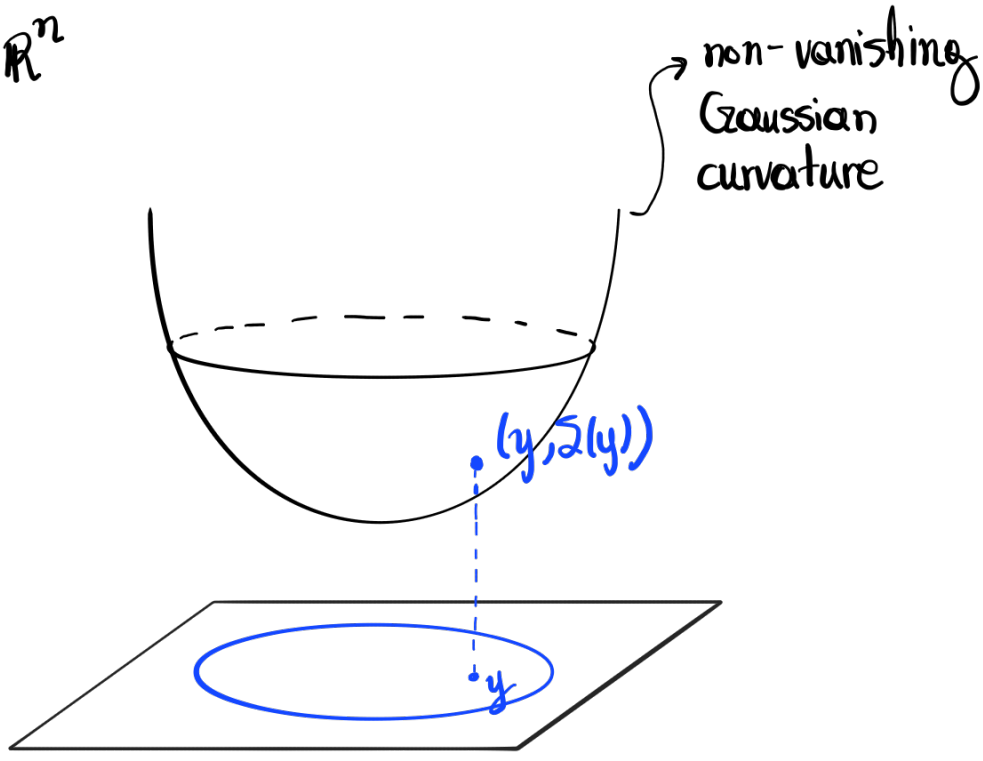
Hörmander-type operators and Schrödinger propagators: Beyond universal bounds

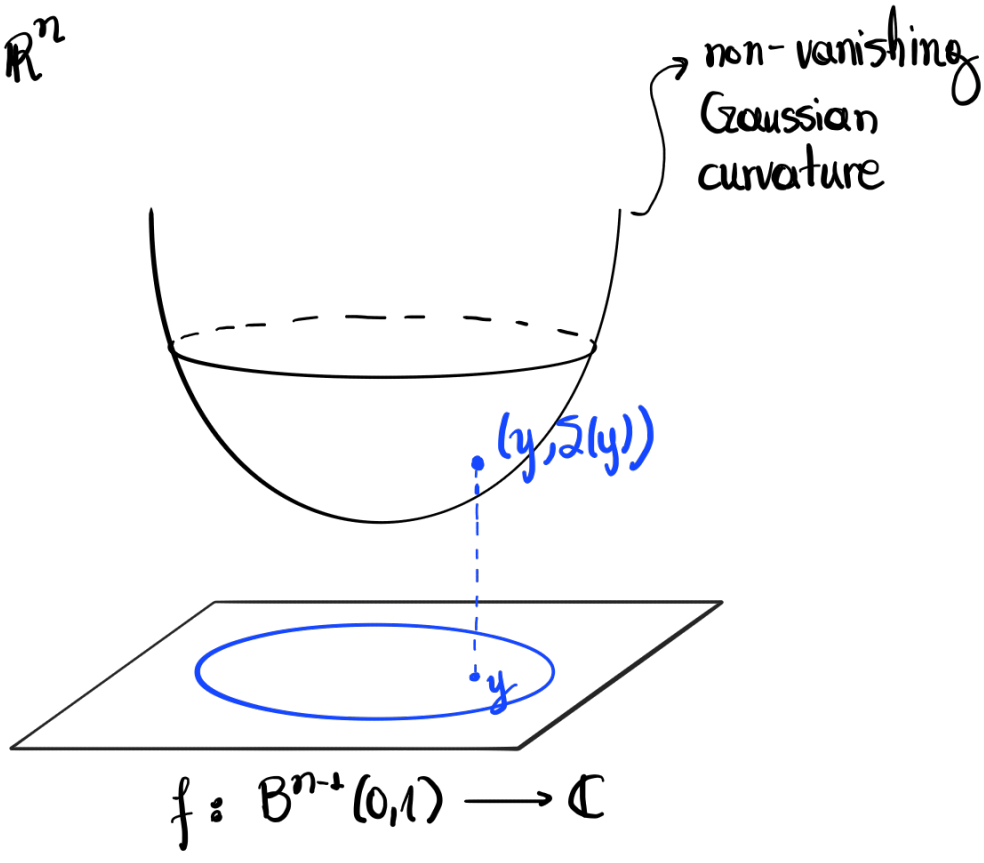
Joint work with:

Mingfeng Chen
Shengwen Gan
Shaoming Guo
Jonathan Hickman
James Wright

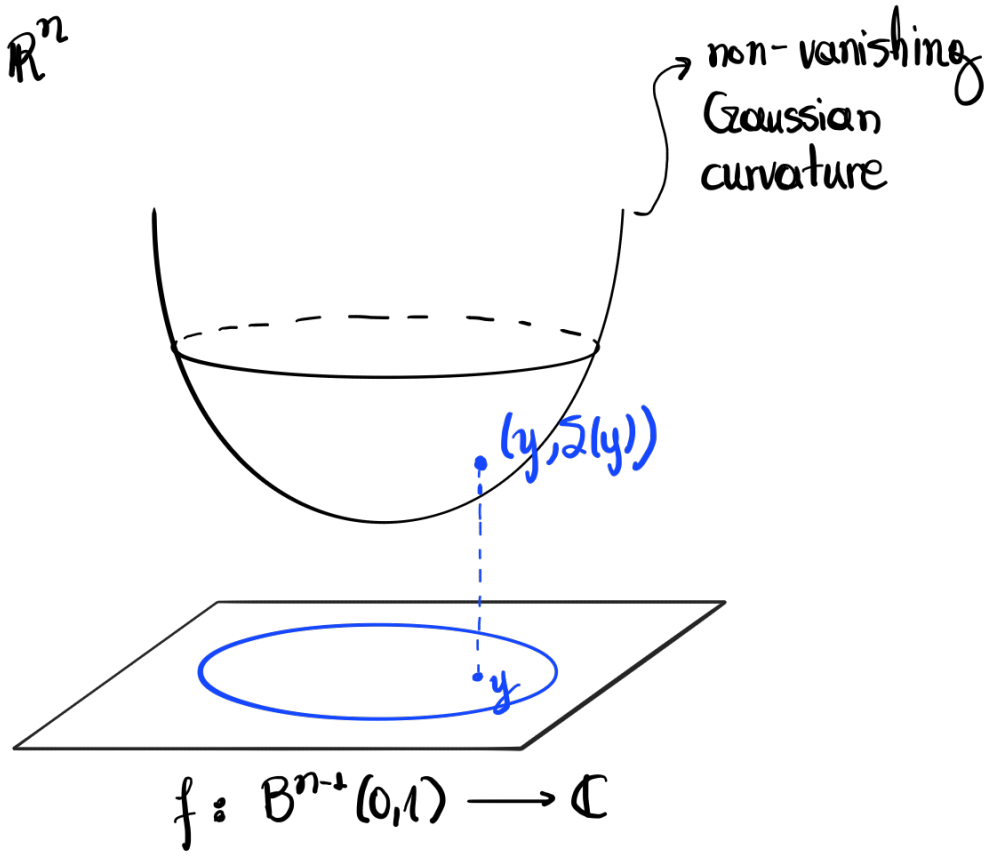
The extension operator:

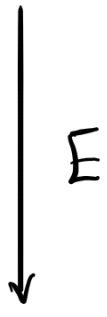


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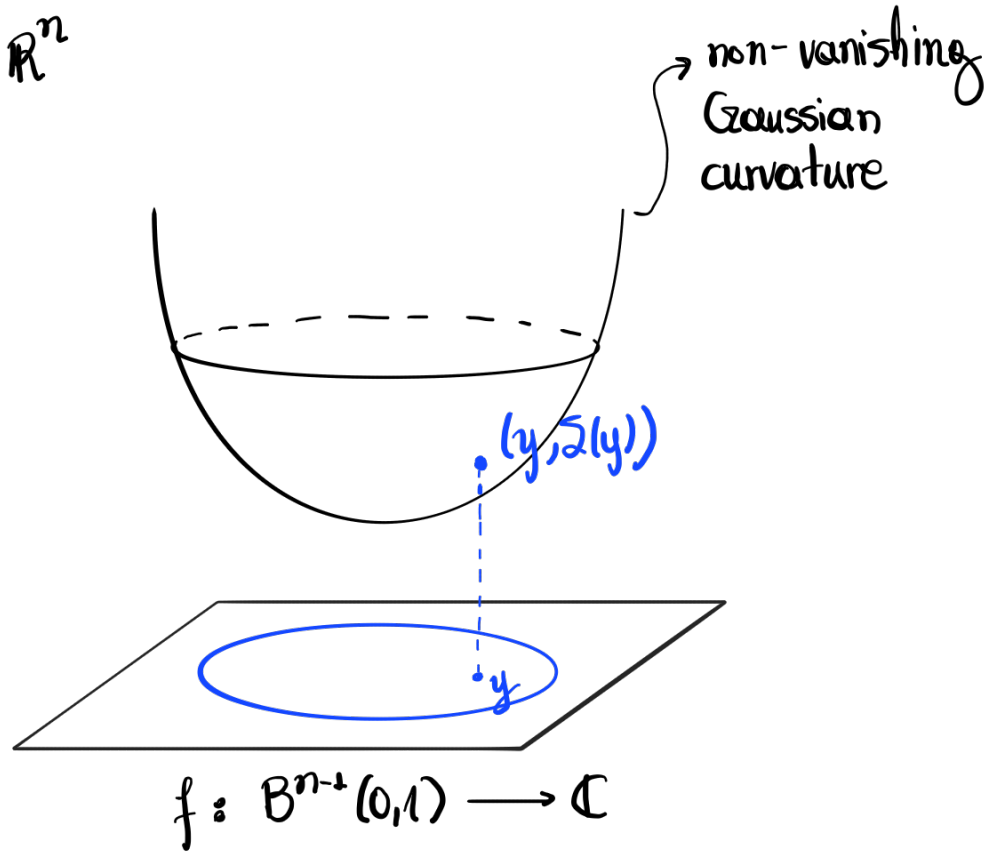
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$\mathbb{R}^n$

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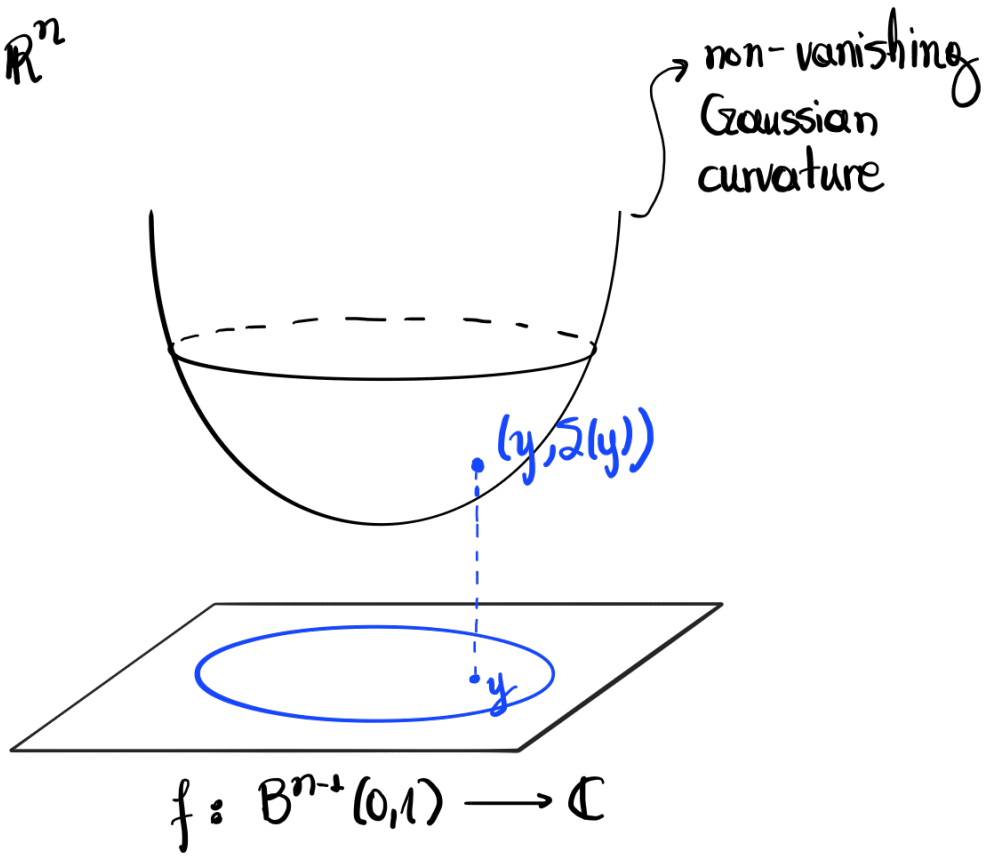


E

$\mathbb{R}^n$

$\cdot (x,t)$

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$\mathbb{R}^n$

$\cdot (x,t)$

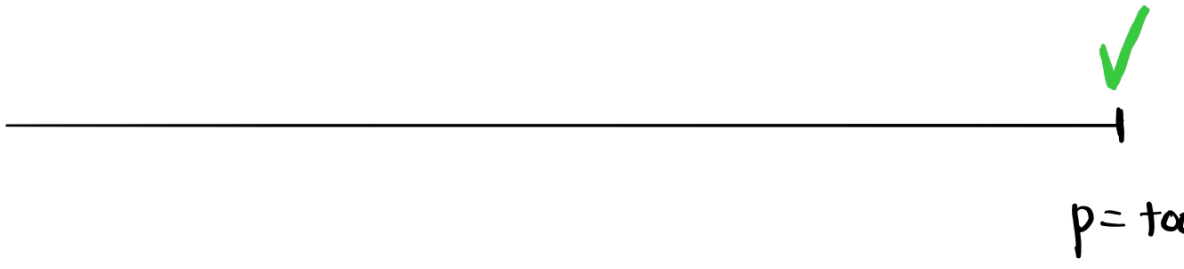
$$Ef(x,t) := \int_{y \in B^{n-1}(0,1)} e^{i\langle (x,t), (y,\Sigma(y)) \rangle} f(y)\,dy$$

