Fourier decay of multiplicative convolutions

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Heuristic Principle

Combining addition and multiplication creates growth (unless we are trapped in a subring).

Conjecture

Let $A \subset \mathbb{Z}$ be a finite set. Then $\max\{|A+A|, |A \cdot A|\} \gtrsim_\varepsilon |A|^{2-\varepsilon}$.

Remark

The best known exponent is slightly larger than $4/3$.

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Convolution of measures

- Given two Borel probability measures μ, ν on $\mathbb{R}$, their convolution $\mu * \nu$ (also a measure on $\mathbb{R}$) is defined as

$$\int f \, d(\mu * \nu) = \iint f(x+y) d\mu(x) d\nu(y).$$

- Equivalently,

$$\mu * \nu(A) = \int \mu(A-x) d\nu(x) = \int \nu(A-y) d\mu(y)$$

- If μ, ν have densities f, g, then $\mu * \nu$ has density $f * g$, where $f * g$ is the usual convolution of functions.
- Convolution should be thought of as "addition" for measures.

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Definition

The Fourier transform of a measure μ on $\mathbb{R}$ is the function

$$\hat{\mu}(\xi) = \int e^{-2\pi i \langle x, \xi \rangle} d\mu(x), \quad \xi \in \mathbb{R}.$$

Lemma (Convolution formula, consequence of $e^{x+y} = e^x e^y$)

If μ, ν are measures on $\mathbb{R}$, then

$$\widehat{\mu * \nu} = \hat{\mu} \cdot \hat{\nu}.$$

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A measure μ on $\mathbb{R}$ is said to have power Fourier decay if there exist $C, \sigma > 0$ such that

$$|\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\sigma}, \quad \forall \xi \in \mathbb{R}.$$

- Power Fourier decay is a key property of a measure. For example, if μ has power Fourier decay, then for every k there is a convolution power μ^{*N}, which has a C^k density.
- On the other hand, μ has power Fourier decay if and only if μ^{*N} has power Fourier decay for every $N \geq 1$.

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Given finite measures μ, ν on $\mathbb{R}$, their multiplicative convolution $\mu \boxtimes \nu$ is defined as the measure on $\mathbb{R}$ given by

$$\int f \, d(\mu \boxtimes \nu)(A) = \iint f(xy) d\mu(x) d\nu(y).$$

- Note that

$$\mathcal{F}(\mu \boxtimes \nu)(\xi) = \int e^{-2\pi i \, xy \, \xi} d\mu(x) d\nu(y).$$

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Bourgain's Theorem on multiplicative convolutions

Theorem (J. Bourgain 2010)

For any $s > 0$, there is $N \in \mathbb{N}$ such that the following holds.

Let μ be a measure on $[1, 2]$ satisfying the Frostman condition

$$\mu(B(x, r)) \leq C\, r^s, \quad \forall x \in \mathbb{R},\ r > 0.$$

Then, $\mu^{\boxtimes N}$ has power Fourier decay.

Why multiplicative convolutions?

- Bourgain, Glibichuk and Konyagin (2006) proved a similar result in the finite field setting $\mathbb{F}_p$. Applied to a multiplicative subgroup $G \subset \mathbb{F}_p^*$, this yields a result on exponential sums, indicating a connection with analytic number theory:

$$\mathcal{F}(\mathbf{1}_G^{\boxtimes N})(\xi) = \sum_{x \in G^N} e^{-2\pi i\, x_1 \cdots x_N \cdot \xi} = |G|^{N-1} \widehat{\mathbf{1}_G}(\xi).$$

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Bourgain's Theorem and sum-product philosophy

- Bourgain's Theorem is a manifestation of the sum-product philosophy: multiplicative convolution creates "additive smoothness".
- The theorem fails over $\mathbb{C}$: if μ is length measure on $[1,2] \subset \mathbb{C}$, then $\mu^{\boxtimes N}$ is supported on $\mathbb{R}$ and so cannot have any Fourier decay at all.
- In fact, Bourgain used a result that is more transparently of sum-product nature as the key ingredient in the proof.
- A corollary of Bourgain's sum-product theorem that morally captures the key idea is: for every $s > 0$ there is $\varepsilon > 0$ such that if $A \subset [1,2]$, $\dim_{\mathrm{H}}(A) = s$, then

$$\dim_{\mathrm{H}}(A \cdot A + A \cdot A) \geq \dim_{\mathrm{H}}(A) + \varepsilon.$$

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Quantitative Bourgain?

- An obvious question is: how does N (the number of multiplicative convolutions needed to obtain power Fourier decay) depend on s (the Frostman exponent)?
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Bourgain's conjectured threshold

Conjecture (J. Bourgain 2010)

Let $s > 0$. If $Ns > 1$, then for any measure μ on $[1,2]$ satisfying the Frostman condition

$$\mu(B(x,r)) \leq C\, r^s, \quad \forall x \in \mathbb{R},\ r > 0,$$

$\mu^{\boxtimes N}$ has power Fourier decay.

