Solvability of the Neumann problem for elliptic PDE's in chord-arc domains with very big pieces of good superdomains.

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HEZKUNTZA UNIBERTSITATE ETA IKERKETA SAILA

The Dirichlet Problem

$\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a bounded domain with n-Ahlfors regular $\partial\Omega$, i.e.,

$$c^{-1}\, r^n \leq \mathcal{H}^n(B(\xi, r) \cap \partial\Omega) \leq c\, r^n, \quad \text{for all } \xi \in \partial\Omega, r \in (0, \operatorname{diam} \partial\Omega).$$

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Let $g \in C_c(\partial\Omega)$.
The **Dirichlet problem** for the Laplacian on Ω with boundary data g is solvable if there exists

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such that

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

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A singular analogue of the maximum principle

Maximum Principle (known)

If $\Delta u = 0$ in Ω, $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, and $u|_{\partial\Omega} = f$, then

$$\max_{x\in\Omega} |u(x)| \leq \max_{\xi\in\partial\Omega} |f(\xi)|.$$

$\sigma := \mathcal{H}^n|_{\partial\Omega}$ — Surface measure on $\partial\Omega$.

$\gamma(\xi) := \{x \in \Omega : |x - \xi| < 2\,\mathrm{dist}(x, \partial\Omega)\}$ — Cone with vertex $\xi \in \partial\Omega$ and aperture 1.

$\mathcal{N}(u)(\xi) := \sup_{x\in\gamma(\xi)} |u(x)|$ — Non-tangential maximal function.

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If $\Delta u = 0$ in Ω and $u|_{\partial\Omega} \to f \in L^p(\sigma)$ non-tangentially σ-a.e., then

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L^p solvability of the Dirichlet, regularity, and Neumann problems

L^p-Dirichlet and regularity problems (D_p) and (R_p)

We say that the Dirichlet problem is L^p-solvable if

$$\|\mathcal{N}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega)} \quad \text{for all Dirichlet data } f \in C(\partial\Omega).$$

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Corkscrew and chord-arc domains

Corkscrew domain

We say that Ω is a corkscrew domain if for all $\xi \in \partial\Omega$ and $0 < t \leq \operatorname{diam}(\Omega)$ there exists a ball $B \subset B(\xi, t) \cap \Omega$ such that $r(B) \approx t$.

Chord-arc domain

$\Omega \subset \mathbb{R}^{n+1}$ is a chord-arc domain if

- $\partial\Omega$ is n-Ahlfors-regular,
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- Ω satisfies the Harnack chain condition (quantitative scale-invariant connectivity).

2-sided chord-arc domain

$\Omega \subset \mathbb{R}^{n+1}$ is a two-sided chord-arc domain if both Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ are chord-arc domains.

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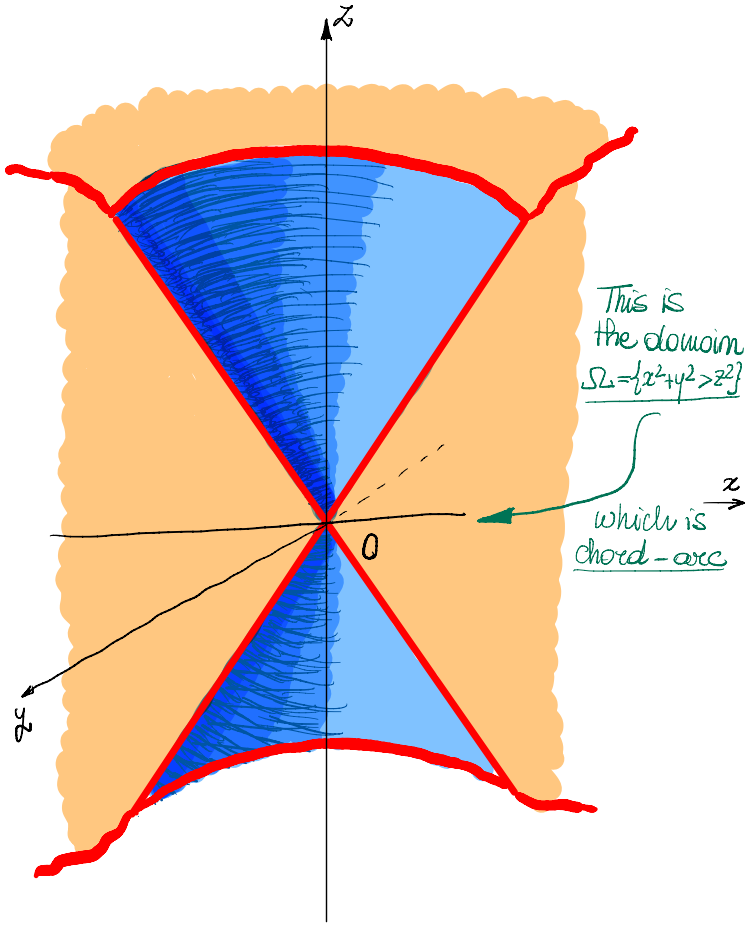
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A chord-arc domain in $\mathbb{R}^3$ which is not Lipschitz