Question: For which p do we have

$$\|Ef\|_{L^p(B_R)} \lessapprox \|f\|_p \quad ?$$

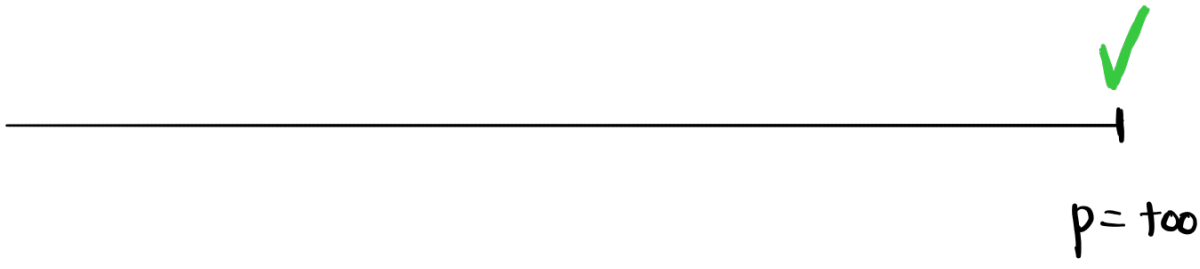
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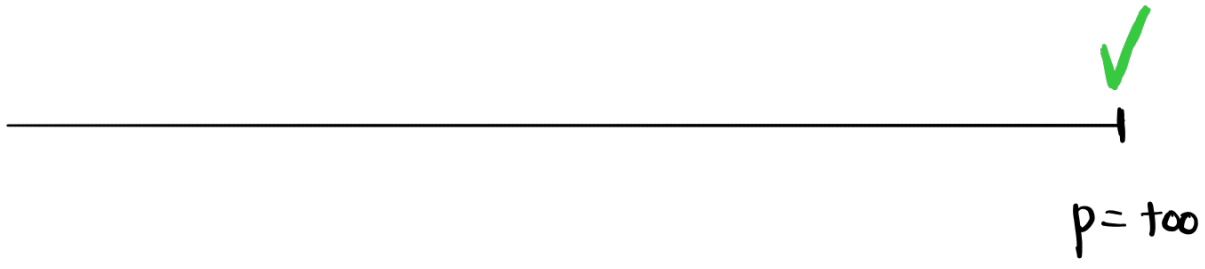
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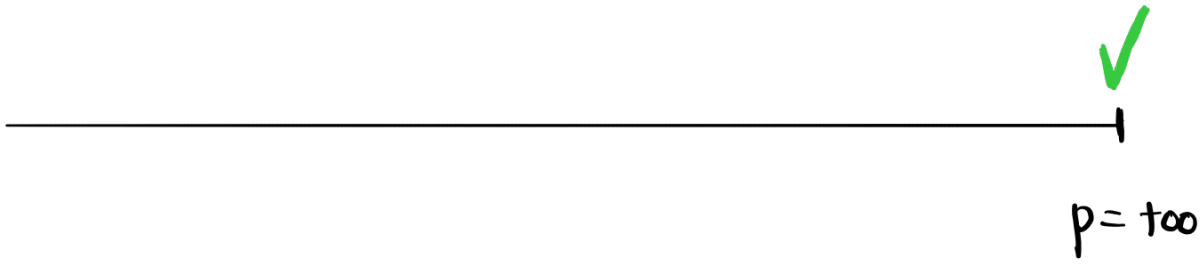
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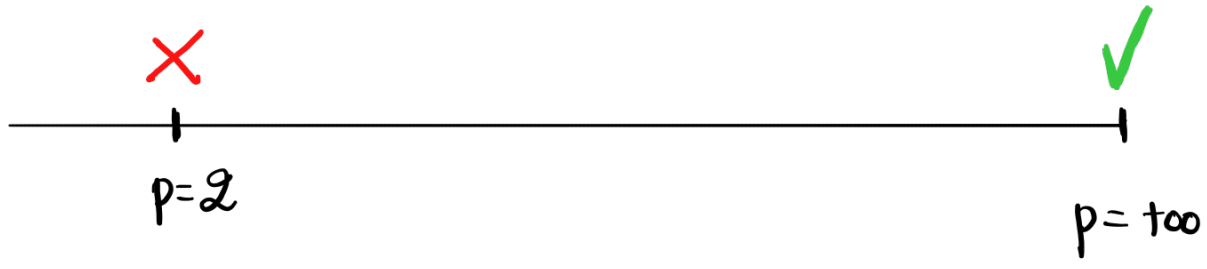
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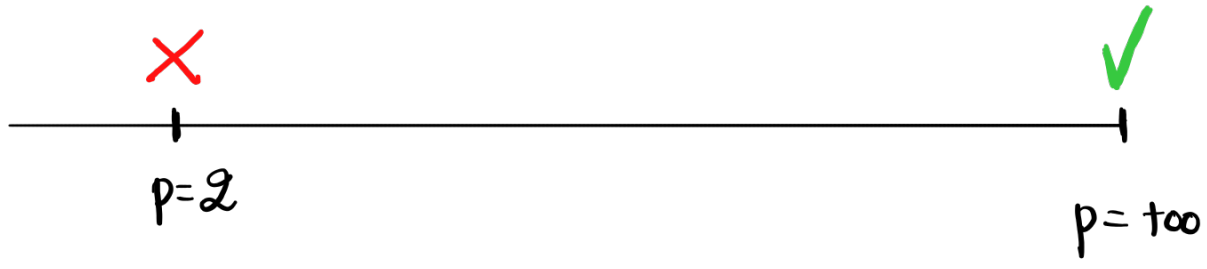
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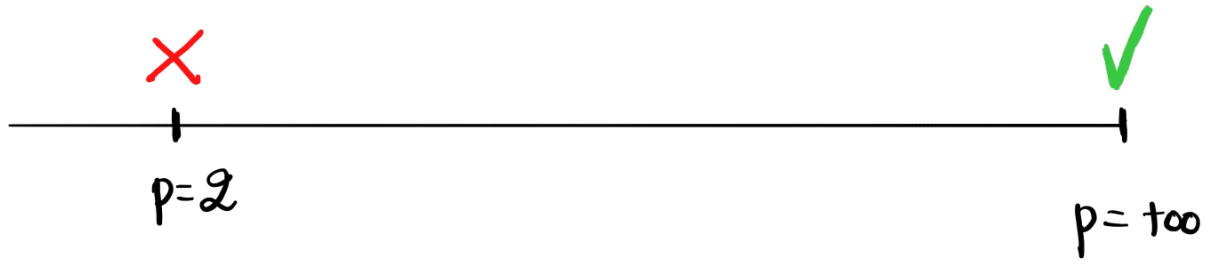
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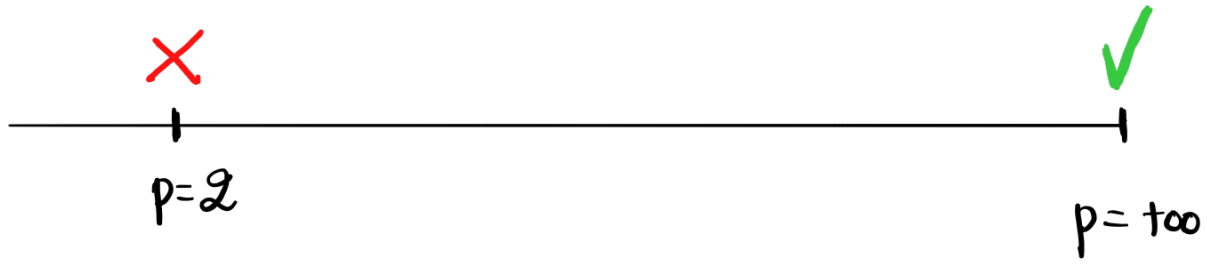
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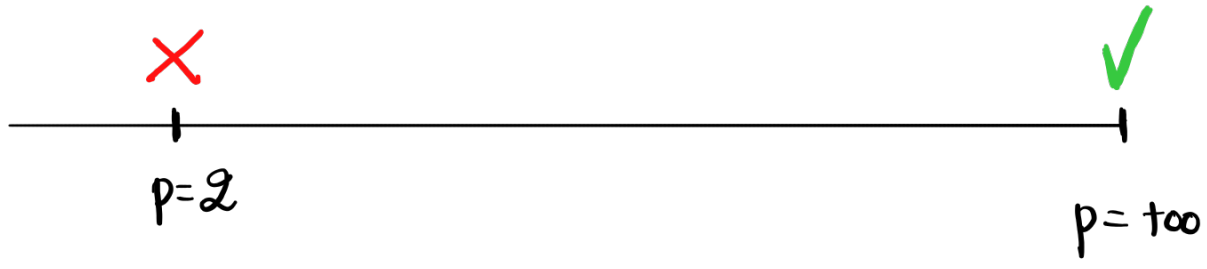


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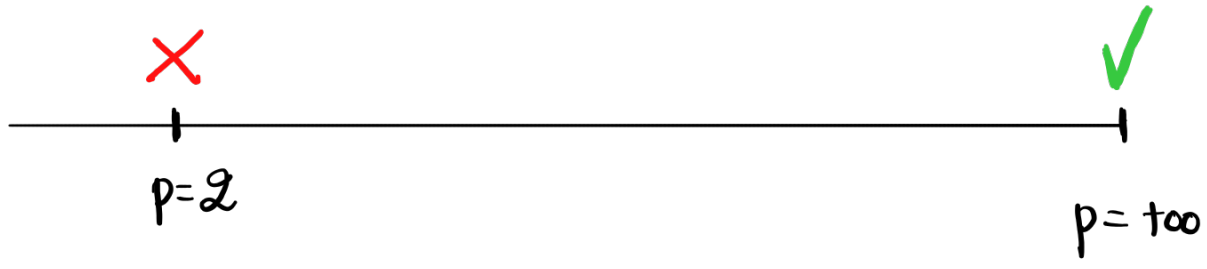
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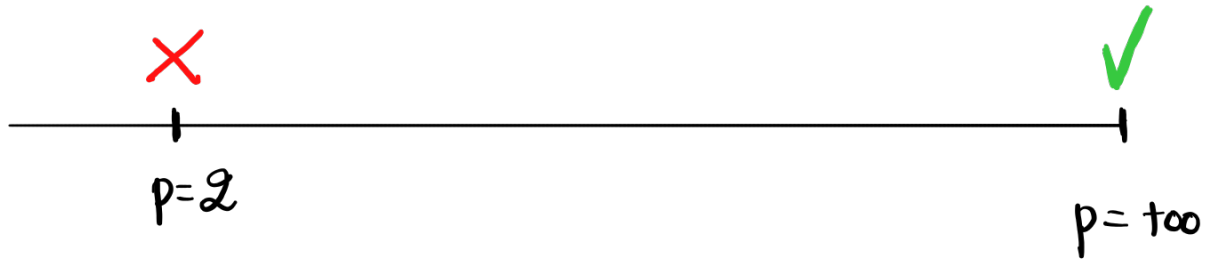
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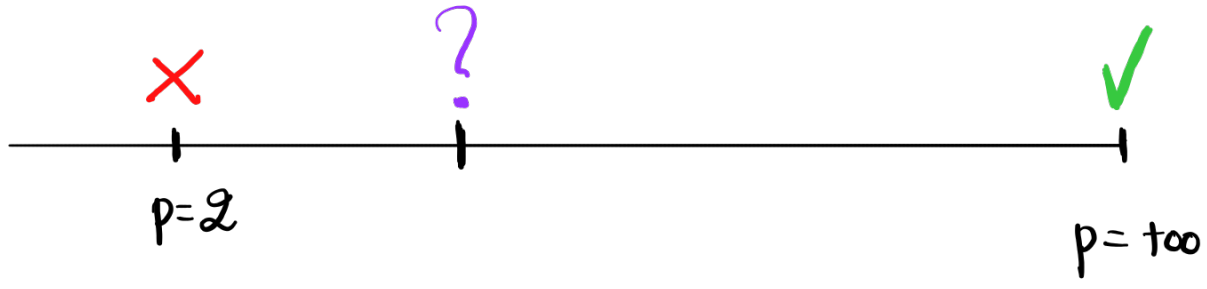
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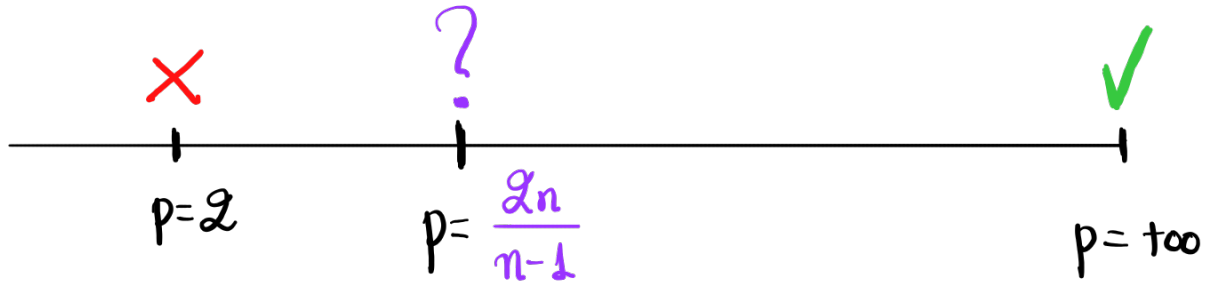
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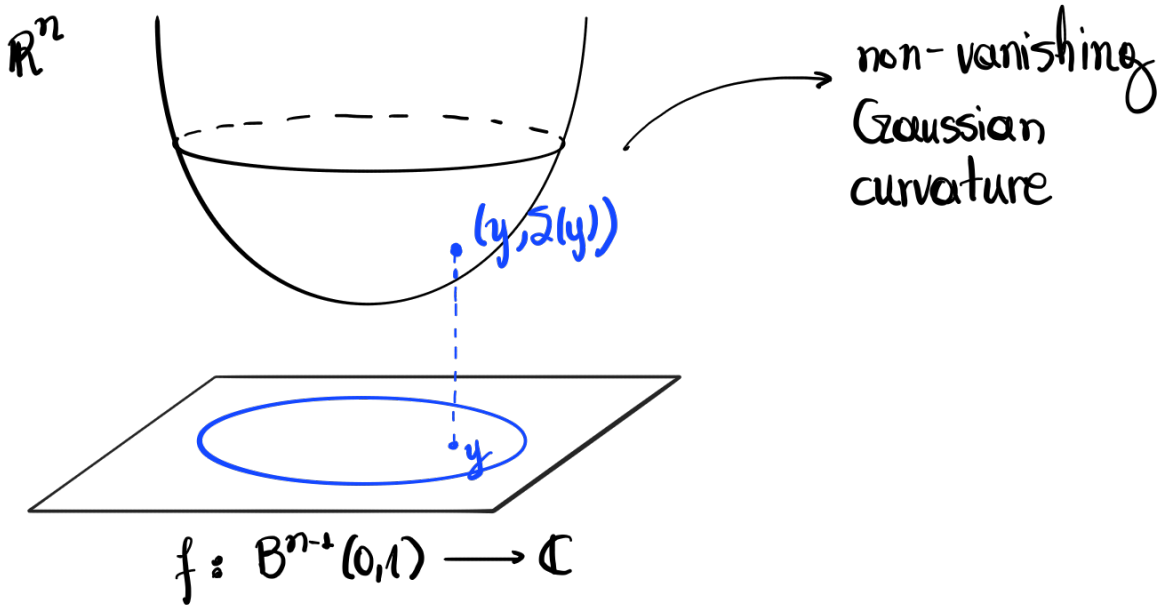
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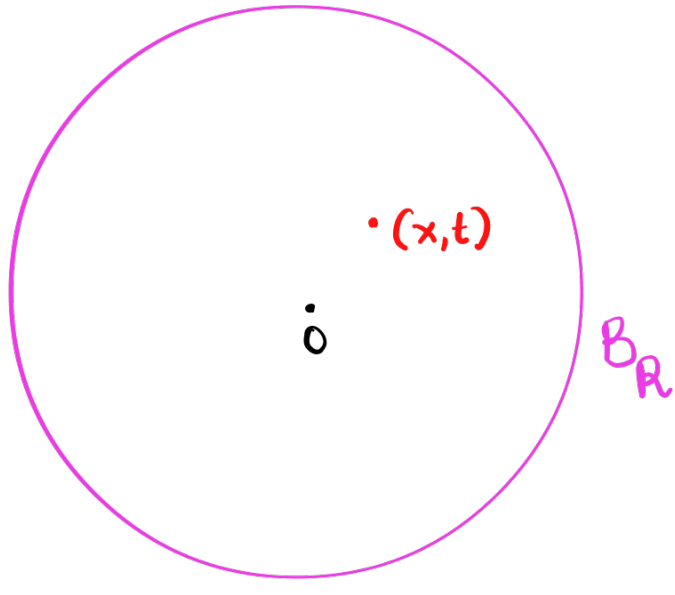
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Extension operators:



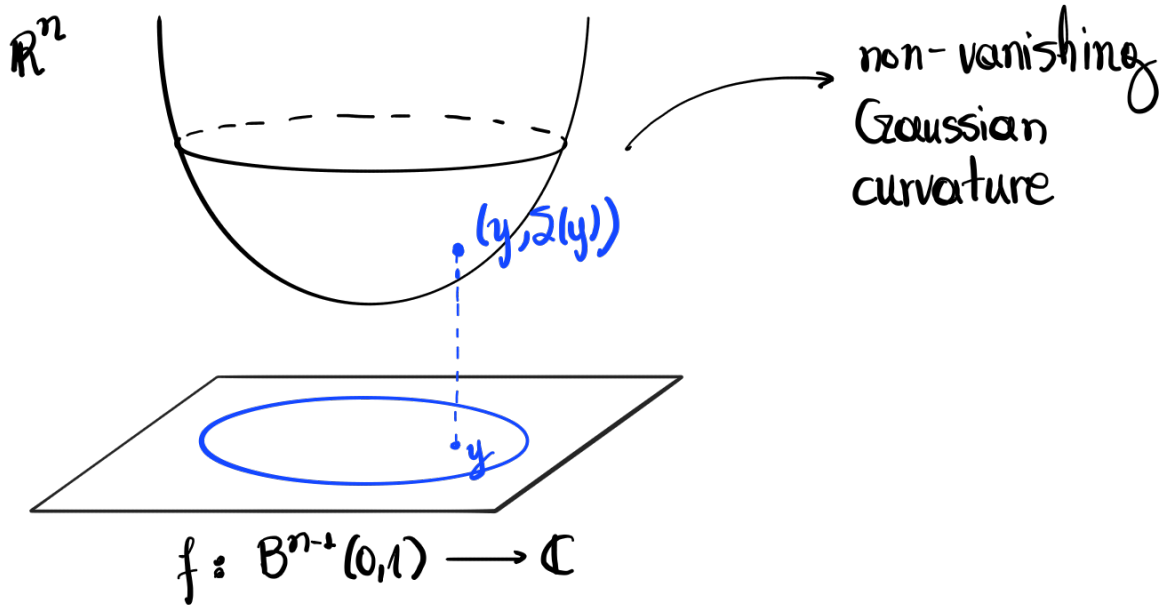
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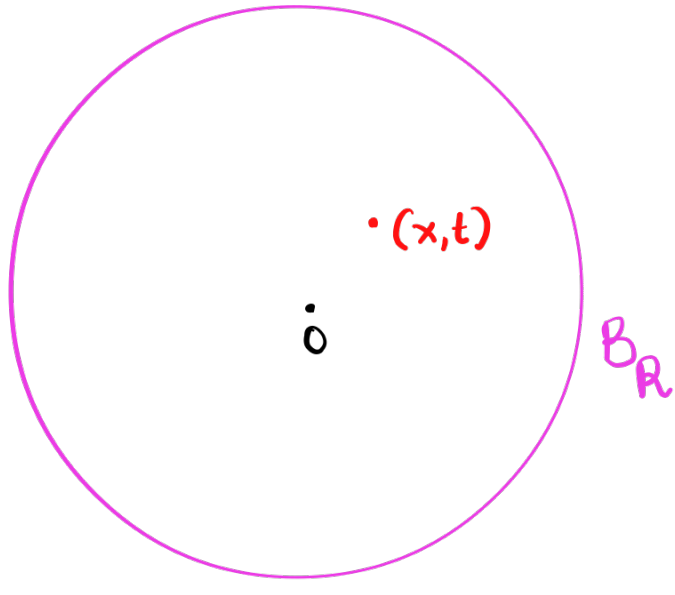
Restriction conjecture (Stein, '60s):

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Hörmander-type operators:



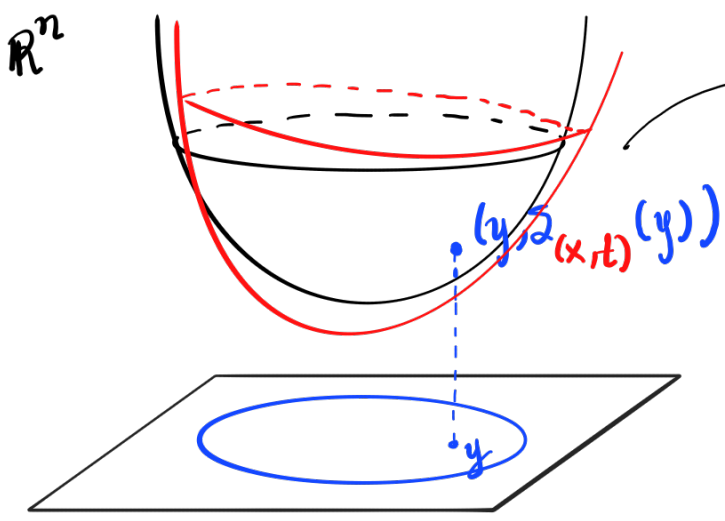
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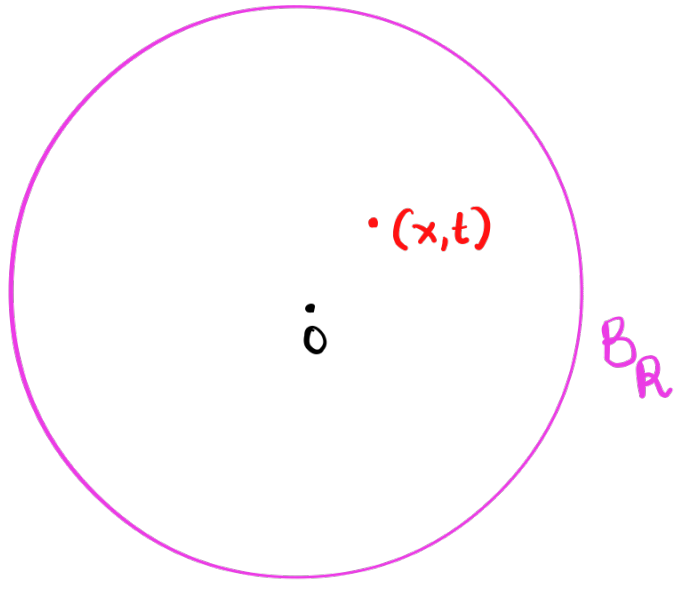
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non-vanishing Gaussian curvature

$$f : B^{n-1}(0,1) \longrightarrow \mathbb{C}$$

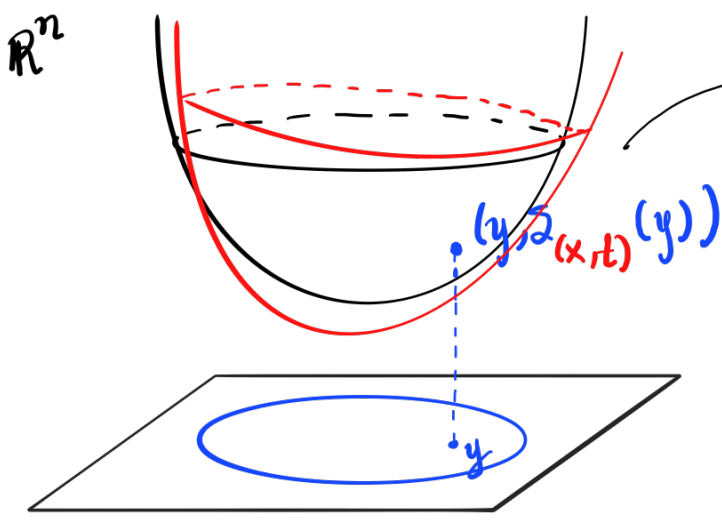
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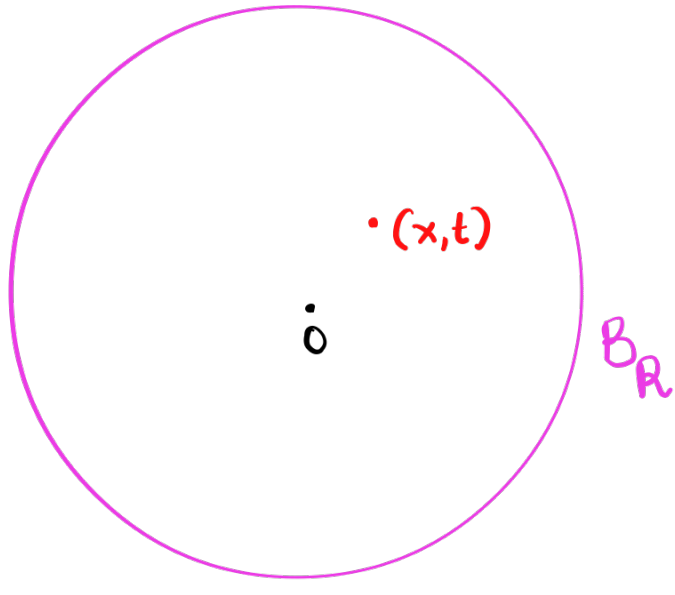
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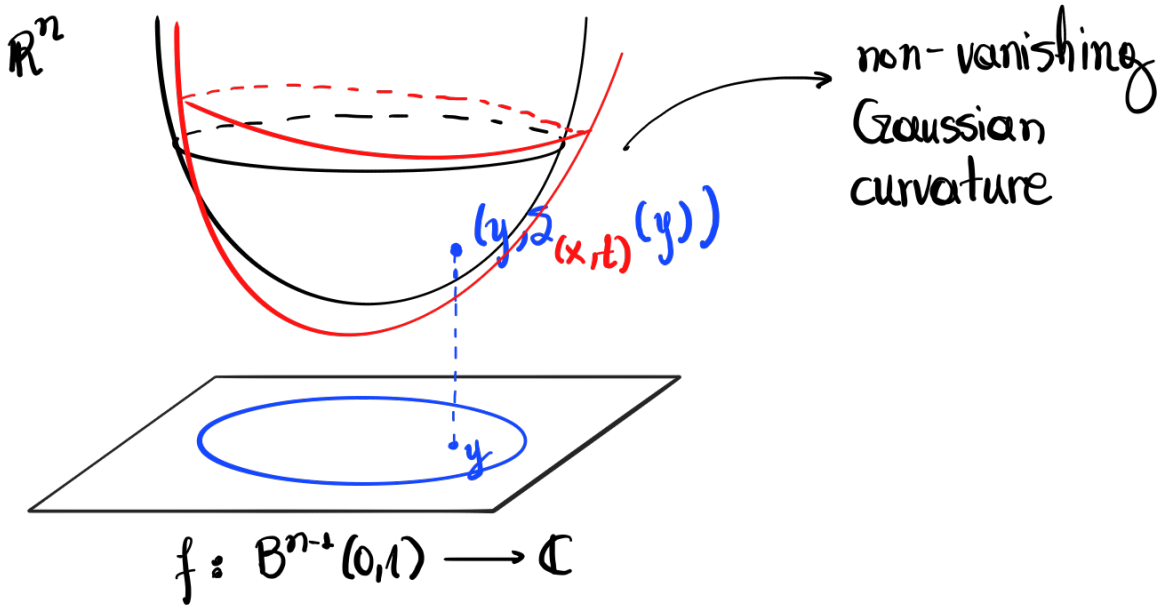
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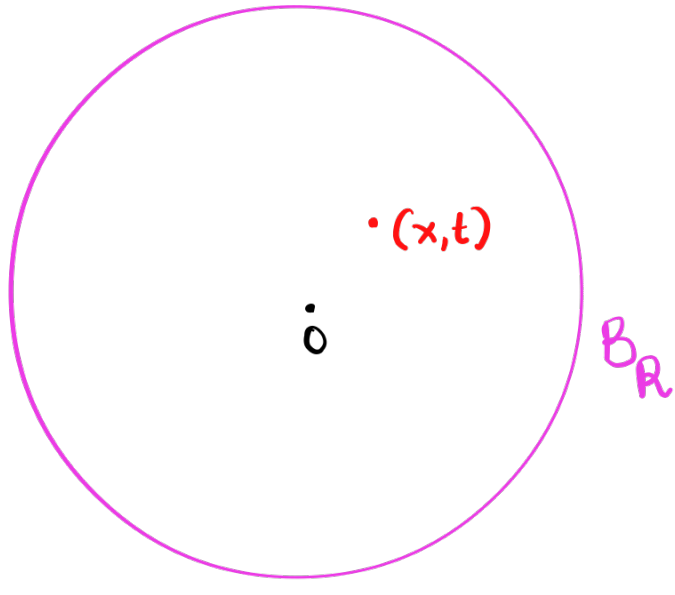
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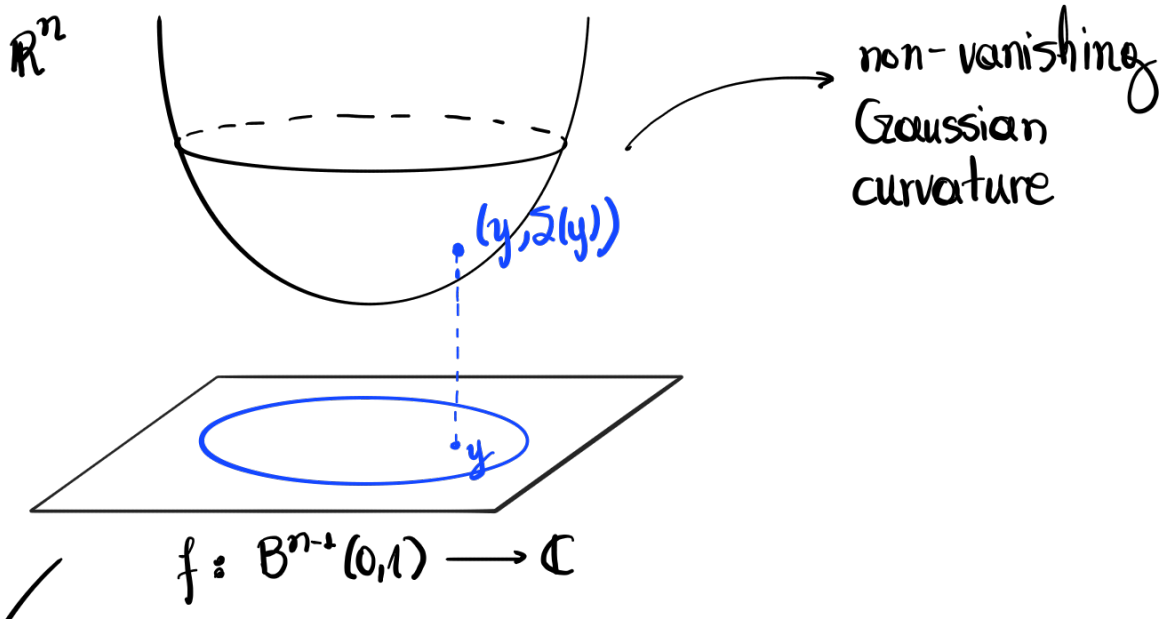
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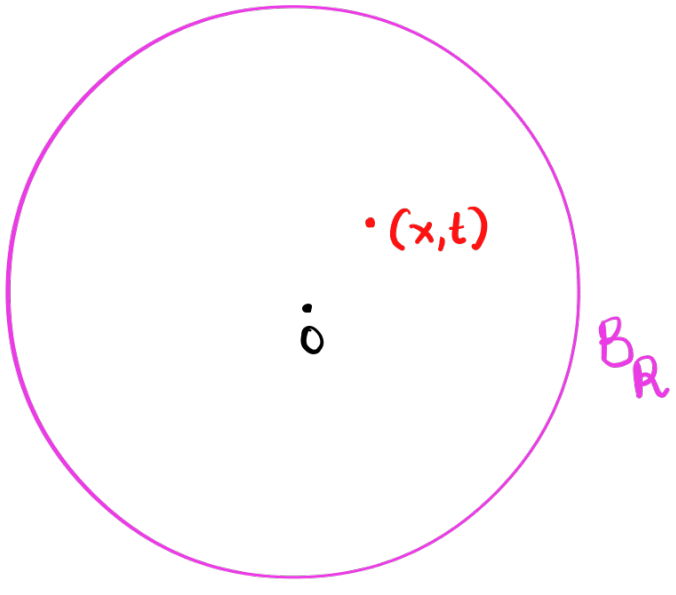
Hörmander's question ('73):

¿ $\|H^R f\|_{L^p(B_R)} \lesssim \|f\|_p$, $\forall\, p \geq \frac{2n}{n-1}$?

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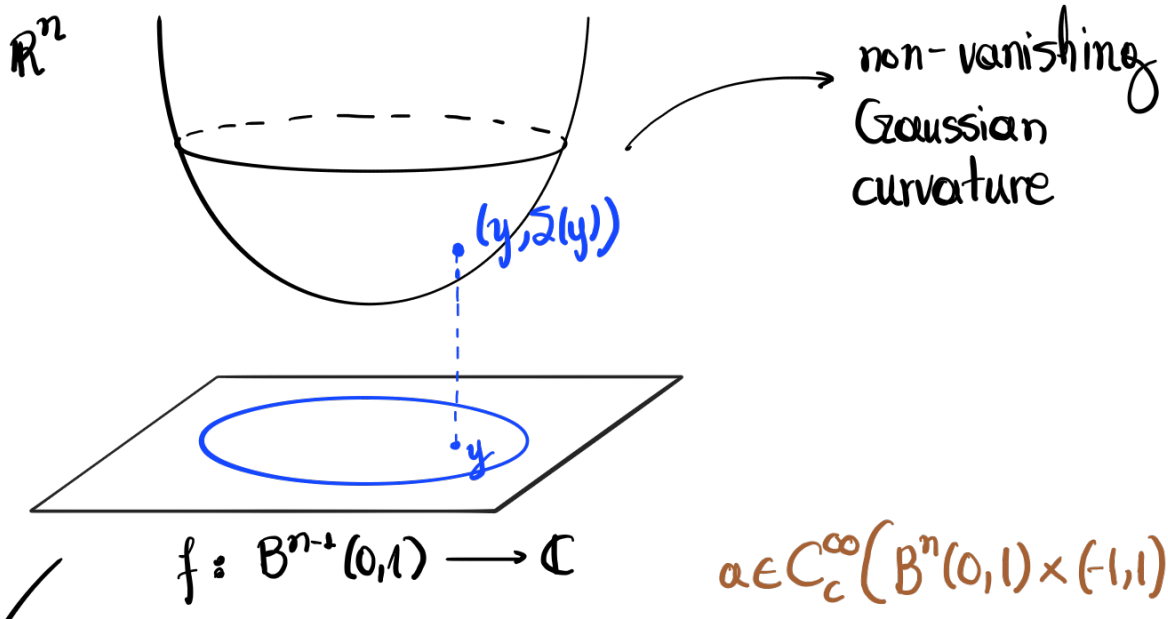
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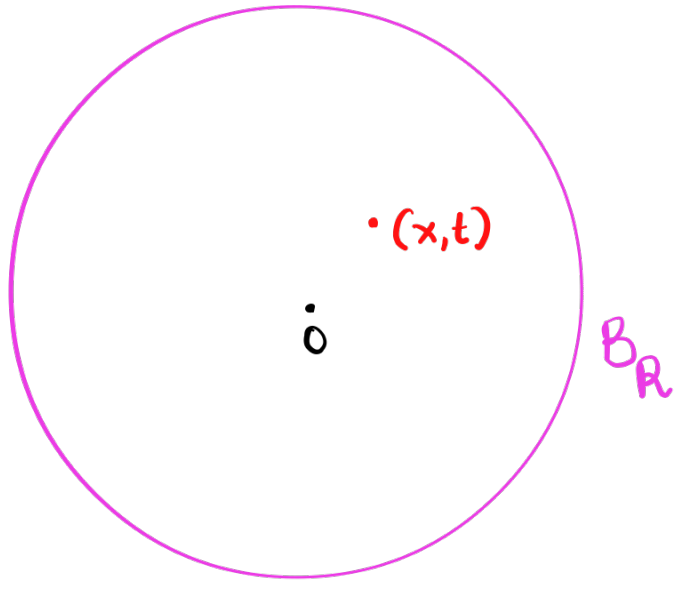
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$$a \in C_c^\infty\left(B^n(0,1) \times (-1,1) \times B^{n-1}(0,1)\right)$$

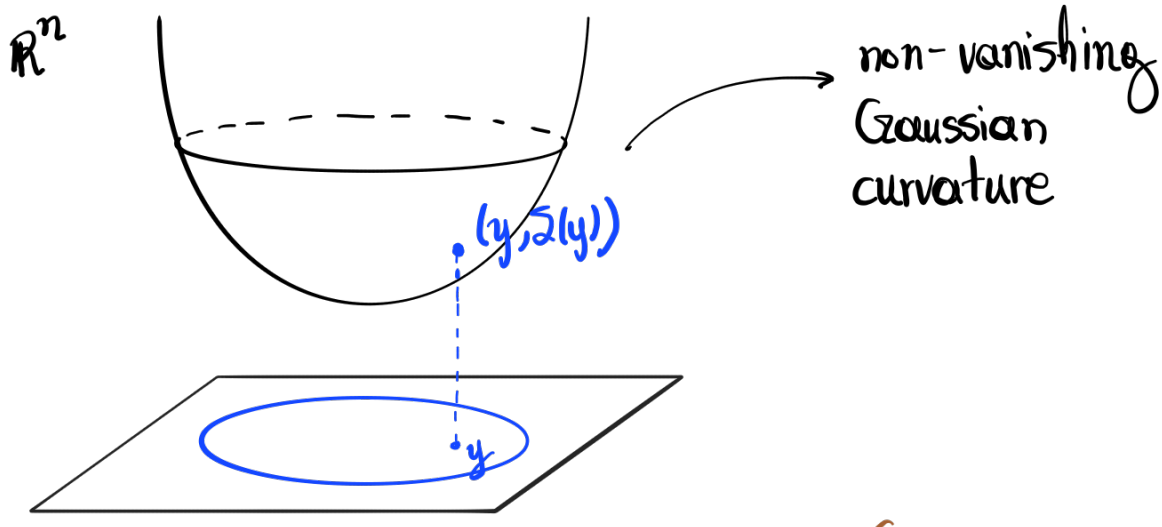
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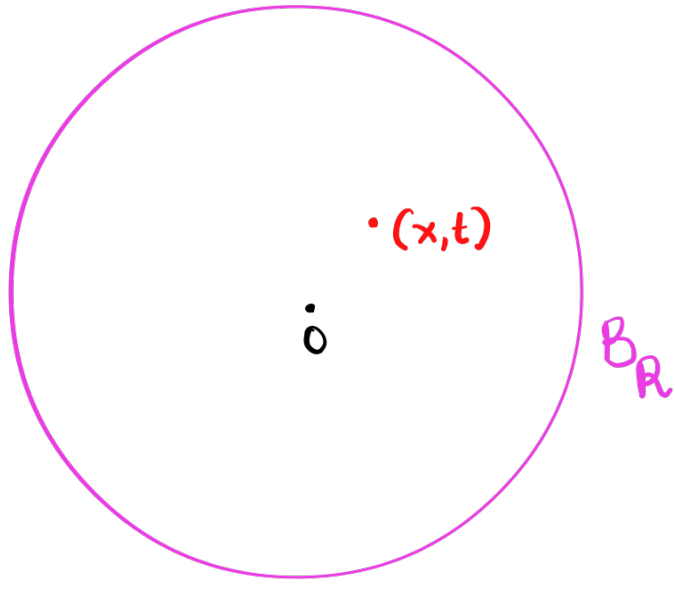
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$f: \mathbb{R}^{n-1} \longrightarrow \mathbb{C}$, $\quad a \in C_c^\infty\left(B^n(0,1)\times(-1,1)\times B^{n-1}(0,1)\right)$