- Bourgain proved the conjecture for $s > 1/2$ and $N = 2$.
- This would be sharp up to the endpoint: if $Ns < 1$, then one can construct a measure with Frostman exponent s such that $\mu^{\boxtimes N}$ does **not** have power Fourier decay.

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Theorem (T. Orponen, N. de Saxcé, P.S. 2024+)

Let $\mu_1, \ldots, \mu_N$ be measures on $[1,2]$ satisfying the Frostman conditions

$$\mu_i(B(x,r)) \leq C_i\, r^{s_i}, \quad \forall x \in \mathbb{R},\ r > 0.$$

If $\sum_{i=1}^{N} s_i > 1$, the multiplicative convolution

$$\mu_1 \boxtimes \cdots \boxtimes \mu_N$$

has power Fourier decay (depending only on s_i).

Corollary

Let $A_1, \ldots, A_N \subset \mathbb{R}$ be Borel sets with $\sum_{i=1}^{N} \dim_H(A_i) > 1$. Then, the additive subgroup generated by $A_1 \cdots A_N$ is $\mathbb{R}$.

Proof.

- Let μ_i be Frostman measures on A_i.
- By the Main Result I, the measure $\mu_1 \boxtimes \cdots \boxtimes \mu_N$ has power Fourier decay and is supported on $A_1 \cdots A_N$.
- By the convolution formula, there is k such that $\nu = (\mu_1 \boxtimes \cdots \boxtimes \mu_N)^{\boxplus k}$ satisfies $|\widehat{\nu}(\xi)| \leq C(1 + |\xi|)^{-2}$.
- Then ν has a continuous density, so $\mathrm{supp}(\nu)$ has nonempty interior. But $\mathrm{supp}(\nu)$ is contained in the additive subgroup generated by $A_1 \cdots A_N$, so this subgroup must be $\mathbb{R}$.

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Main result II

In the main result, we do not know how fast the power Fourier decay is. However, with more convolutions we can achieve a quantitative decay rate.

Theorem (T. Orponen, N. de Saxcé, P.S. 2024+)

There is an absolute constant $C > 0$ such that the following holds.

Let $\sigma > 0$. Let $\mu_1, \dots, \mu_N$ be measures on $[1,2]$ satisfying the Frostman conditions with exponent σ, where

$$N\sigma > C.$$

Then,

$$|\mathcal{F}(\mu_1 \boxtimes \cdots \boxtimes \mu_N)(\xi)| \leq C(1+|\xi|)^{-\exp(C\sigma^{-1})}, \quad \forall \xi \in \mathbb{R}.$$

An application

- Recently, G. Leclerc, S. Paukkonen and T. Sahlsten established power Fourier decay for a wide class of dynamically defined measures on the line under weak smoothness and/or expansivity assumptions.
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A sum-product estimate

The main technical tool of the proof is a sum-product estimate. The actual version is quite technical to state, but it is morally captured by the following Hausdorff dimension version:

Theorem (T. Orponen-P.S. 2023+)

Let A, B be Borel sets in $\mathbb{R}$. Then

$$\dim_H(A \cdot B + A \cdot B + A \cdot B + A \cdot B) \geq \min\{\dim_H(A) + \dim_H(B), 1\}.$$

Radial projections

Let $F \subset \mathbb{R}^2$.

- Given $x \in \mathbb{R}^2$, the radial projection $\Delta^x(F)$ is the set of directions between x and points in F.
- The direction set $\Delta(F)$ is the union of all radial projections $\Delta^x(F)$, $x \in F$. That is,

$$\Delta(F) = \left\{ \frac{y-x}{\|y-x\|} : x, y \in F, x \neq y \right\}.$$

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First step: application of radial projections

Theorem (Radial projections, T. Orponen-P.S.-H. Wang 2024)

Let $E, F \subset \mathbb{R}^2$ be Borel sets with E not contained in a line. Then,

$$\sup_{x \in E} \dim_{\mathrm{H}}(\Delta^x(F)) \geq \min\{\dim_{\mathrm{H}}(E), \dim_{\mathrm{H}}(F), 1\}.$$

Corollary

$$\dim_{\mathrm{H}}\left(\Delta^0((A-A)\times(B-B))\right) \geq \min\{\dim_{\mathrm{H}}(A)+\dim_{\mathrm{H}}(B), 1\}.$$

Proof.

Apply radial projections to $E = F = A \times B$. □

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Second step: application of Kaufman

Theorem (R. Kaufman 1968)

Let $E, F \subset \mathbb{R}^2$. Then,

$$\sup_{x \in E} \dim_H(x \cdot F) \geq \min\{\dim_H(\Delta^0(E)), \dim_H(F)\}.$$

Theorem (Sum product estimate again)

$$\dim_H\big(2(A \cdot B) - 2(A \cdot B)\big) \geq \min\{\dim_H(A) + \dim_H(B), 1\} =: t.$$

Proof.

- Let $E = (A - A) \times (B - B)$, $F = B \times A$.
- By radial projections, $\dim_H(\Delta^0(E)) \geq t$.
- By Kaufman, $\sup_{x \in E} \dim_H(x \cdot F) \geq t$.
- But $E \cdot F \subset 2(A \cdot B) - 2(A \cdot B)$, so we are done.