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > z^2\}$$



Uniform rectifiability

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$E \subset \mathbb{R}^{n+1}$ is uniformly n-rectifiable if it is n-Ahlfors-regular and it has Big Pieces of Lipscitz Images, i.e., there are $M, \theta > 0$ such that for all $\xi \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g_{\xi,r} : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^{n+1}, \qquad \operatorname{Lip}(g_{\xi,r}) \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(\xi,r) \cap g_{\xi,r}(B_n(0,r))\big) \geq \theta\, r^n.$$

Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

If Ω is a chord-arc domain, then $\partial\Omega$ is uniformly n-rectifiable.

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Main Question

For which domains $\Omega \subset \mathbb{R}^{n+1}$ are (D_p), (R_p), and (N_p) solvable?

The L^p Dirichlet problem:

1977 Dahlberg showed that (D_p) is solvable for $p \in [2-\varepsilon, \infty]$ in Lipschitz domains, using harmonic measure.

1990 David and Jerison, and independently Semmes, showed that in chord-arc domains (D_p) is solvable for all p big enough.

2020 Azzam, Hofmann, Martell, M., and Tolsa showed that if Ω is a corkscrew domain with n-Ahlfors regular $\partial\Omega$, then (D_p) is solvable for p big enough iff Ω has interior big pieces of chord-arc subdomains.

\+ Many other important contributions and generalizations to elliptic operators.

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1981 For Lipschitz domains with connected $\partial\Omega$, Jerison and Kenig proved that (R_2) and (N_2) are solvable.

1984 For Lipschitz domains with connected $\partial\Omega$, Verchota showed that (R_p) is solvable for $1 < p \leq 2$.

1987 For Lipschitz domains with connected $\partial\Omega$, Dahlberg and Kenig showed that (R_p) and (N_p) are solvable for $1 < p < 2 + \varepsilon$.

Why $\partial\Omega$ should be connected?

Let $\Omega := B(0,2) \setminus \overline{B}(0,1)$ and define $f \in \mathrm{Lip}(\partial\Omega)$ by $f = 1$ on $\partial B(0,1)$ $f = 2$ on $\partial B(0,2)$. The solution of the Dirichlet problem is given by $u(x) = c_n |x|^{1-n} + (1 - c_n)$. So $|\nabla u(x)| > 0$ and $\nabla_t f = 0$. Similar example for (N_p).

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L^p solvability of the regularity and Neumann problems

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(R_p) is solvable in rough domains

M. and Tolsa (2021)

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded corkscrew domain with n-Ahlfors-regular $\partial\Omega$ that supports a weak p-Poincaré inequality. Then, if $\frac{1}{p} + \frac{1}{p'} = 1$,

$$(D_{p'}) \text{ is solvable } \iff (R_p) \text{ is solvable.}$$

For $n \geq 3$, we constructed a chord-arc domain Ω with connected boundary for which (R_p) is not solvable.

