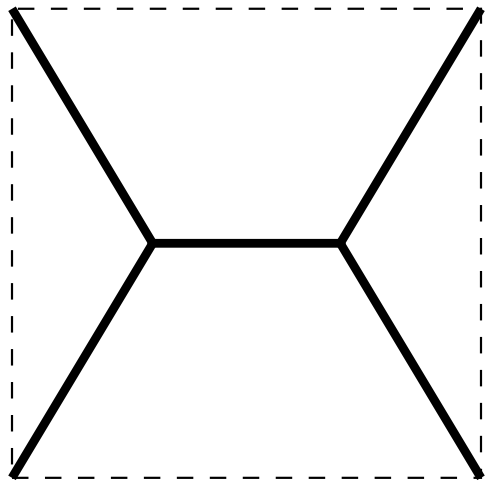
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Length $= 3$

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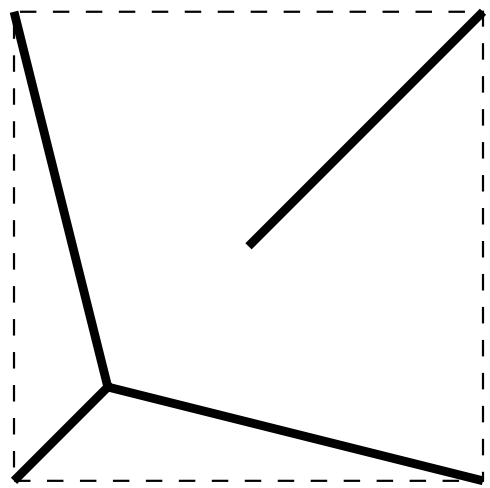
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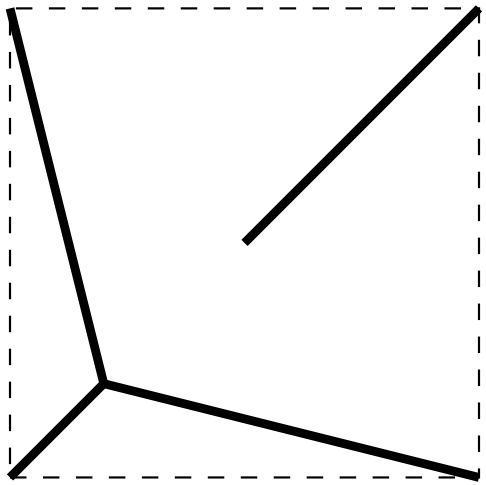
Length ~ 2.732

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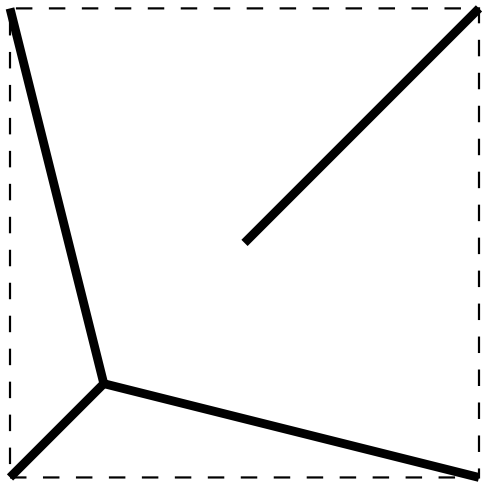


Length ~ 2.639



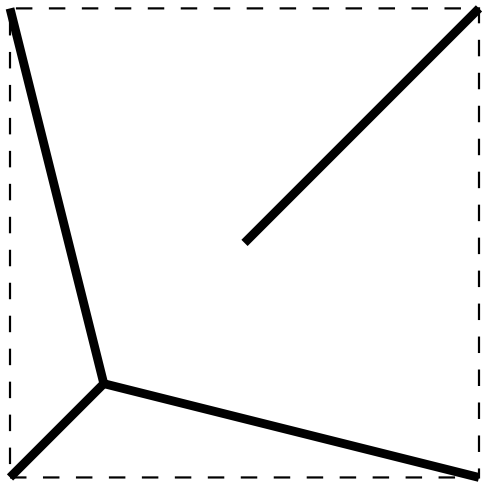
Lower Bound (Jones 1962)

For any convex $\Omega \subset \mathbb{R}^2$, then length is at least half the boundary.