$\operatorname{supp} f \subseteq B^{n-1}(0,1)$

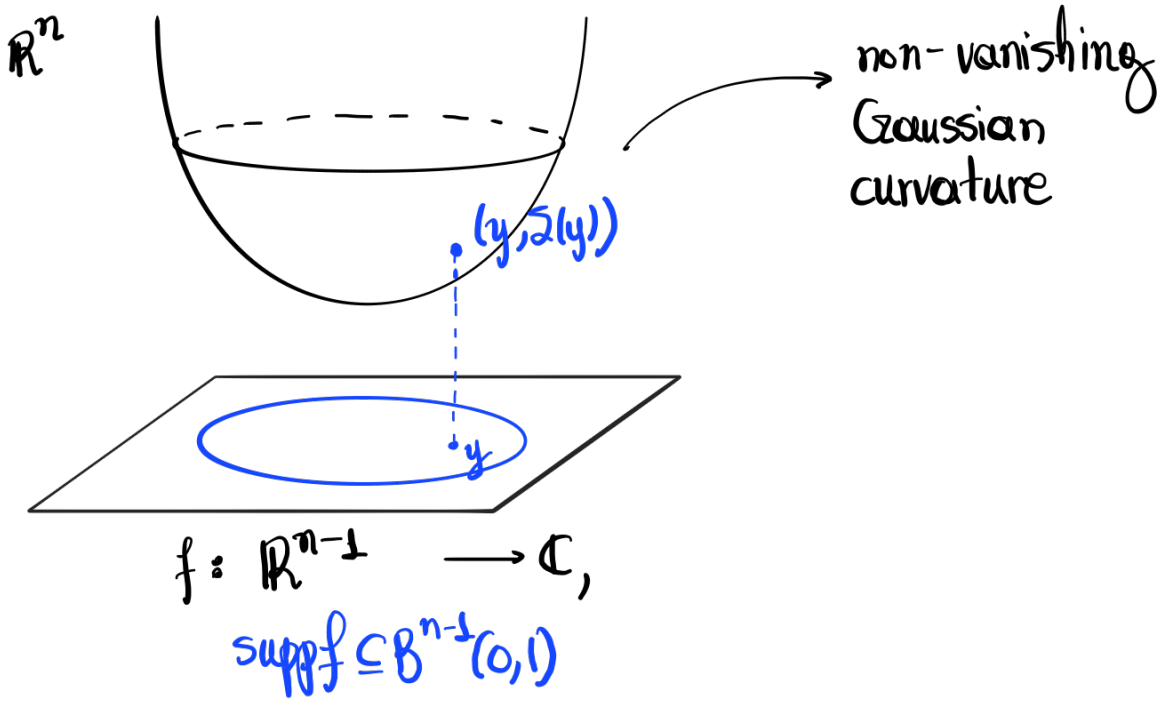
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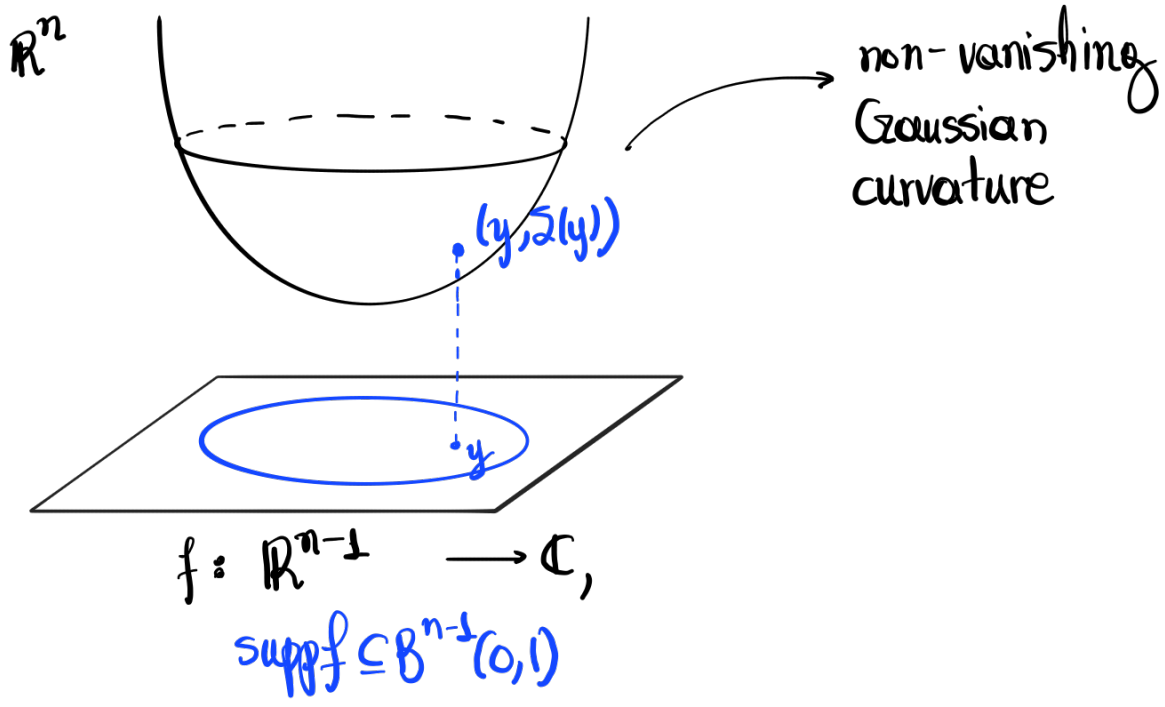


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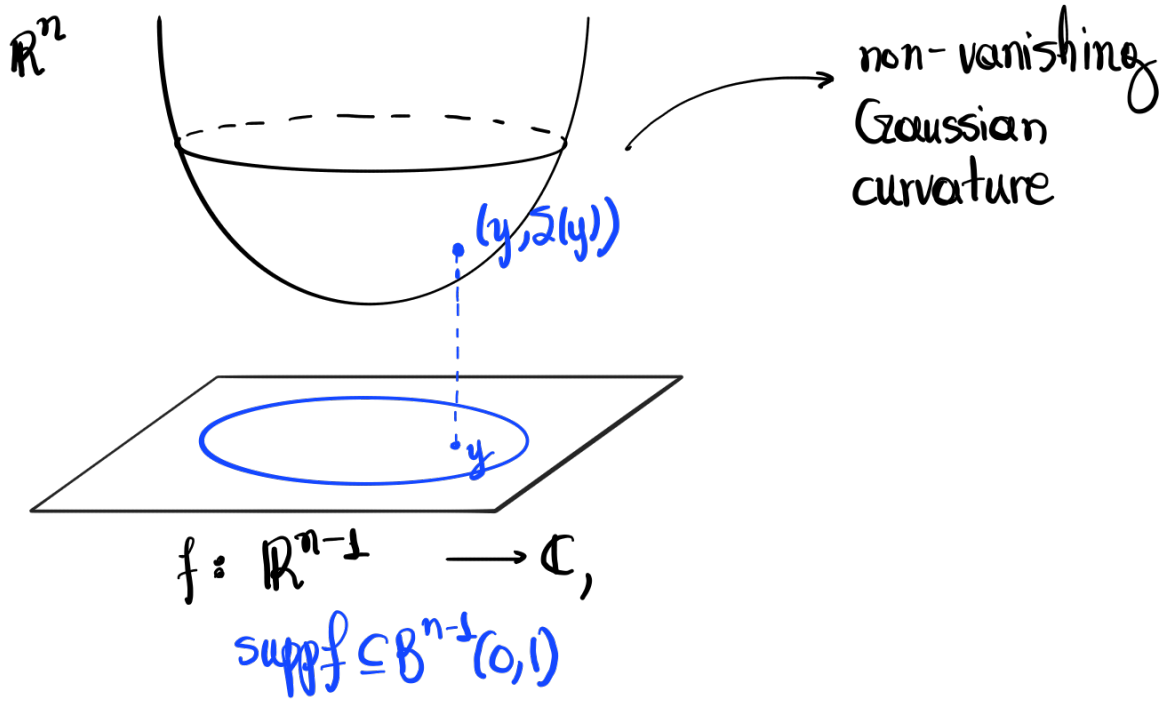
$$Ef(x,t) := \int_{y\in\mathbb{R}^{n-1}} e^{i\underbrace{\langle (x,t),(y,\Sigma(y))\rangle}_{\| \\ R\cdot\langle (\frac{x}{R},\frac{t}{R}),(y,\Sigma(y))\rangle \\ \| \\ R\cdot\phi(\frac{x}{R},\frac{t}{R};y)}} a^R(x,t;y)\, f(y)\,dy,$$

for $\phi \in C^\infty\left(B^n(0,1)\times(-1,1)\times B^{n-1}(0,1)\right)$

Restriction conjecture (Stein, '60s):

$$\|Ef\|_{L^p(B_R)} \lesssim \|f\|_p \,, \quad \forall\, p \geq \frac{2n}{n-1}$$

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$$\| \; \phi^R(x,t;y)$$

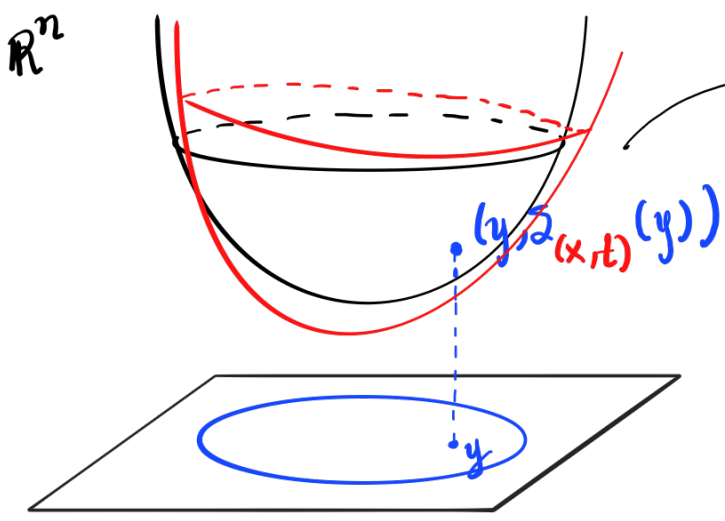
$$:= \boxed{R\cdot\phi\left(\tfrac{x}{R},\tfrac{t}{R};y\right)},$$

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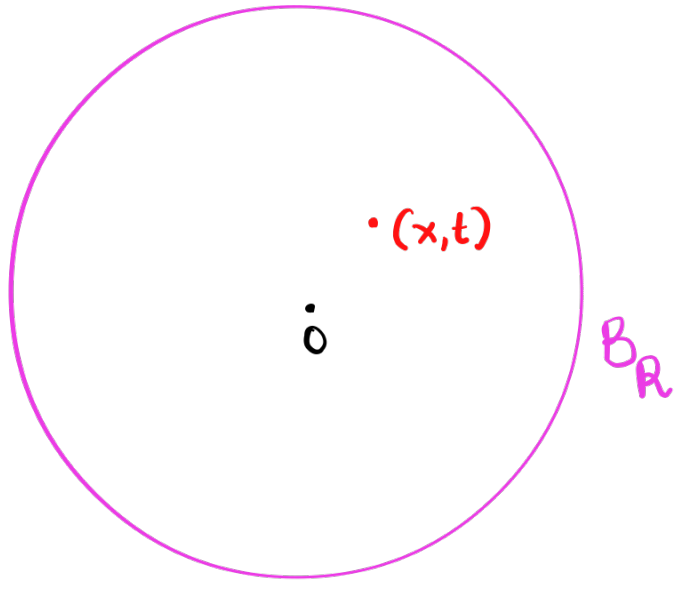
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Hörmander-type operators:



non-vanishing Gaussian curvature

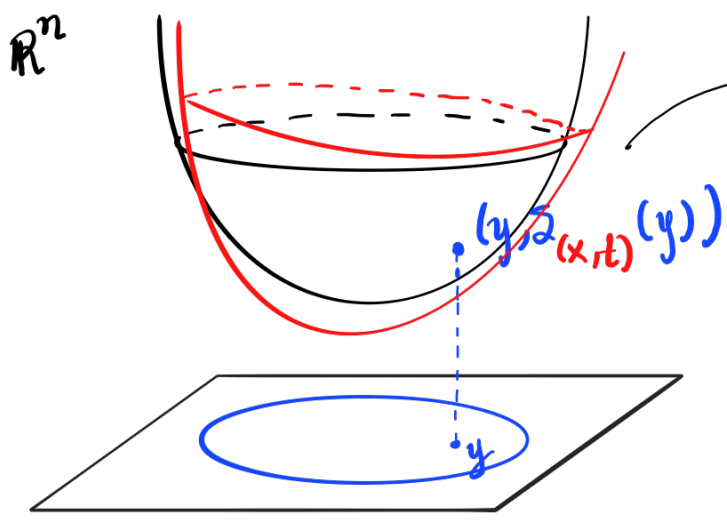
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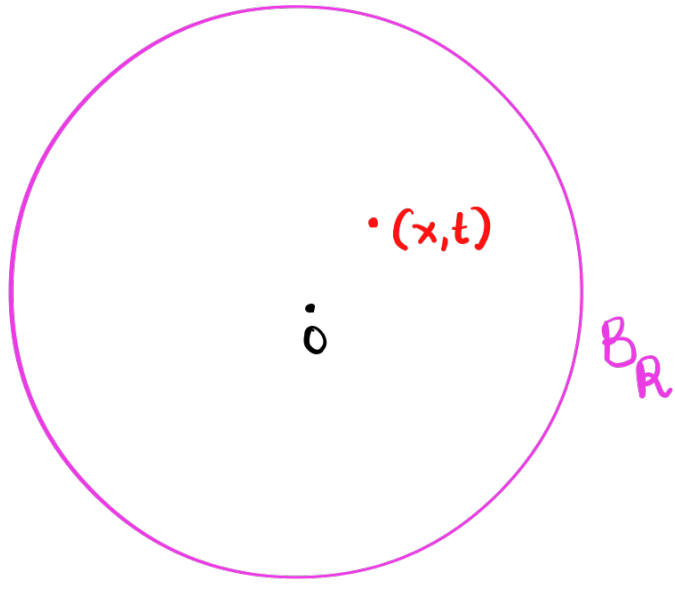


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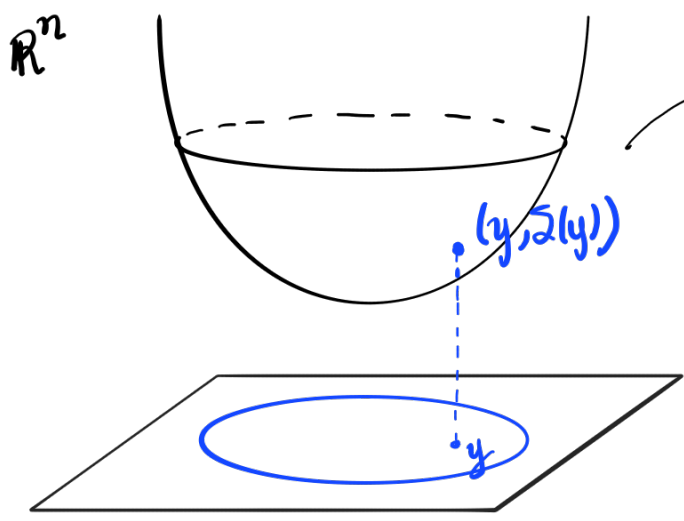
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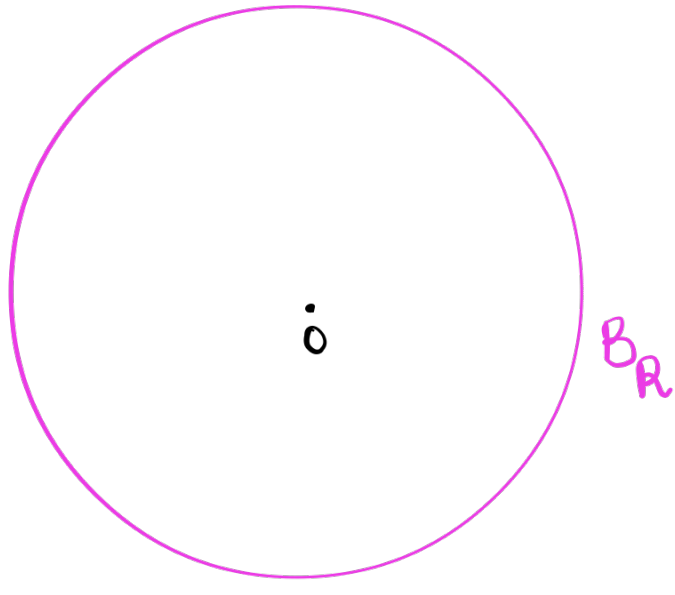