Remarks

- $(R_p) \Longrightarrow (D_{p'})$ holds for general $L = -\operatorname{div} A\nabla$.
- $(D_{p'}) \Longrightarrow (R_p)$ holds without weak p-Poincaré inequality if instead of $\nabla_t f$ we use the Hajlasz upper gradient $\nabla_H f$.
- M., Poggi, and Tolsa (2022) extended " $\Rightarrow$ " to $L = -\operatorname{div} A\nabla$ s.t. $\operatorname{dist}(\cdot, \Omega^c)|\nabla A|^2$ is a Carleson measure if $\partial\Omega$ is n-UR. Simultaneously, Dindoš, Hofmann, Pipher in Lipschitz graph domains.

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A few words about the boundary Poincaré inequality

Boundary Poincaré

We say that $\partial\Omega$ supports the weak (q,p)-Poincaré inequality if

$$\Big(\fint_{B(\xi,r)\cap\partial\Omega} |f-m_{\sigma,B}f|^q\,d\sigma\Big)^{\frac{1}{q}} \lesssim r\Big(\fint_{B(\xi,Cr)\cap\partial\Omega} |\nabla_t f|^p\,d\sigma\Big)^{\frac{1}{p}} \quad (1)$$

for any Lipschitz function f and all $\xi\in\partial\Omega$, $r>0$.

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If $\partial\Omega$ is n-Ahlfors regular supporting a weak $(1,n)$-Poincaré inequality then

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The main theorem in the most general form

Theorem 1 (M. and Tolsa (2024))

Assumptions:

- $\Omega \subset \mathbb{R}^{n+1}$ *is a bounded chord-arc domain.*
- $\partial\Omega$ *supports a* p*-Poincaré inequality for some* $p \in (1, 2)$*.*
- (R_q) *is solvable in* Ω *for some* $q > p$*.*
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  That is, for every $\xi \in \partial\Omega$ *and* $0 < r \leq \operatorname{diam}(\Omega)$*, there exists a* C*-chord-arc domain* $U_{\xi,r}$ *such that* $\Omega \cap B(\xi, r) \subset U_{\xi,r}$*,* (N_q) *is solvable in* $U_{\xi,r}$ *uniformly on* ξ *and* r*, and*

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Assumptions:

- $\star$ $\Omega \subset \mathbb{R}^{n+1}$ *is a bounded chord-arc domain.*
- $\star$ $\partial\Omega$ *supports a* p*-Poincaré inequality for some* $p \in (1, 2)$*.*
- $\star$ (R_q) *is solvable in* Ω *for some* $q > p$*.*
- $\star$ Ω *has Very Big Pieces of chord-arc superdomains where* (N_q) *is solvable.*

  That is, for every $\xi \in \partial\Omega$ *and* $0 < r \leq \operatorname{diam}(\Omega)$*, there exists a* C*-chord-arc domain* $U_{\xi,r}$ *such that* $\Omega \cap B(\xi, r) \subset U_{\xi,r}$*,* (N_q) *is solvable in* $U_{\xi,r}$ *uniformly on* ξ *and* r*, and*

$$\mathcal{H}^n(B(\xi, r) \cap \partial\Omega \setminus \partial U_{\xi,r}) \leq \varepsilon\, \mathcal{H}^n(B(\xi, r) \cap \partial\Omega),$$