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Lower Bound (Kawamura-Moriyama-Otachi-Pach, 2019)

For the square at least ≥ 2.00002.

The Averaging Problem

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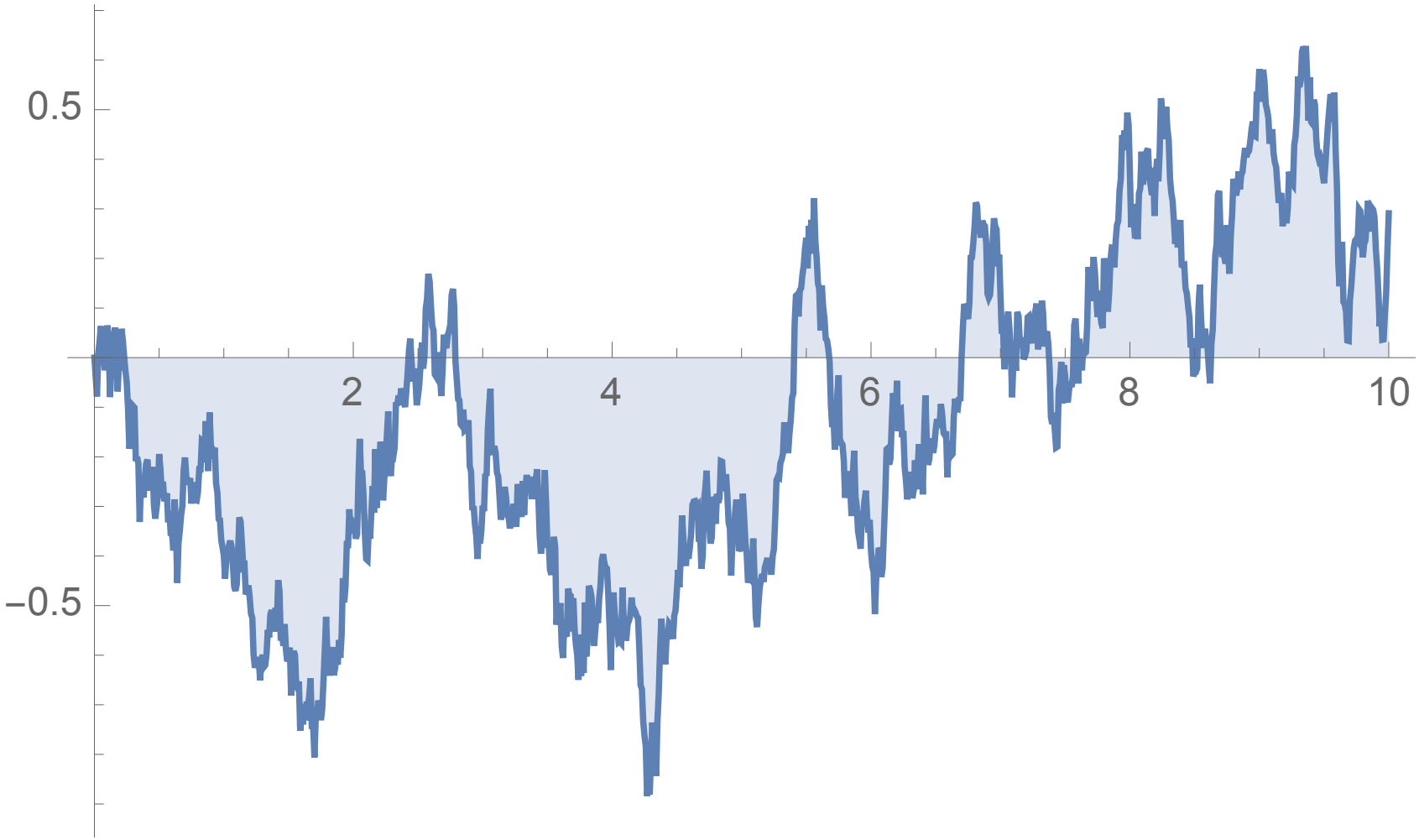
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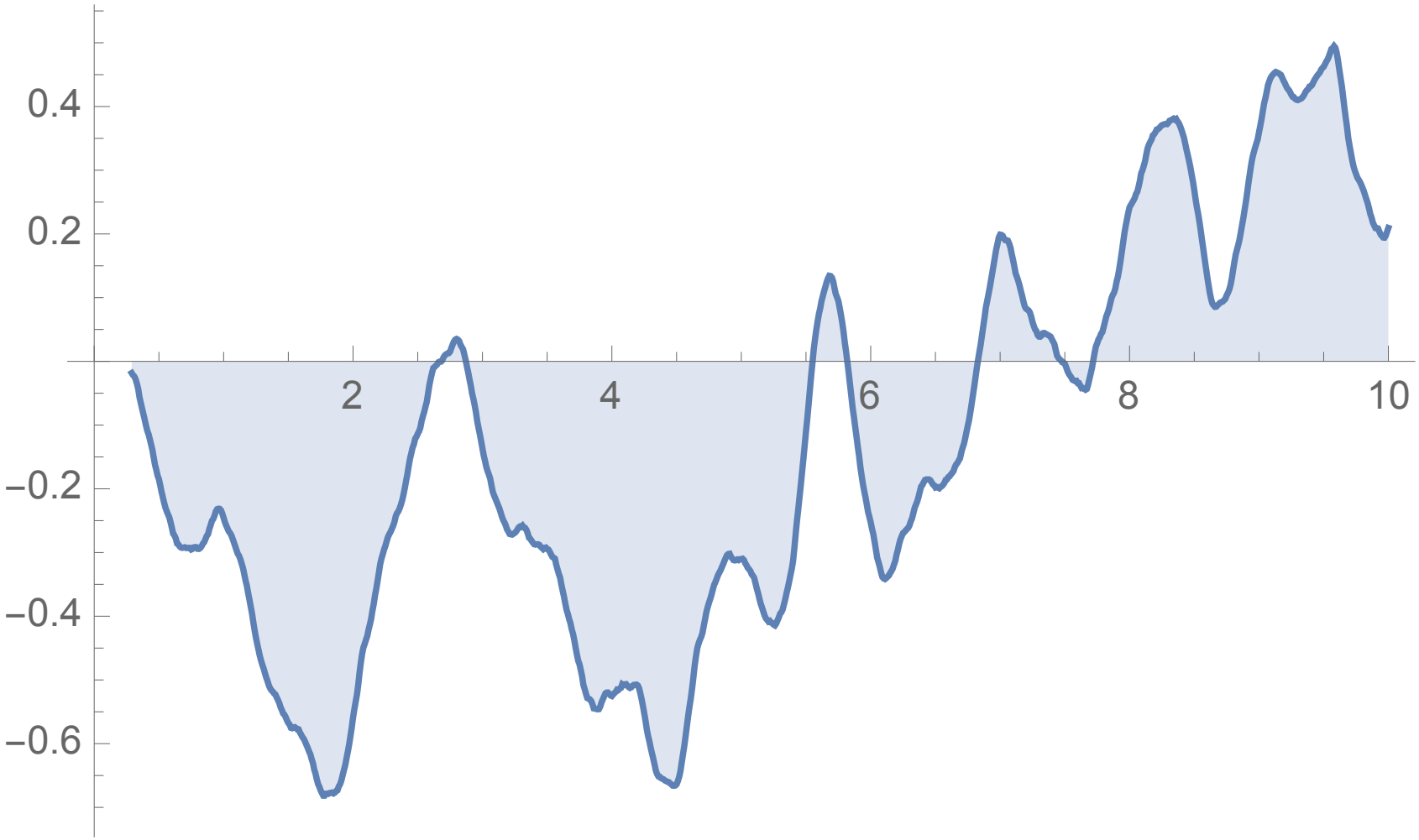
The Averaging Problem