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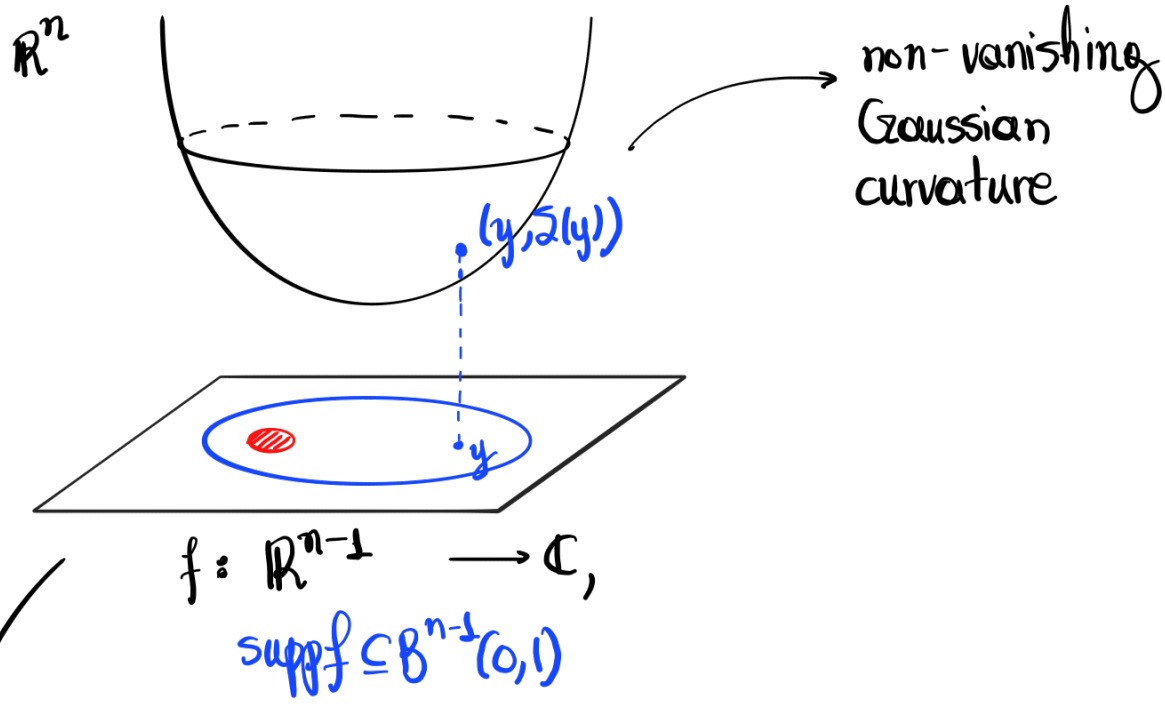
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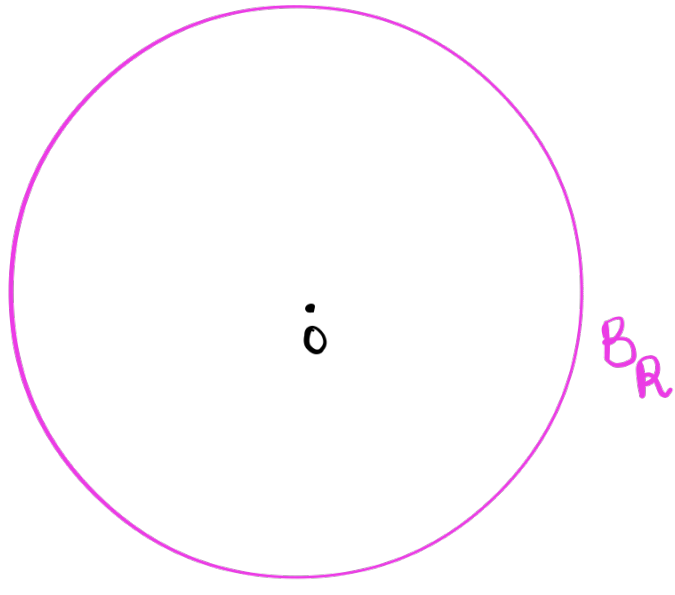
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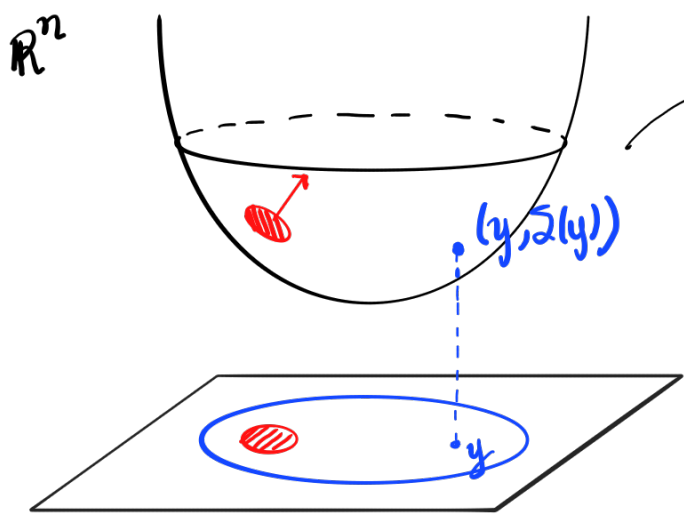
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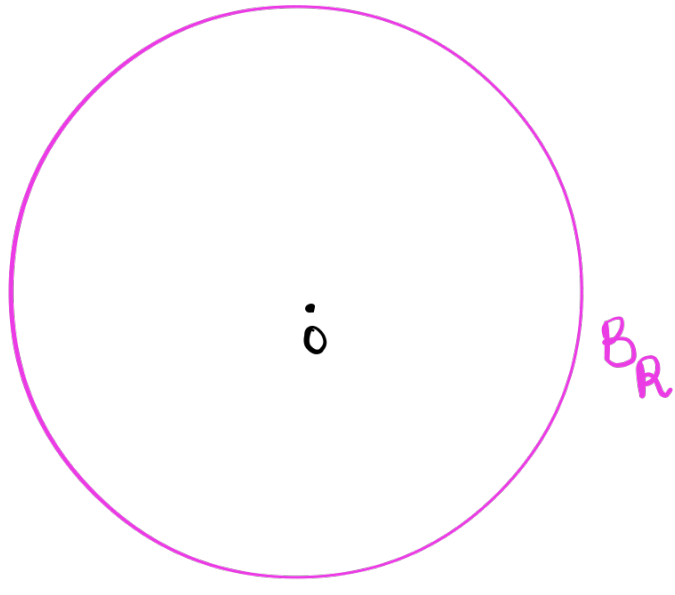


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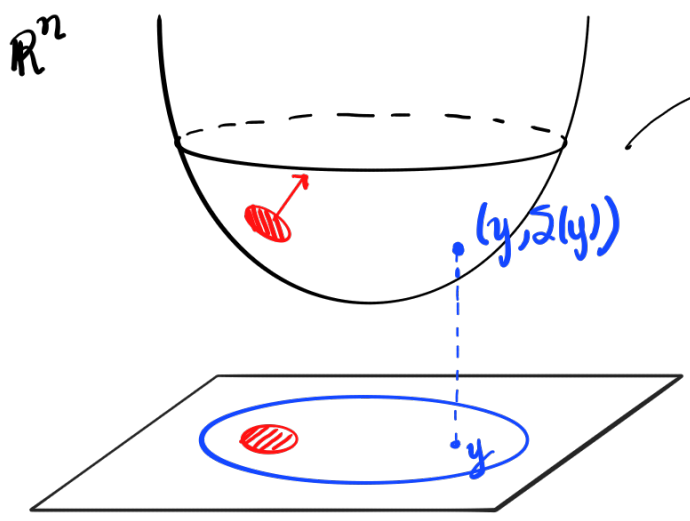
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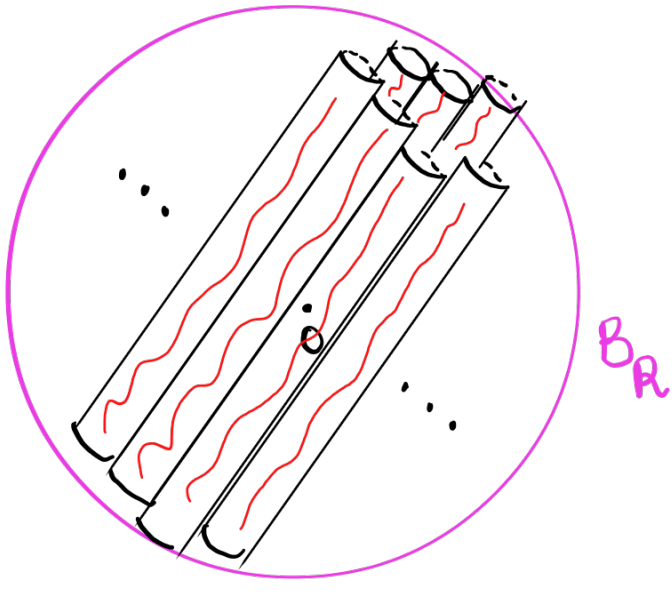


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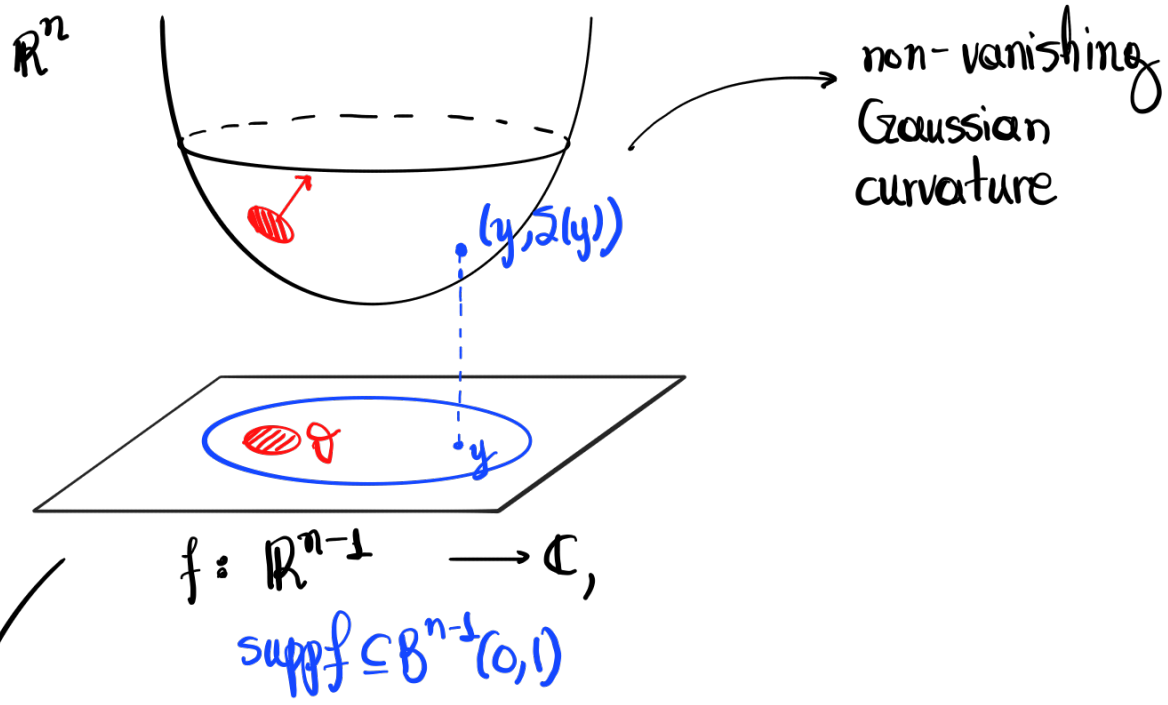
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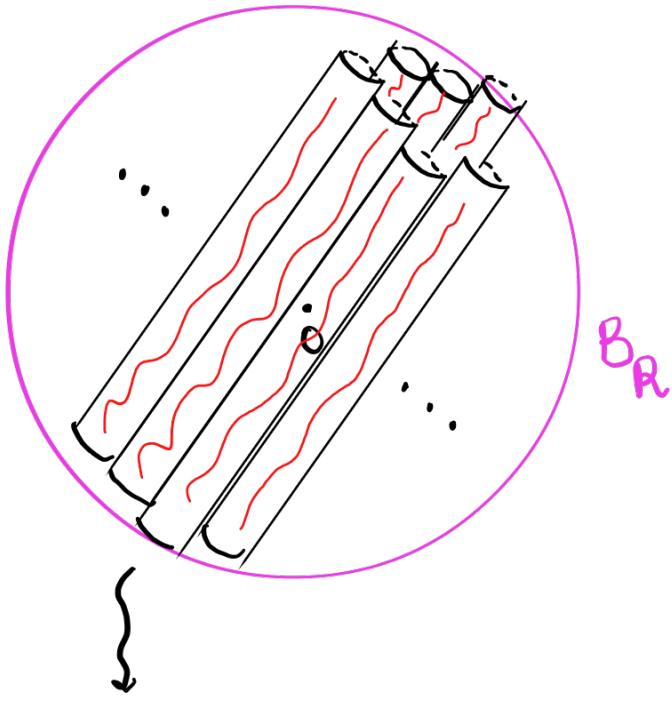
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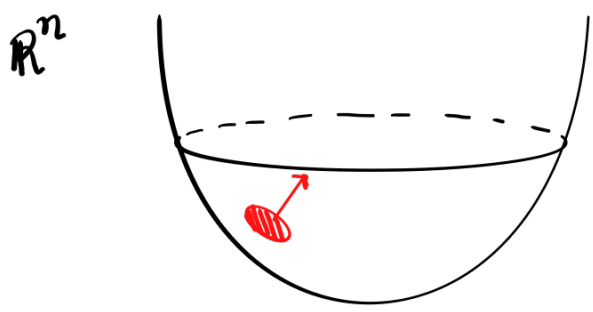


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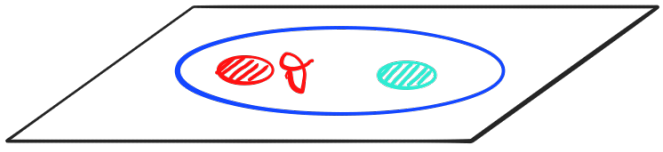


Central lines: $\Gamma_{\theta,v} = \{\partial_y \phi\,(\cdot, y(\theta)) = v\}$

Extension operators:



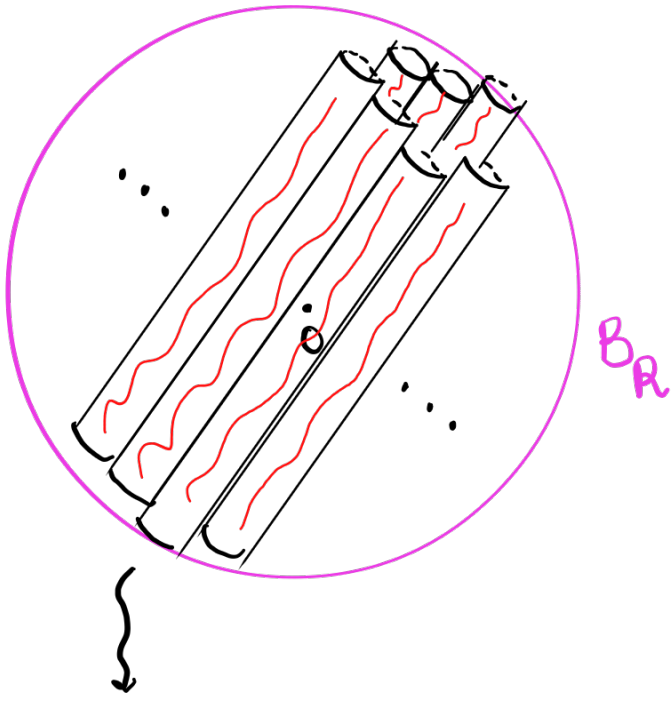
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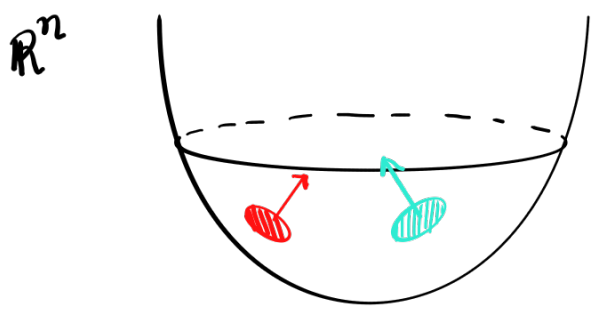
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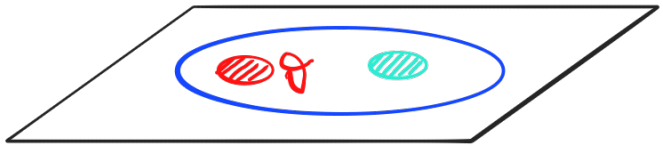


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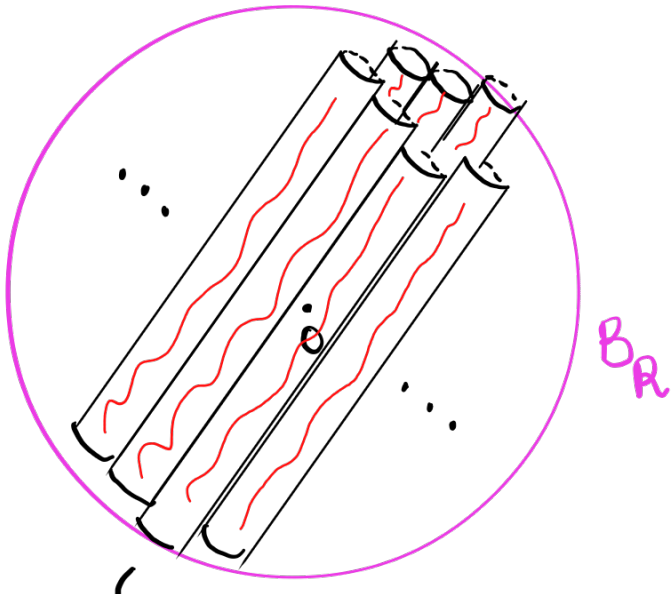
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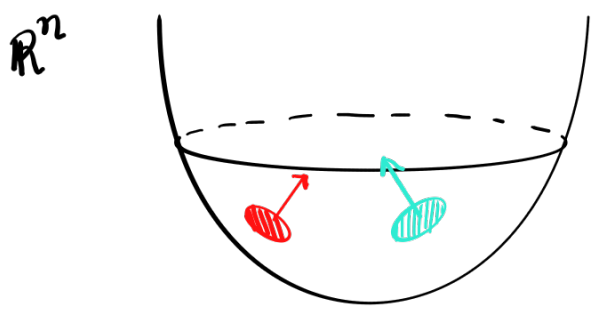
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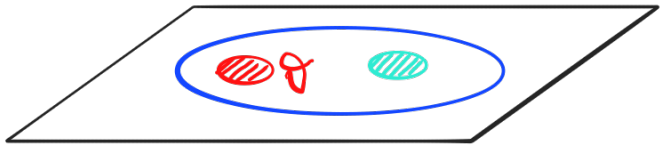