  for some $\varepsilon > 0$ *small enough.*

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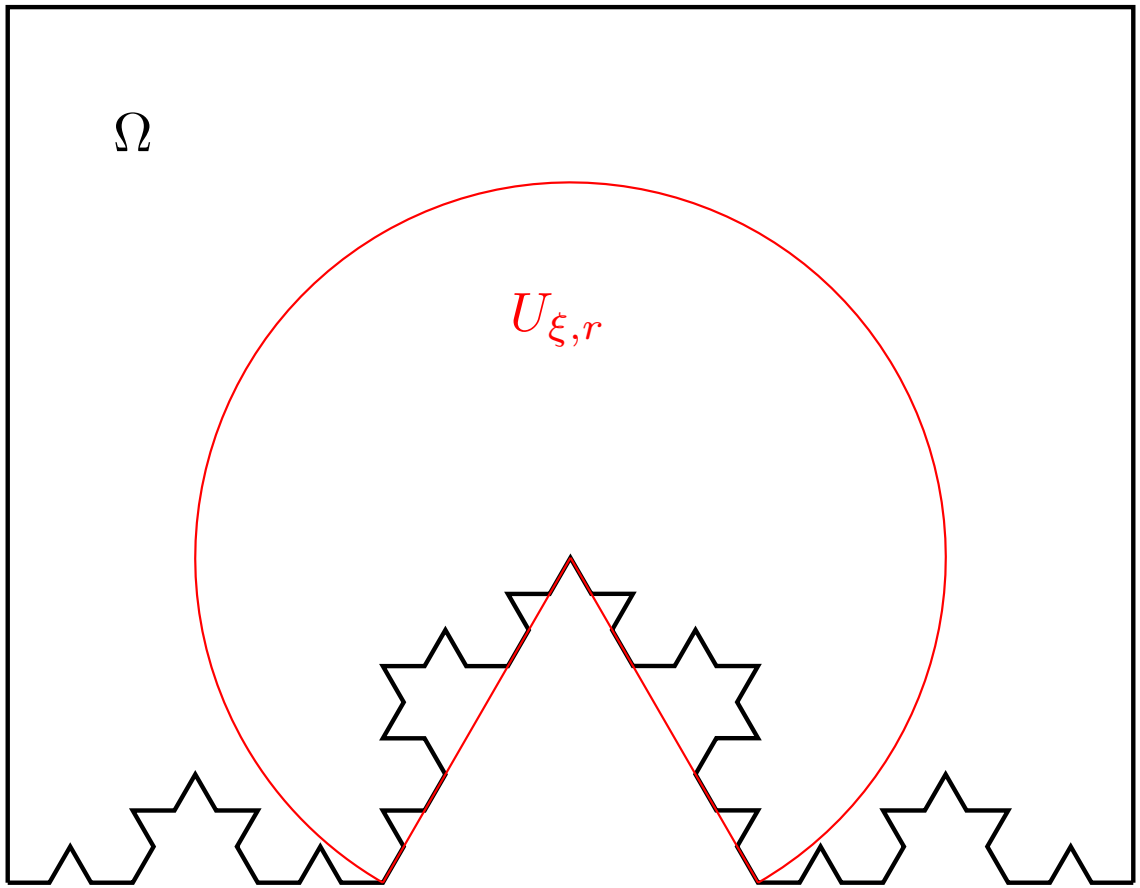
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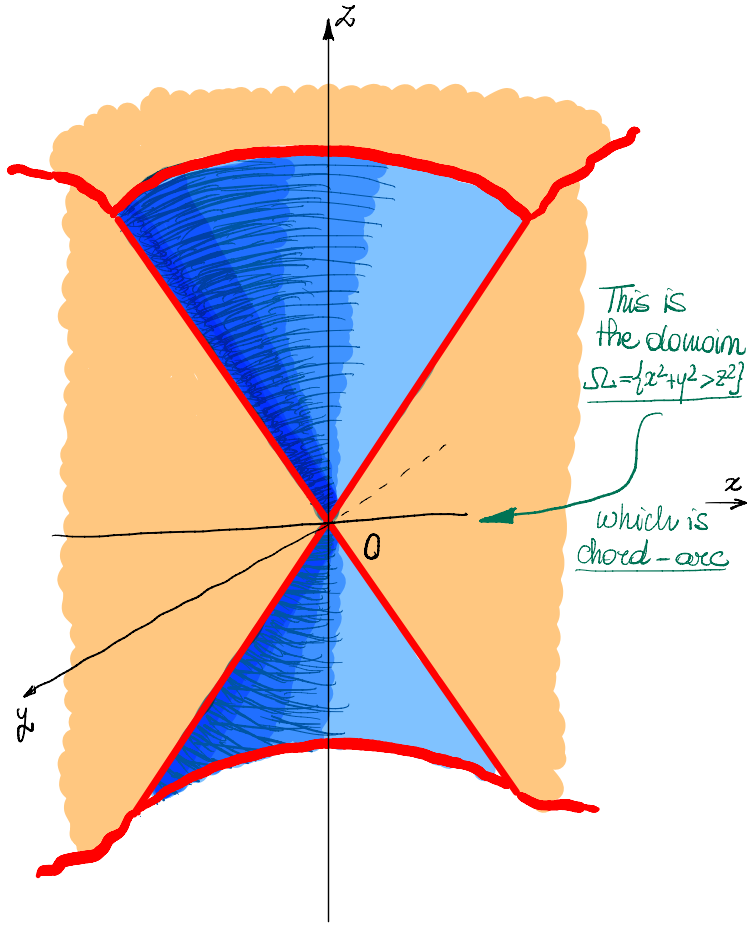
A domain and a superdomain