What is the 'best' average?

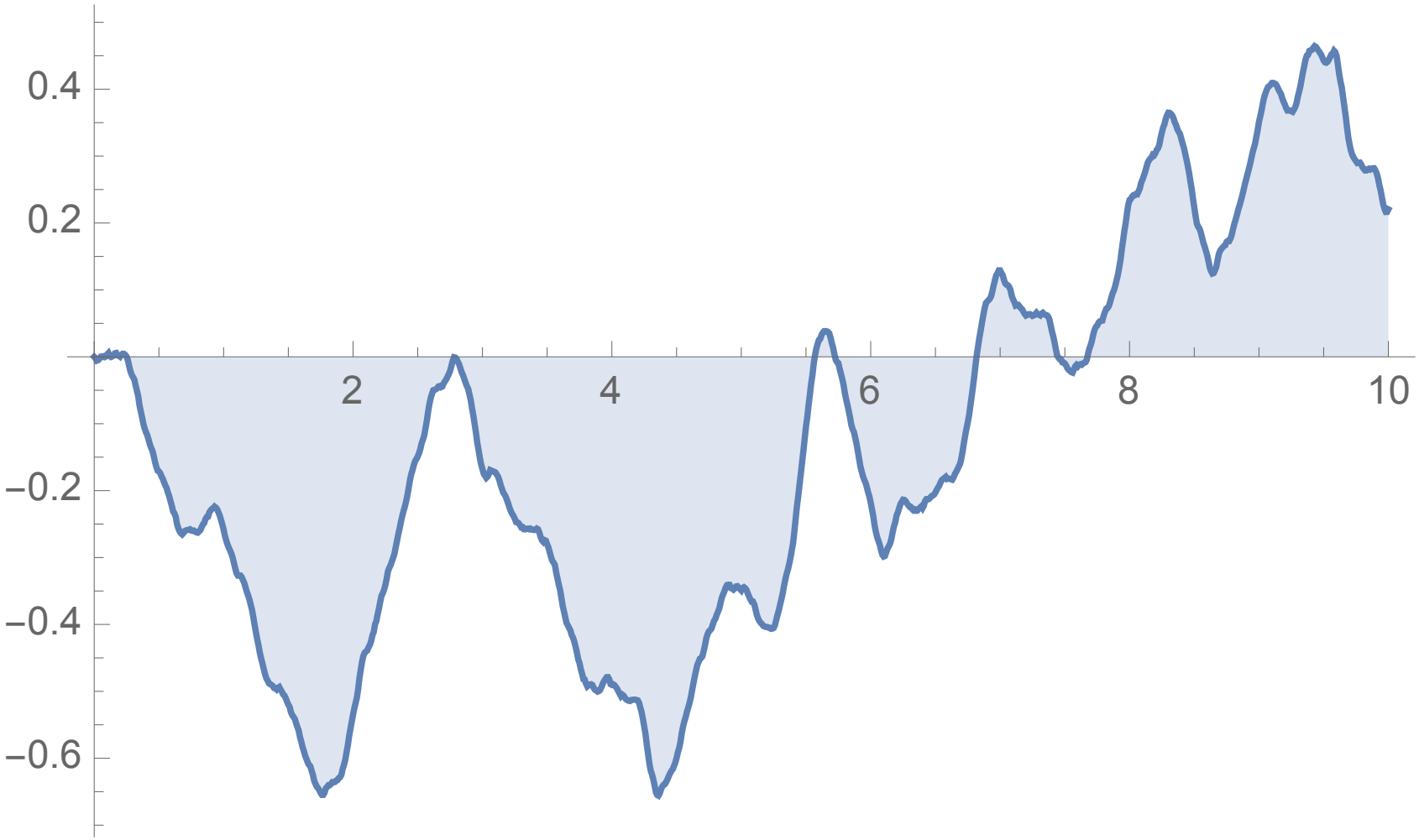
An example: we want understand local averages of