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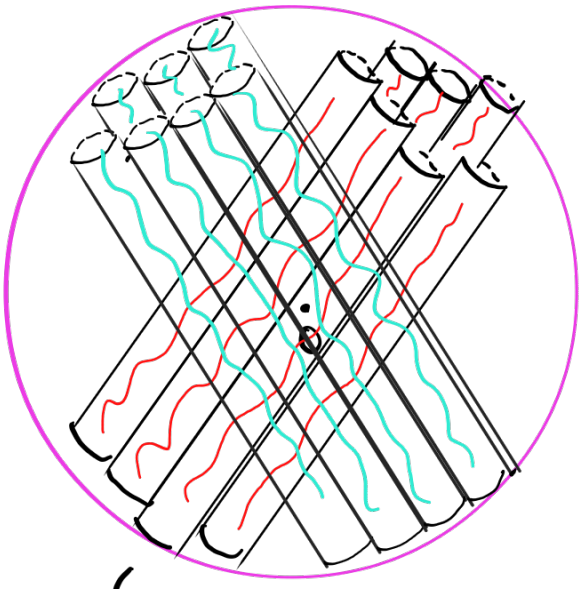
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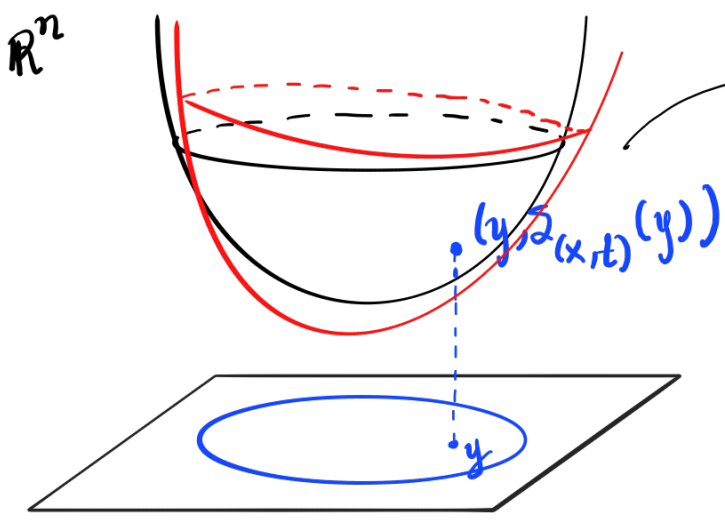
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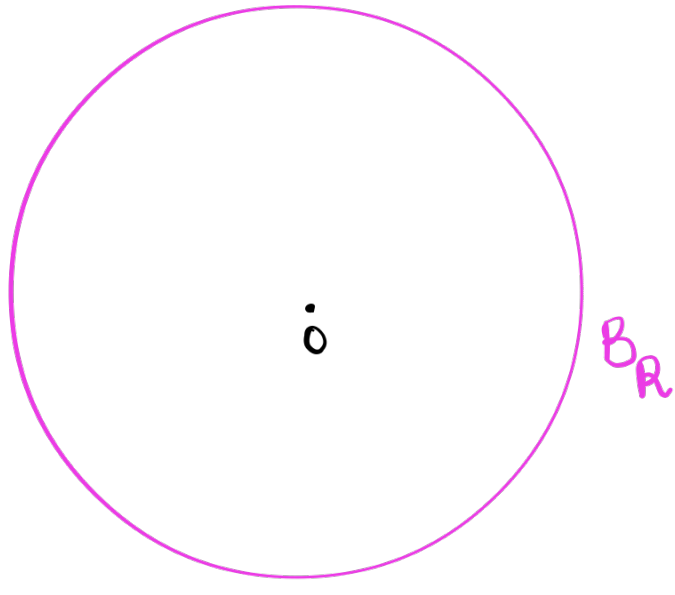


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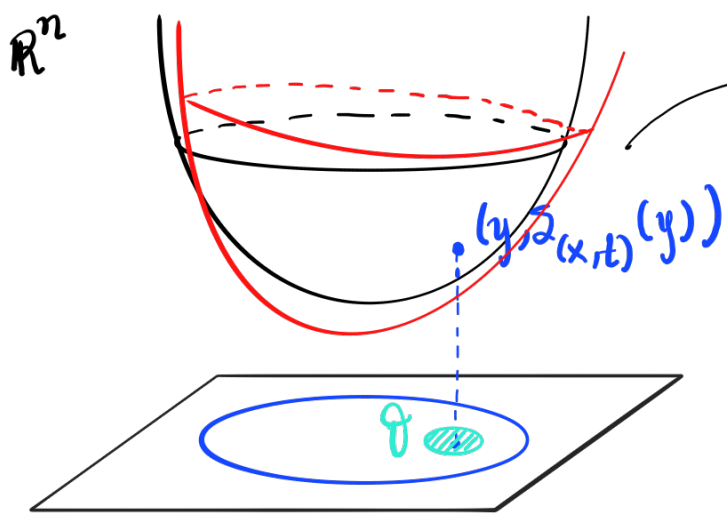
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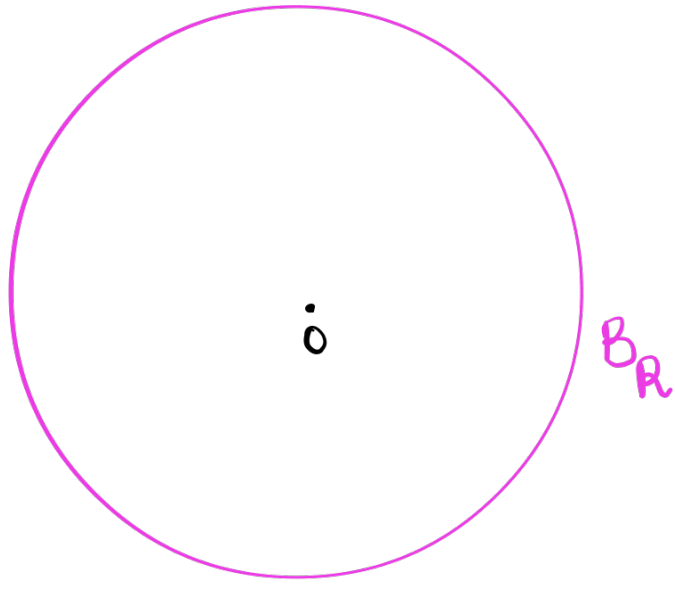


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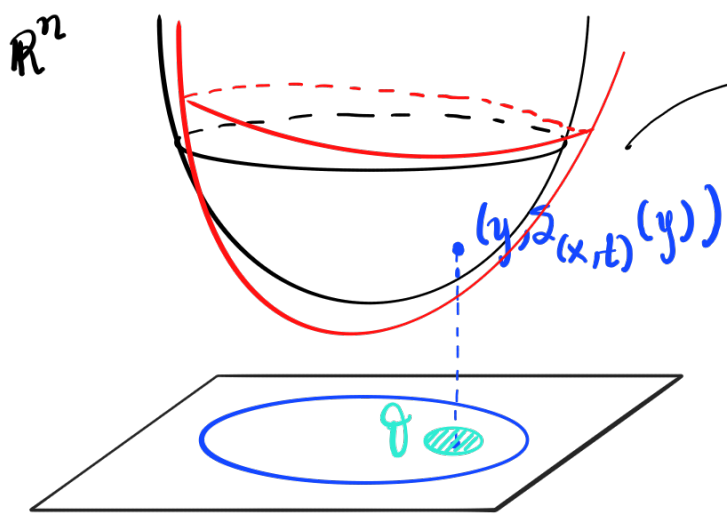
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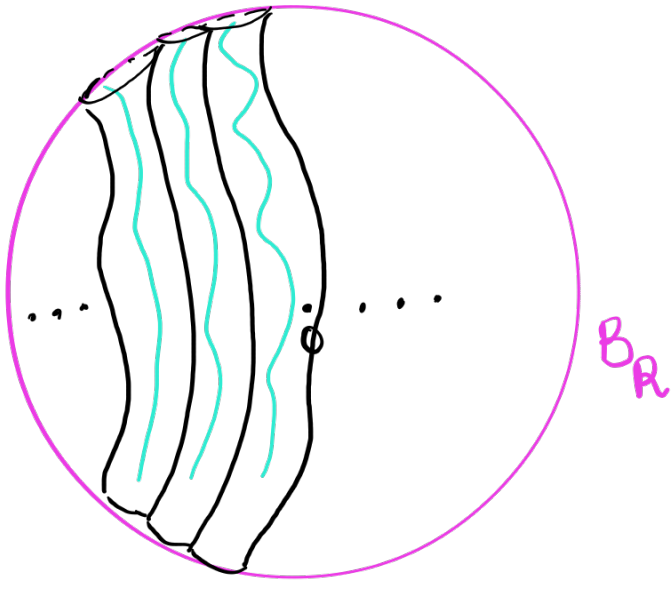


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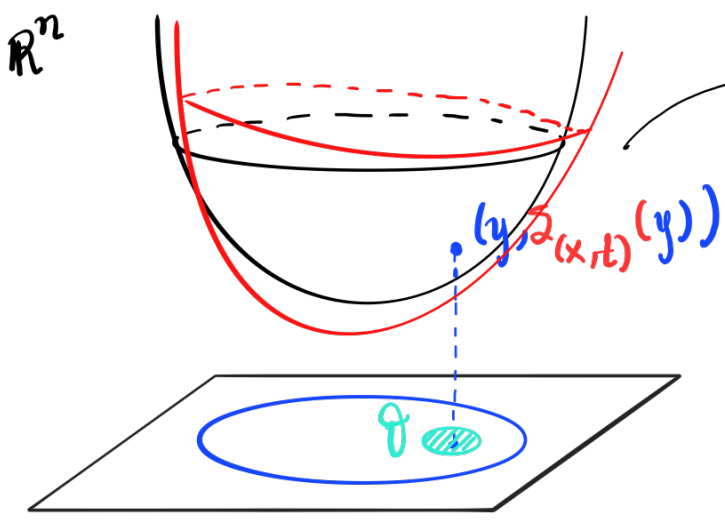
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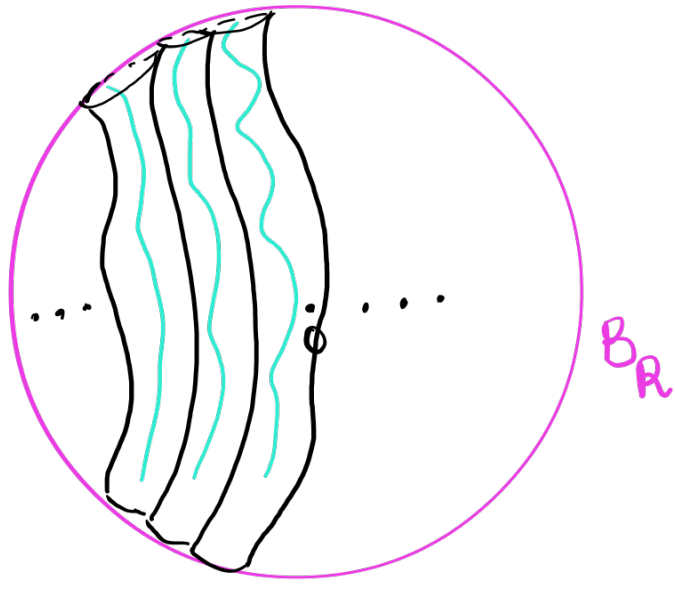


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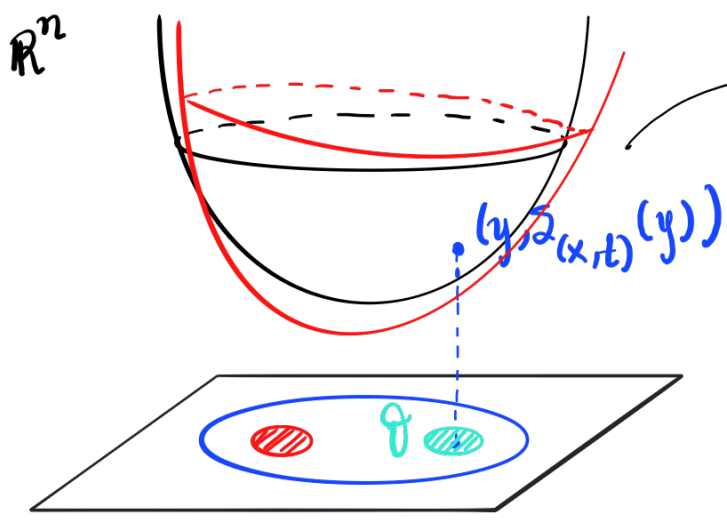
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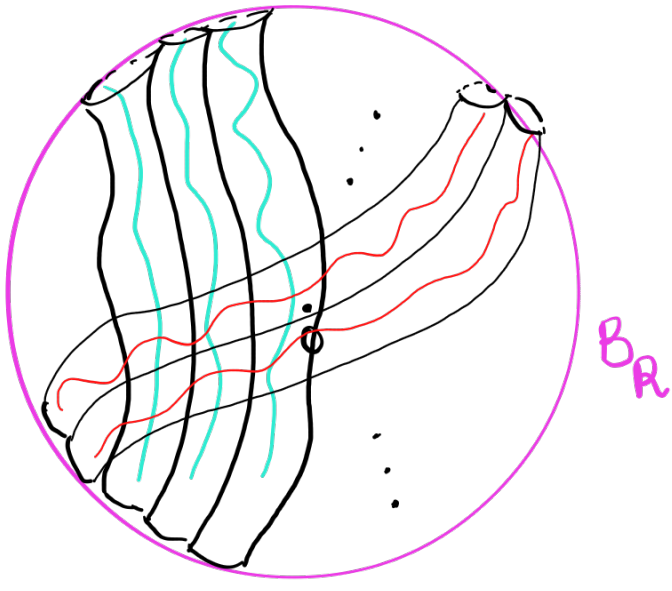


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$\sigma = \max(n-1)$ — Guth - Hickman - I. '17 → sharp

$\sigma = \max - 1$

⋮

$\sigma = 3$

Hickman - I. '20 → sharp

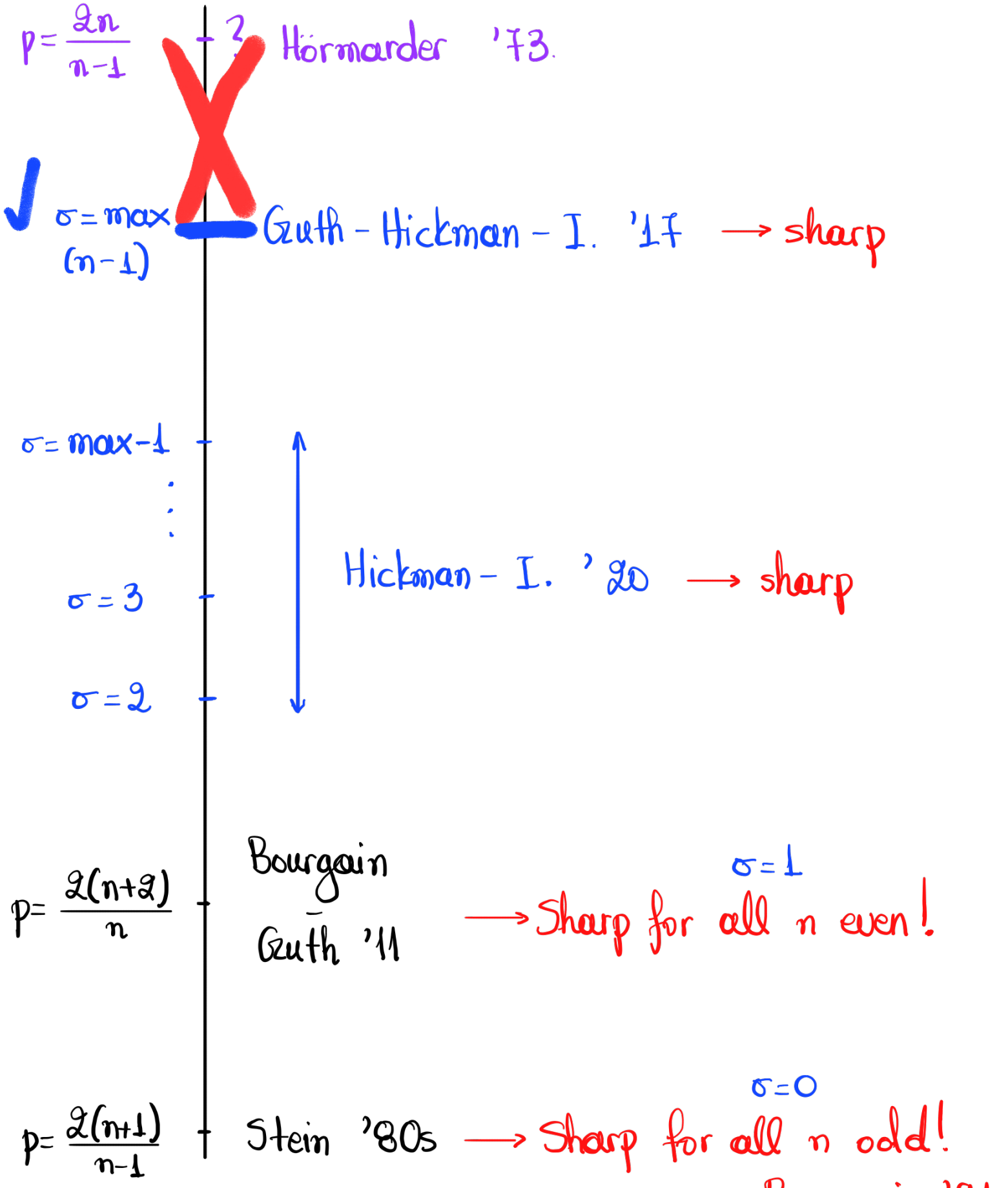
$\sigma = 2$

$p = \frac{2(n+2)}{n}$ — Bourgain, Guth '11 → Sharp for all n even! ($\sigma = 1$)