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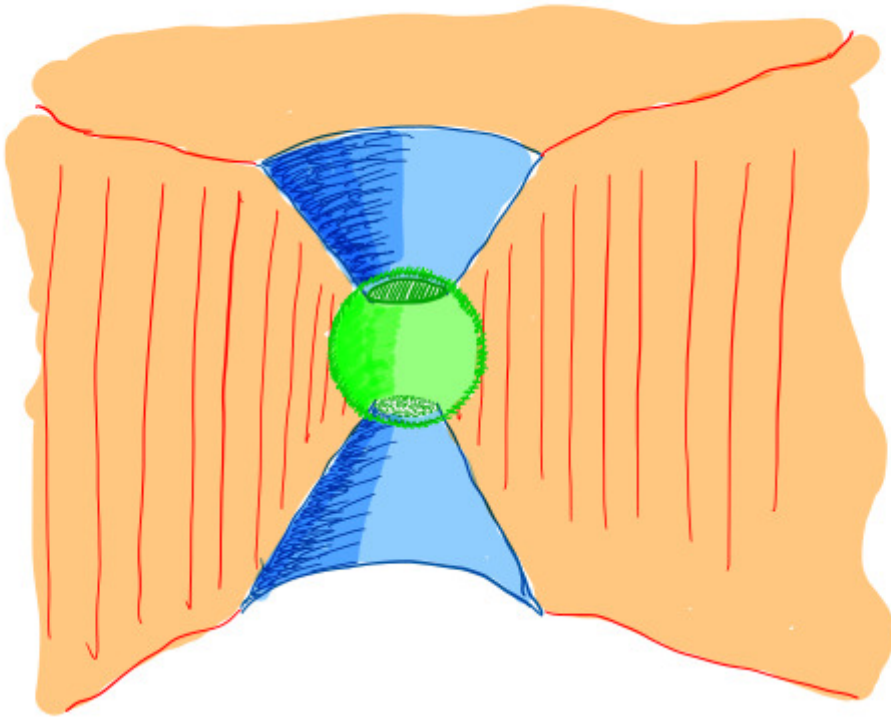


A domain and a superdomain

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > z^2\}$$



A domain and a superdomain



Very Big Pieces of Lipschitz superdomains

In the ball $B(0,r)$ define domain

$$\Omega_B = B(0,r) \cap (\Omega \cup B(0,\eta r))$$

Comments on the main theorem

Remarks

- ⋆ The $U_{\xi,r}$'s are superdomains for Ω relative to $B(\xi, r)$.
- ⋆ The smallness of ε depends on n, $|p-q|$, the chord-arc constants of $U_{\xi,r}$, the solvability of (R_q) in Ω, and the uniform solvability of (N_q) in the $U_{\xi,r}$'s.
- ⋆ We do not require any kind of flatness of $\partial\Omega$.
- ⋆ The theorem is also valid for general elliptic PDE's in divergence form with bounded coefficients with mild regularity.