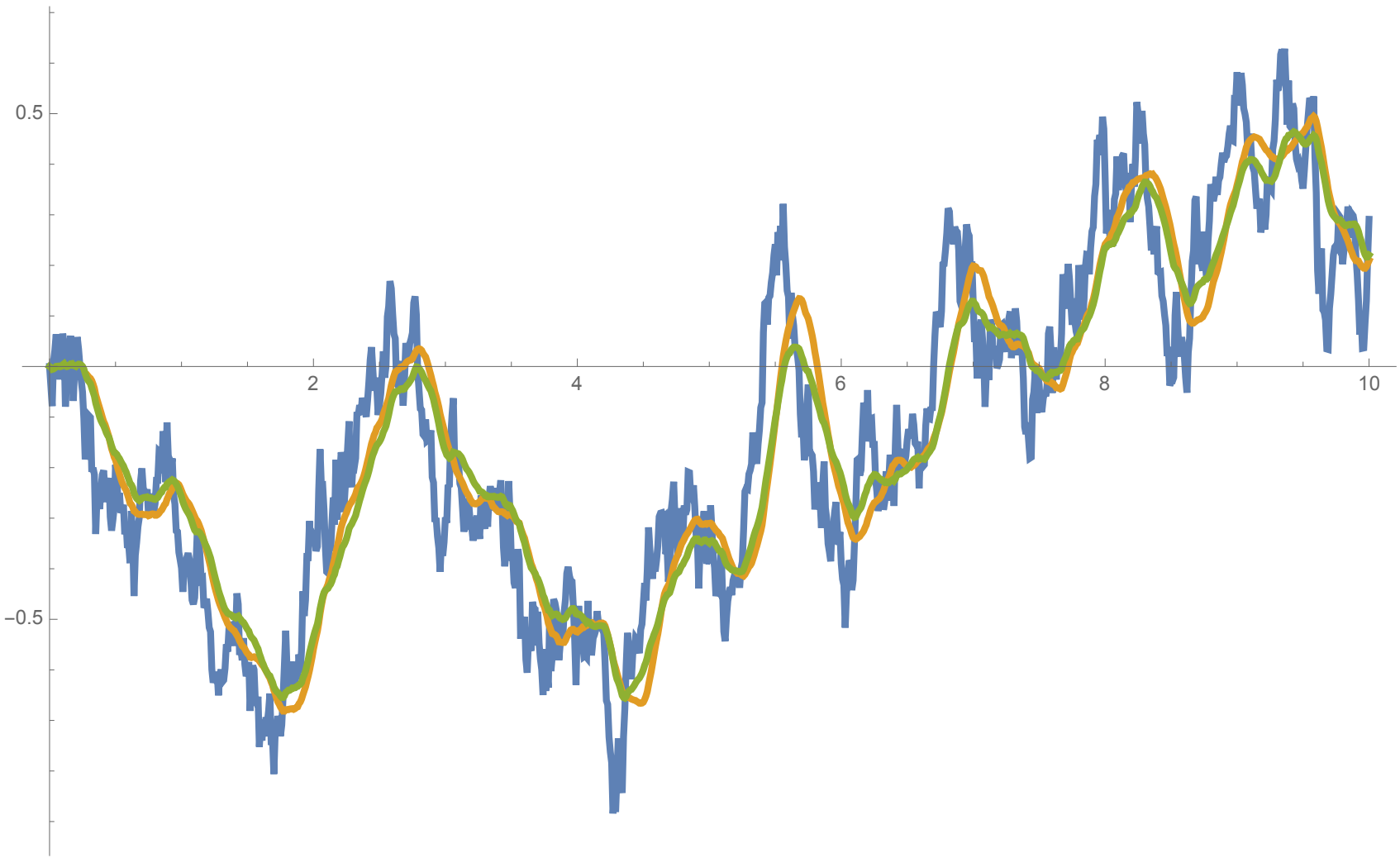
Averaging with characteristic functions



Averaging with exponential distribution



So which one is the right one?



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Some New Uncertainty Principles

Theorem (S, Math Res Lett, 2021)

For $\alpha > 0$ and $\beta > n/2$, there exists $c_{\alpha,\beta,n} > 0$ such that for all $u \in L^1(\mathbb{R}^n)$

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- I think the **extremizers** are going to be interesting.

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Theorem (Cho, Park)

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Theorem (Noah Kravitz and S, Bull. London Math Soc, 2021)

Let $u : \{-n, \ldots, n\} \to \mathbb{R}$ be a symmetric function with normalization $\sum_{k=-n}^{n} u(k) = 1$.

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Note. This is the discrete analogue of the characteristic function (like in the continuous case!)

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Open problem. For $\ell = 3, 4, \ldots$

$$\sup_{0 \neq f \in \ell^2(\mathbb{Z})} \frac{\|\Delta^{\ell}(f * u)\|_{\ell^2(\mathbb{Z})}}{\|f\|_{\ell^2}} \geq ?$$

Kozma-Oravecz Inequalities

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$$f(x) = \cos(2\pi \langle x, k \rangle)$$

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Theorem (Taikov, 1965)

The function $f : \mathbb{T} \to \mathbb{R}$ given by

$$f(x) = \sum_{k=A}^{A+B} a_k \cos(kx)$$

has a root in every interval of length $(B+1)/(2A+B)$.

Theorem (Kozma-Oravecz, 2003)

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Open problem. The *sum* should be wrong scaling. Maybe $(\sum_{\lambda \in \{\|k\| : k \in S\}} \frac{1}{\lambda^2})^{1/2}$?



Thank you!