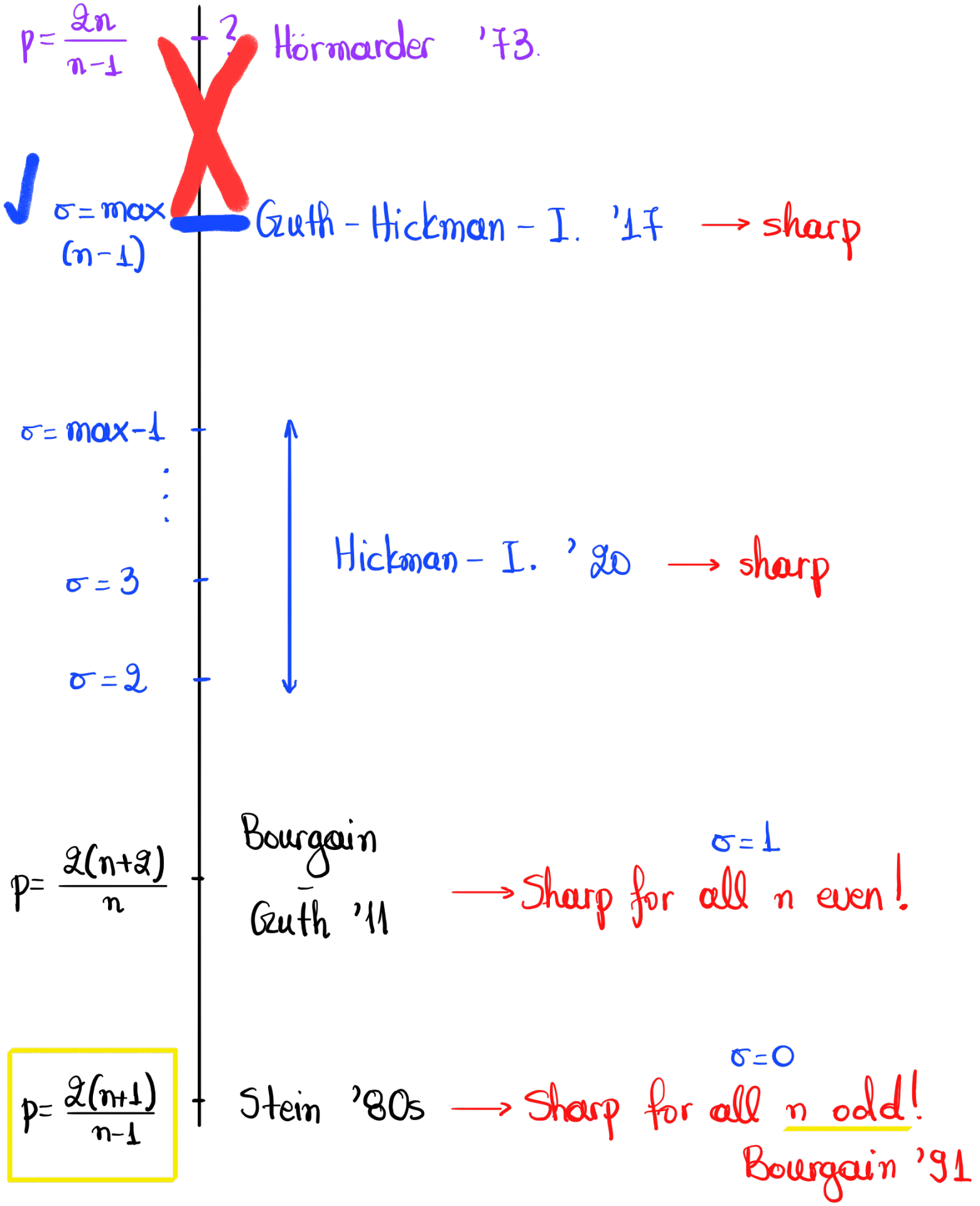
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Bourgain '91

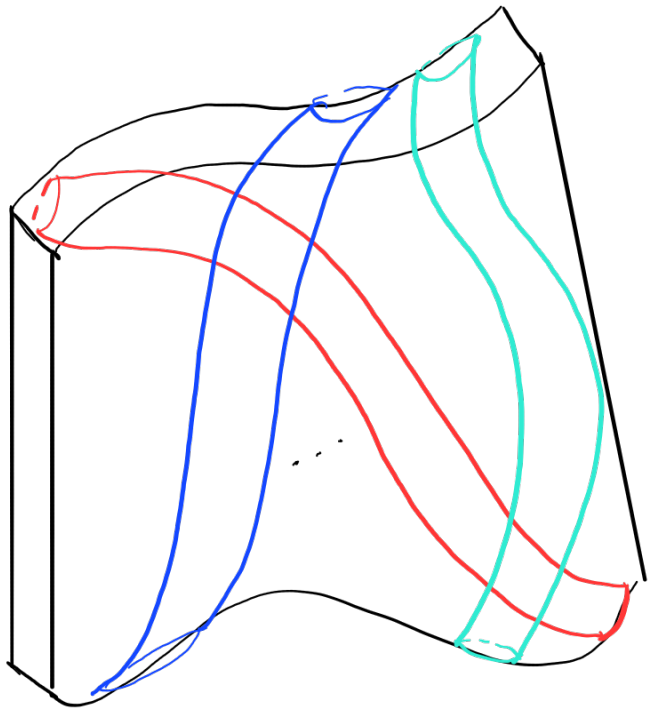
Hörmander ('73): $\|H^R f\|_{L^p(B_R)} \lesssim \|f\|_p$, $\forall\, p \geq \frac{2n}{n-1}$?



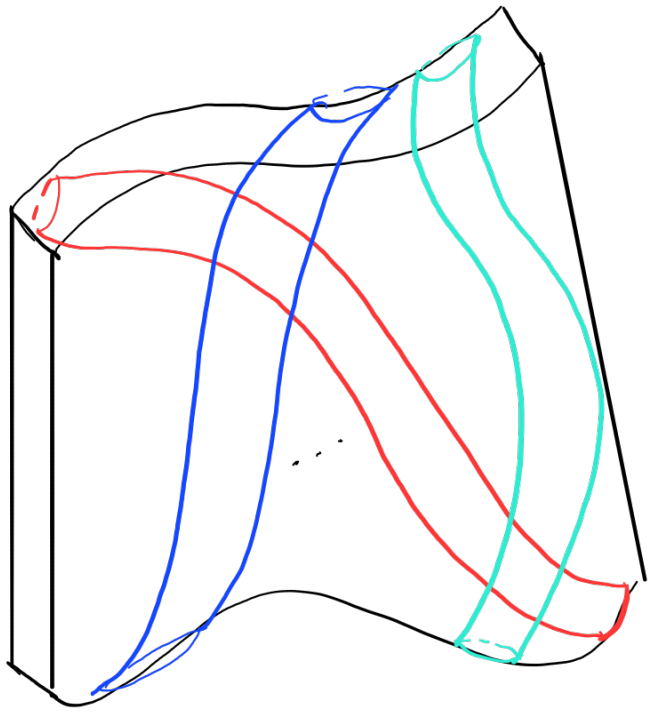
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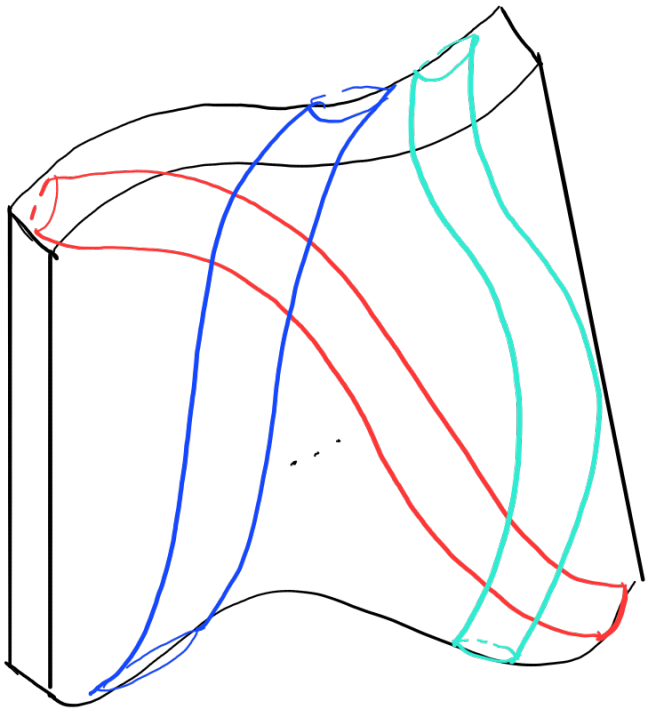
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Bourgain: 1 curved tube in each "direction",

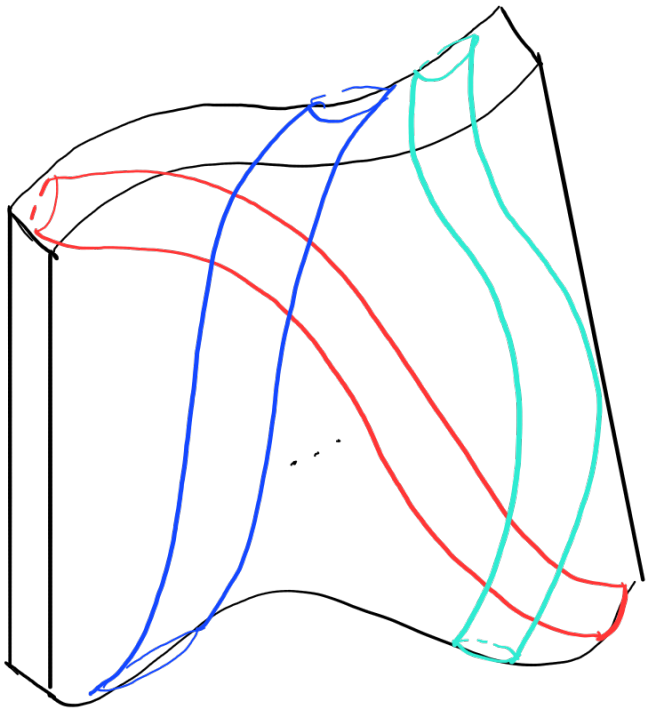


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Z: low-dim variety

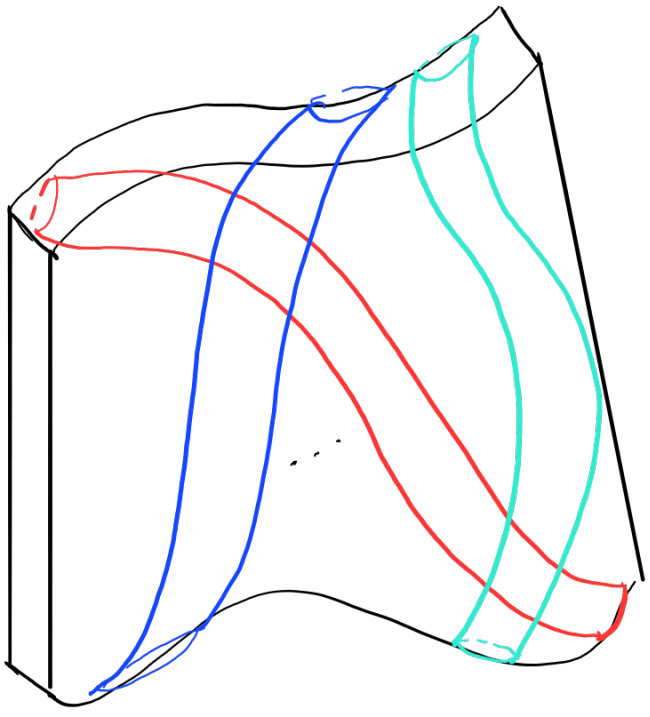
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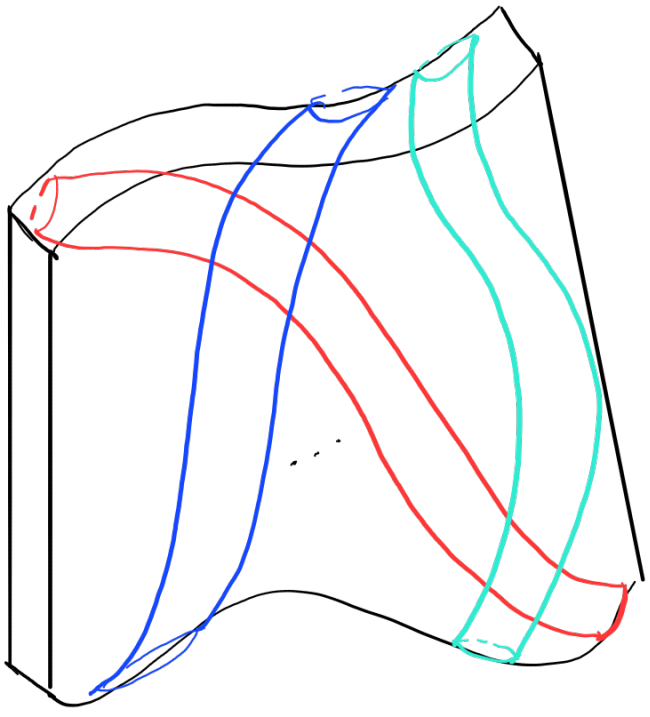


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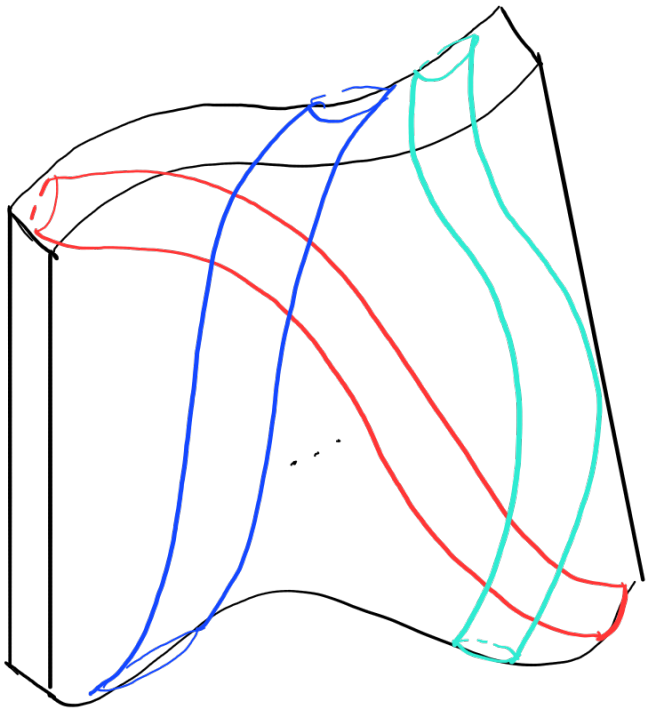


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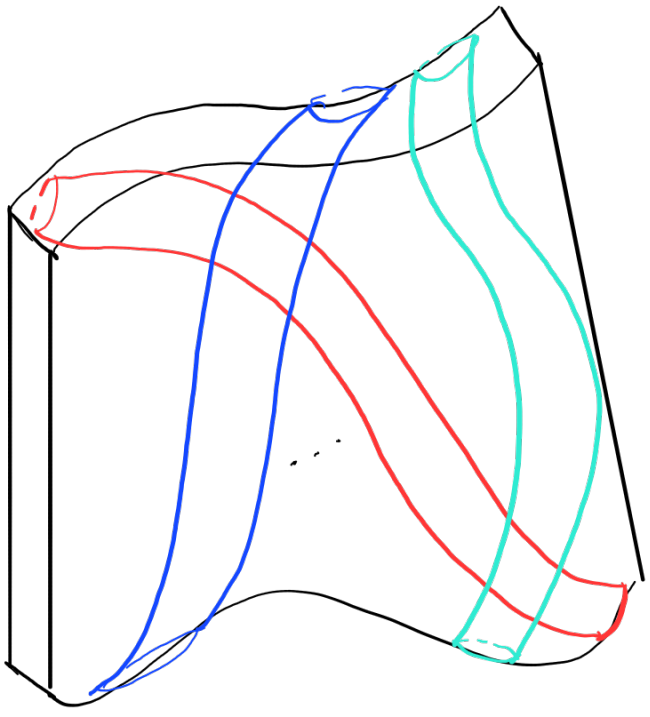
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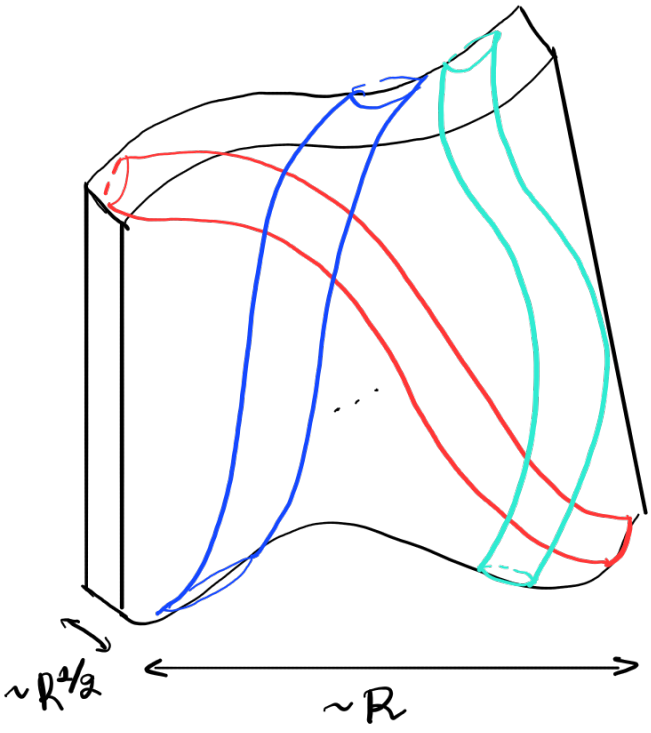


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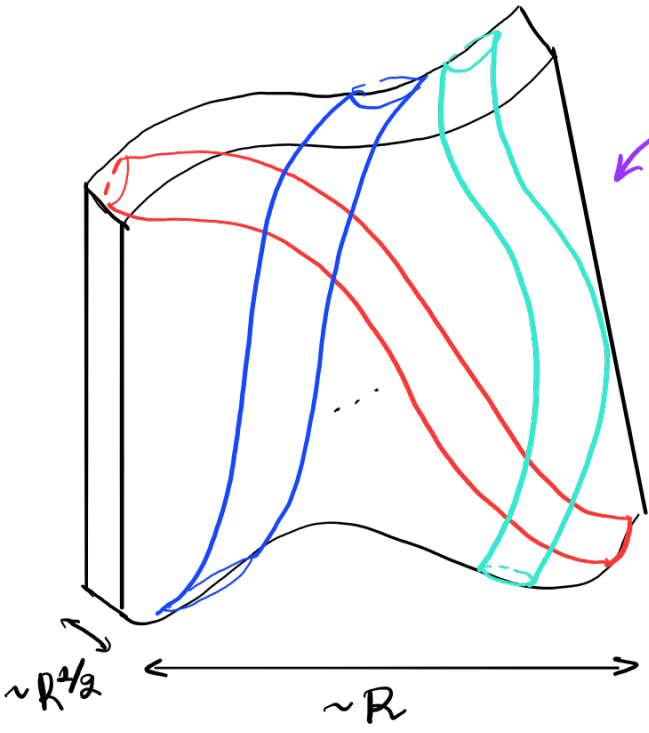
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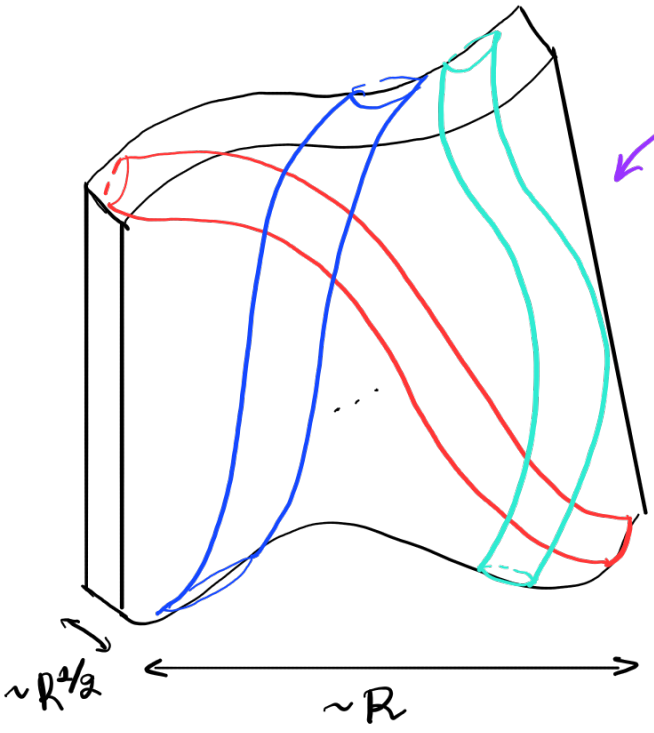


G: grain of dimension m



Why is this clustering bad?