Corollary

Let Ω be a bounded chord-arc domain whose boundary supports a weak 1-Poincaré inequality (e.g. a two-sided chord-arc domain). Then, if Ω has Very Big Pieces of Lipschitz superdomains, there exists $p \in (1, 2)$ such that (N_p) is solvable in Ω.

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General chord-arc domains

David-Jerison and independently Semmes (1990)

If Ω is a chord-arc domain, then it has Big Pieces of Lipschitz subdomains and superdomains.
That is, there exists $\varepsilon \in (0,1)$ such that for every $\xi \in \partial\Omega$ and $0 < r \leq \operatorname{diam}(\Omega)$ there exists C-Lipschitz domains $I(\xi, r)$ and $U_{\xi,r}$ such that $\mathrm{I}_{\xi,r} \subset \Omega \cap B(\xi, r)$ and $\Omega \cap B(\xi, r) \subset U_{\xi,r}$ and

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Remarks

- ⋆ If the previous corollary were valid for general $\varepsilon \in (0,1)$, then we would have that (N_p) is solvable in Ω for p small enough.
- ⋆ Difficult to arrive to ε close to 1 by iteration of the theorem because of the dependence of ε on (N_q) constant for $U_{\xi,r}$.

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Proof of the theorem: the tangential derivatives of Hofmann, Mitrea, Taylor

Definition of HMT derivatives

For a Lipschitz function $f : \partial\Omega \to \mathbb{R}$ and $1 \leq j, k \leq n+1$,

$$\partial_{t,j,k} f := \nu_j \, (\partial_k f)|_{\partial\Omega} - \nu_k \, (\partial_j f)|_{\partial\Omega},$$

where the ν_i's are the components of the outer unit normal ν and ∂_i's are the components of ∇_t.

Integration by parts

For "nice enough functions" f, g on $\partial\Omega$,

$$\int_{\partial\Omega} \partial_{t,j,k} f \; g \, d\sigma = - \int_{\partial\Omega} f \, \partial_{t,j,k} g \, d\sigma.$$

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Proof of the theorem: the $L^{p'}$ rough Neumann problem

Rough Neumann problem

We say that the $L^{p'}$ rough Neumann problem $(N^R_{p'})$ is solvable if, for all $1 \leq j,k \leq n+1$, and every Lipschitz function $g \in L^{p'}(\partial\Omega)$, the solution $u \in W^{1,2}(\Omega)$ of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_{t,j,k} g & \text{in } \partial\Omega, \\ \int_{\partial\Omega} \mathrm{Tr}_{\Omega\to\partial\Omega}(u)\, d\sigma = 0 \end{cases} \tag{2}$$

satisfies

$$\|\mathcal{N}(u)\|_{L^{p'}(\sigma)} \lesssim \|g\|_{L^{p'}(\sigma)}.$$

$$\langle \partial_\nu u, \phi\rangle := \ell(\phi) := \int_\Omega \nabla u \cdot \nabla\phi, \qquad \text{for any } \phi \in C_c^\infty(\mathbb{R}^{n+1}.$$

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$L^{p'}$ Rough Neumann problem // $L^{p'}$ Dirichlet problem

Representation of solution of $(D_{p'})$ with data f

$$u(x) = \int_{\partial\Omega} \partial_\nu G(x,\xi)\, f(\xi)\, d\sigma(\xi)$$

Representation of solution of $(N_{p'})$ with data in divergence form

$$w(x) = \int_{\partial\Omega} N(x,\xi)\, h(\xi)\, d\sigma(\xi) \text{“} = \text{”} - \int_{\partial\Omega} \nabla_t N(x,\xi)\, \cdot \vec{g}(\xi)\, d\sigma(\xi)$$

for $\vec{g} \in L^{p'}(\sigma)$ and so, roughly speaking, $h \in W^{-1,p'}(\sigma)$.