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- But $E \cdot F \subset 2(A \cdot B) - 2(A \cdot B)$, so we are done.

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Second step: application of Kaufman

Theorem (R. Kaufman 1968)

Let $E, F \subset \mathbb{R}^2$. Then,

$$\sup_{x \in E} \dim_H(x \cdot F) \geq \min\{\dim_H(\Delta^0(E)), \dim_H(F)\}.$$

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Theorem (T. Orponen, N. de Saxcé, P.S. 2024+)

Let $\mu_1, \ldots, \mu_N$ be measures on $[1,2]$ satisfying the Frostman conditions

$$\mu_i(B(x,r)) \leq C_i\, r^{s_i}, \quad \forall x \in \mathbb{R},\ r > 0.$$

If $\sum_{i=1}^{N} s_i > 1$, the multiplicative convolution

$$\mu_1 \boxtimes \cdots \boxtimes \mu_N$$

has power Fourier decay (depending only on s_i).

Sketch of proof I: discretization

- Fix a small scale $\delta > 0$. We let μ_i^δ be mollifications at scale δ, and aim to prove decay for $\mu_1^\delta \boxtimes \cdots \boxtimes \mu_N^\delta$ at frequencies $|\xi| \sim \delta^{-1}$.
- This works because mollifying μ_i at scale δ preserves the Frostman condition, or the size of the Fourier transform at frequencies $|\xi| \sim \delta^{-1}$.

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Sketch of proof II: the case $N = 2$

Proposition (J. Bourgain 2010)

Let μ, ν be measures on $[1, 2]$. Suppose that

$$\|\mu^\delta\|_{L^2}^2 \leq \delta^{-1+s}, \quad \|\nu^\delta\|_{L^2}^2 \leq \delta^{-1+t}.$$

Then,

$$\left|\mathcal{F}(\mu^\delta \boxtimes \nu^\delta)(\xi)\right| \lesssim \delta^{-(s+t-1)/2}, \quad \forall |\xi| \sim \delta^{-1}.$$

- The L^2 assumption follows from a Frostman condition with exponent s and t, respectively.
- Roughly speaking, the L^2 assumption is a Frostman assumption at scale δ only.

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Sketch of proof III: a Cauchy-Schwarz trick

Lemma

$$\left|\widehat{\mu \boxtimes \nu}(\xi)\right|^2 \leq \mathcal{F}\big((\mu \boxminus \mu) \boxtimes \nu\big)(\xi).$$

Proof.

$$\begin{aligned}
\left|\widehat{\mu \boxtimes \nu}(\xi)\right|^2 &= \left|\iint e^{-2\pi i\, xy\cdot\xi} d\mu(x) d\nu(y)\right|^2 \\
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Sketch of proof IV: a white lie about sum-product for measures

Theorem

$$\dim_H(A \cdot B + A \cdot B - A \cdot B - A \cdot B) \geq \min\{\dim_H(A) + \dim_H(B), 1\}.$$

The following is a reasonable version for measures:

White Lie

Let μ, ν be measures on $[1,2]$ satisfying Frostman conditions with exponents s, t. Let

$$\eta = (\mu \boxtimes \nu) \boxplus (\mu \boxtimes \nu) \boxminus (\mu \boxtimes \nu) \boxminus (\mu \boxtimes \nu).$$

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Sketch of proof V: putting everything together

- Let μ_i be measures with Frostman exponents s_i, $i = 1, 2, 3$, where $s_1 + s_2 + s_3 > 1$ and $s_1 + s_2 \leq 1$.
- Let $\eta = (\mu_1 \boxtimes \mu_2) \boxplus (\mu_1 \boxtimes \mu_2) \boxminus (\mu_1 \boxtimes \mu_2) \boxminus (\mu_1 \boxtimes \mu_2)$.
- By the white lie, $\|\eta^\delta\|_{L^2}^2 \lessapprox \delta^{s_1+s_2-1}$.
- By the Bourgain proposition ($N = 2$), we get

$$|(\eta^\delta \boxtimes \mu_3^\delta)(\xi)| \lessapprox \delta^{-(s_1+s_2+s_3-1)/2}.$$

- Using the Cauchy-Schwarz trick several times, we conclude

$$\left|\mathcal{F}(\mu_1^\delta \boxtimes \mu_2^\delta \boxtimes \mu_3^\delta)(\xi)\right| \lessapprox \delta^{-(s_1+s_2+s_3-1)/8}.$$

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Sketch of proof VI: how false is the white lie?

- We don't know how to prove the white lie, but if instead of four $\boxplus, \boxminus$ operations on $\mu \boxtimes \nu$, we switch to a huge number of operations (depending on the Frostman exponents), then it becomes morally true.
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Summary

- Multiplicative convolutions arise naturally when computing Fourier transforms of measures with some "product invariance".
- The additive structure of the Fourier transform encapsulated by the character formula $e^{x+y} = e^x e^y$ suggests that multiplicative convolutions should have Fourier decay.
- Bourgain and co-authors established versions of this in finite fields and in the Euclidean setting, but the latter was not quantitative.
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