- $\star$ $\partial_\nu G(x,\xi)$ is non-negative and we can use maximum principle.
- $\star$ $\partial_{t,j,k} N(x,\xi)$ does not have a constant sign and maximum principle is not available.

If (R_p) is solvable, then $(N_p) \Leftrightarrow (N^R_{p'})$

Proof of $(N_p) \Rightarrow (N^R_{p'})$.

Let u, g be as in equation (2). Let $\varphi \in L^p(\sigma)$ be Lipschitz, and let $v : \Omega \to \mathbb{R}$ be the solution of the Neumann problem with boundary data $\varphi - m_{\sigma,\partial\Omega}\varphi$.

$$\int_{\partial\Omega} u\,\varphi\,d\sigma = \int_{\partial\Omega} u\,(\varphi - m_{\sigma,\partial\Omega}\varphi)\,d\sigma$$
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So,

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$$\begin{aligned}\int_{\partial\Omega} u\,\varphi\,d\sigma &= \int_{\partial\Omega} u\,(\varphi - m_{\sigma,\partial\Omega}\varphi)\,d\sigma \\ &= \int_{\partial\Omega} u\,\partial_\nu v\,d\sigma = \int_\Omega \nabla u\cdot\nabla v = \int_{\partial\Omega} \partial_\nu u\,v\,d\sigma \\ &= \int_{\partial\Omega} \partial_{t,j,k} g\,v\,d\sigma = -\int_{\partial\Omega} g\,\partial_{t,j,k} v\,d\sigma.\end{aligned}$$

So,

$$\begin{aligned}\left|\int_{\partial\Omega} u\,\varphi\,d\sigma\right| &\leq \|g\|_{L^{p'}(\sigma)}\,\|\partial_{t,k,j} v\|_{L^p(\sigma)} \lesssim \|g\|_{L^{p'}(\sigma)}\,\|\mathcal{N}_\Omega(\nabla v)\|_{L^p(\sigma)} \\ &\lesssim \|g\|_{L^{p'}(\partial\Omega)}\,\|\partial_\nu v\|_{L^p(\sigma)} \lesssim \|g\|_{L^{p'}(\partial\Omega)}\,\|\varphi\|_{L^p(\sigma)}.\end{aligned}$$

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$(N_p) \Rightarrow (N_{p'}^R)$

Therefore, by duality,

$$\|u|_{\partial\Omega}\|_{L^p(\sigma)} \lesssim \|g\|_{L^{p'}(\partial\Omega)}.$$

Since $(R_p) \Rightarrow (D_{p'})$, we get

$$\|\mathcal{N}(u)\|_{L^{p'}(\sigma)} \lesssim \|u|_{\partial\Omega}\|_{L^p(\sigma)} \lesssim \|g\|_{L^{p'}(\sigma)}.$$

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The good λ inequality (1)

We prove the main theorem by a good λ inequality for $(N^R_{p'})$ and a bootstrapping argument.

- ⋆ By the solvability of $(D_{p'})$ in Ω, it suffices to estimate u on $\partial\Omega$.
- ⋆ For the solution u of (N^R) with boundary data $\partial_{t,j,k} g$ and $\lambda > 0$, we set

$$V_\lambda = \big\{\zeta \in \partial\Omega : M_\sigma(u)(\zeta) > \lambda\big\}.$$

- ⋆ We decompose V_λ into Whitney boundary cubes and we estimate $V_{A\lambda}$ for some $A > 2$.
- ⋆ We consider the solvability of rough Neumann from $L^{p'}(\sigma)$ to $L^{p',\infty}(\sigma)$.
- ⋆ Then, we show that its "norm" C_* satisfies

$$C_*^{p'} \lesssim C(A)\big(\widetilde{C}_q^{p'}\, C(K) + C_*^{p'}\, \gamma(K,\varepsilon)^{p'}\big),$$

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Thank you for your attention!