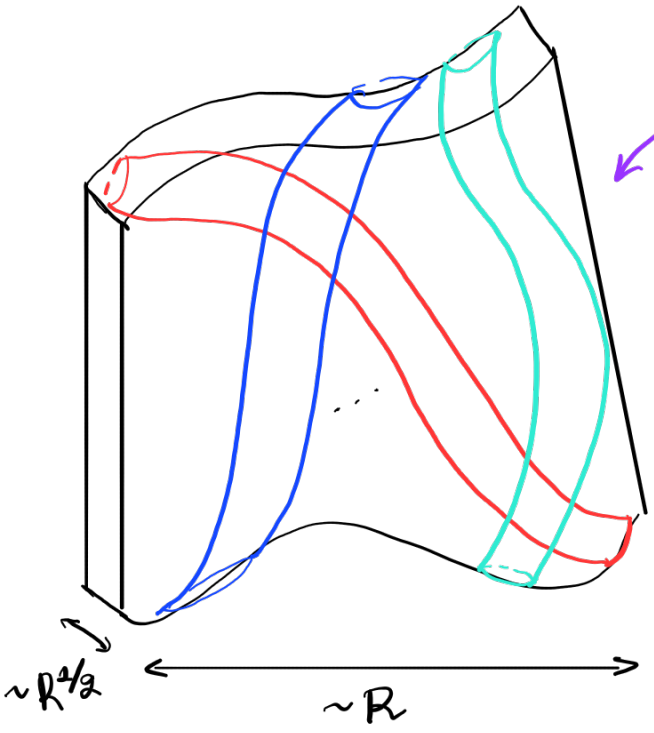
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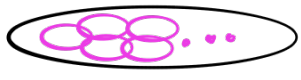


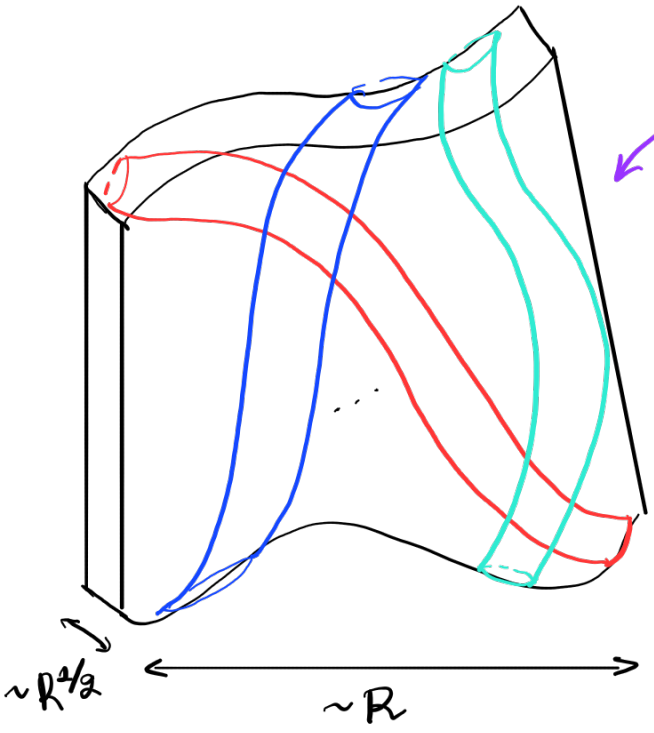
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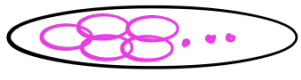
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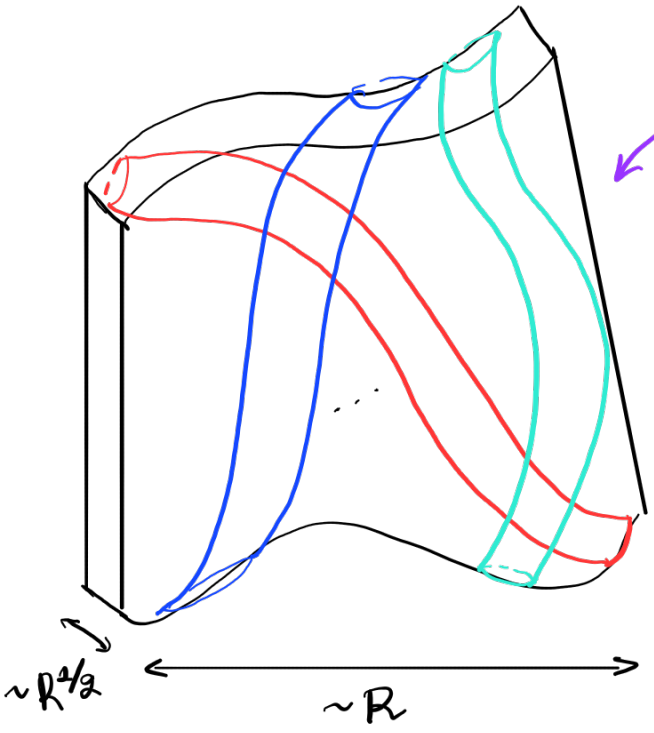
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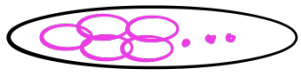
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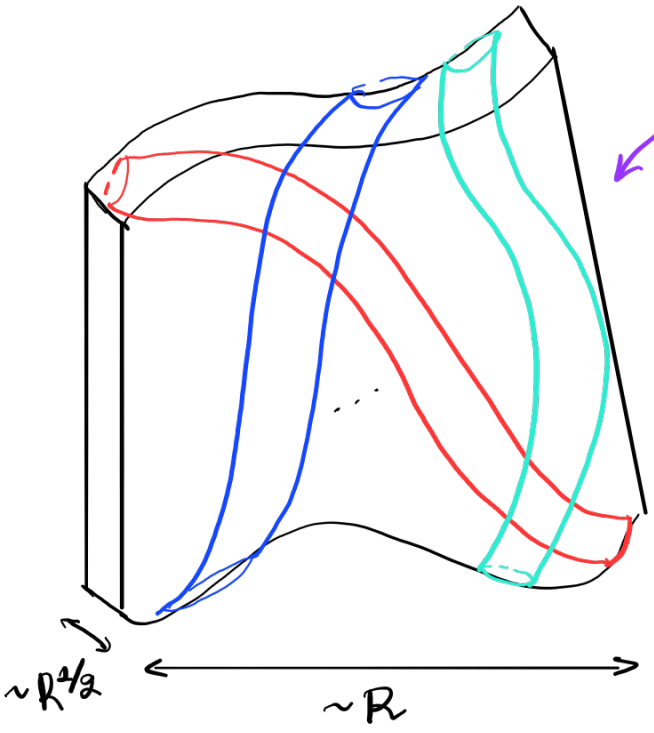
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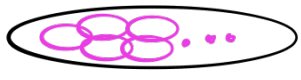
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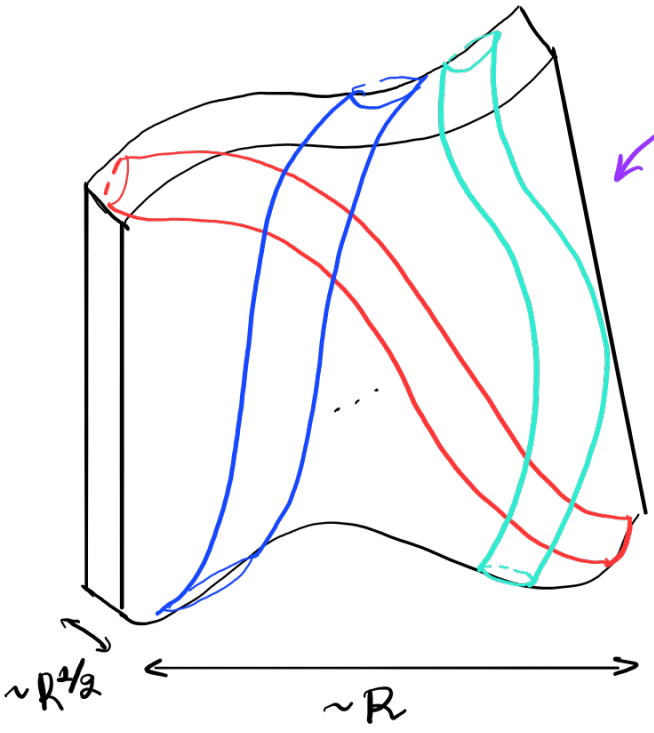
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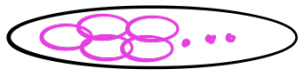
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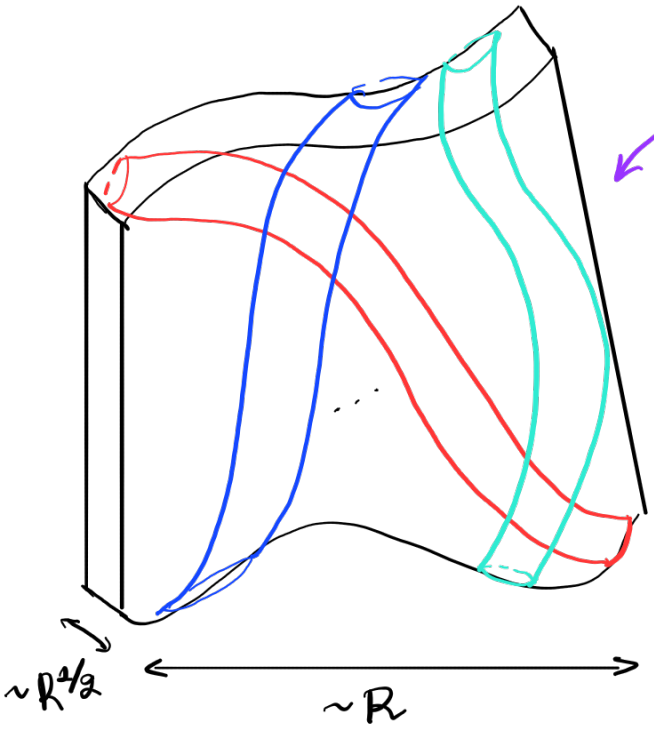
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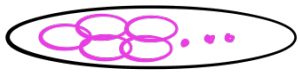
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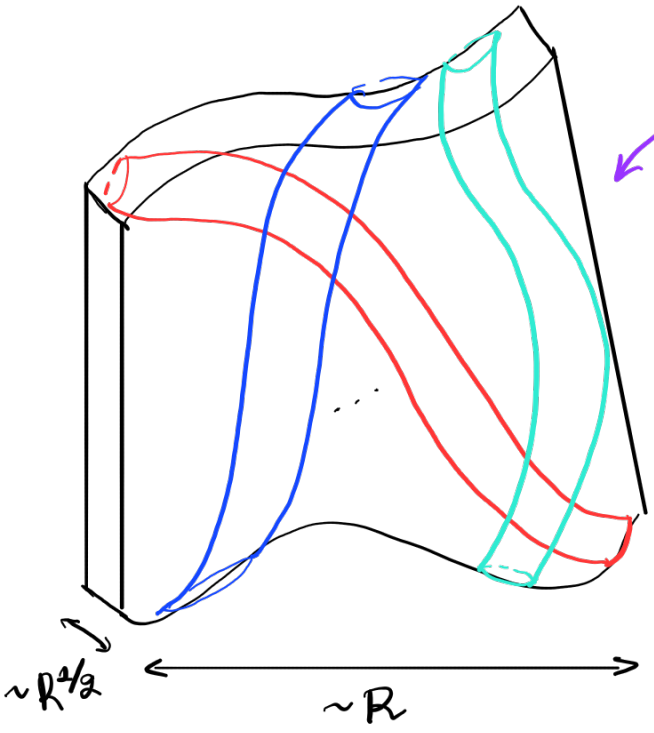
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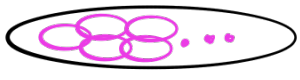
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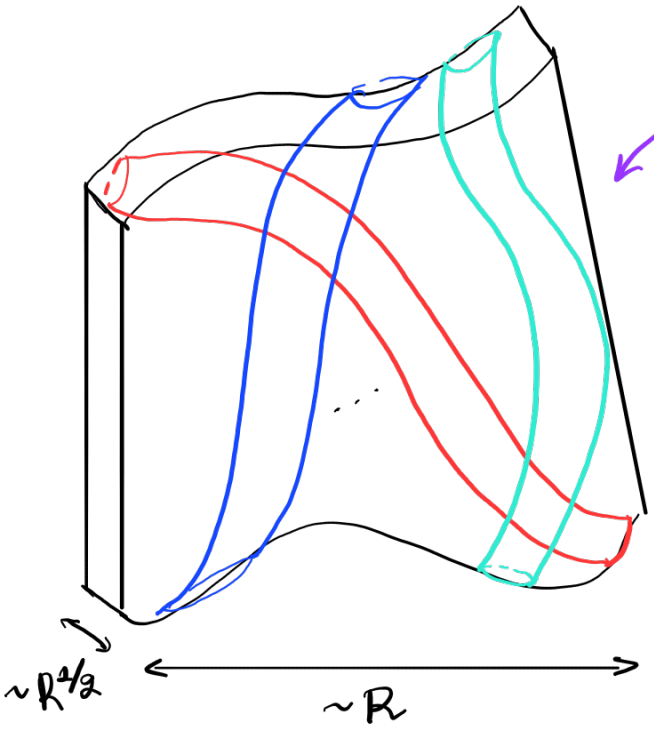
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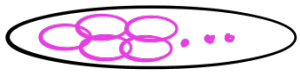
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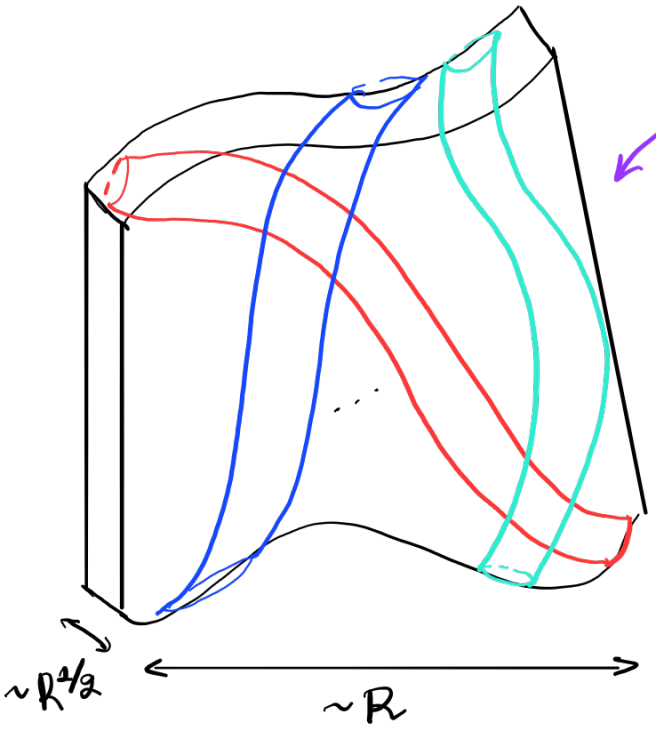


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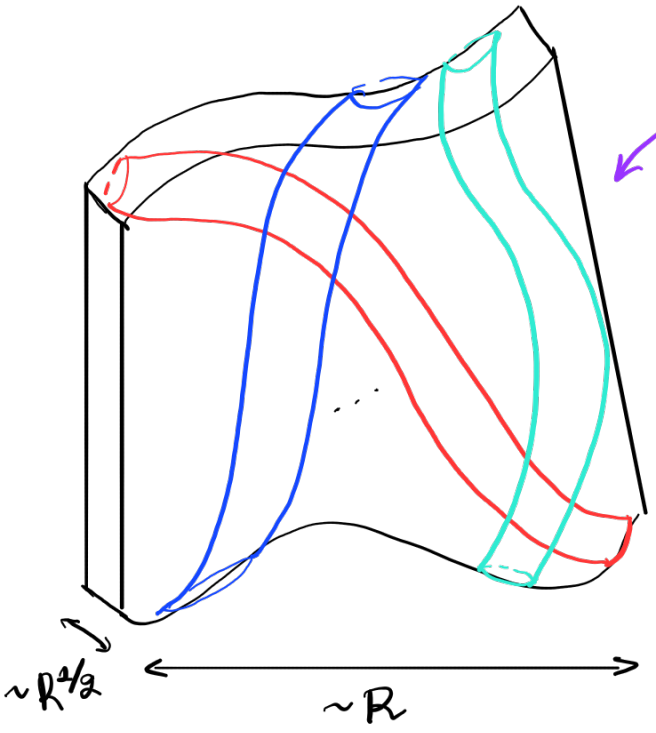
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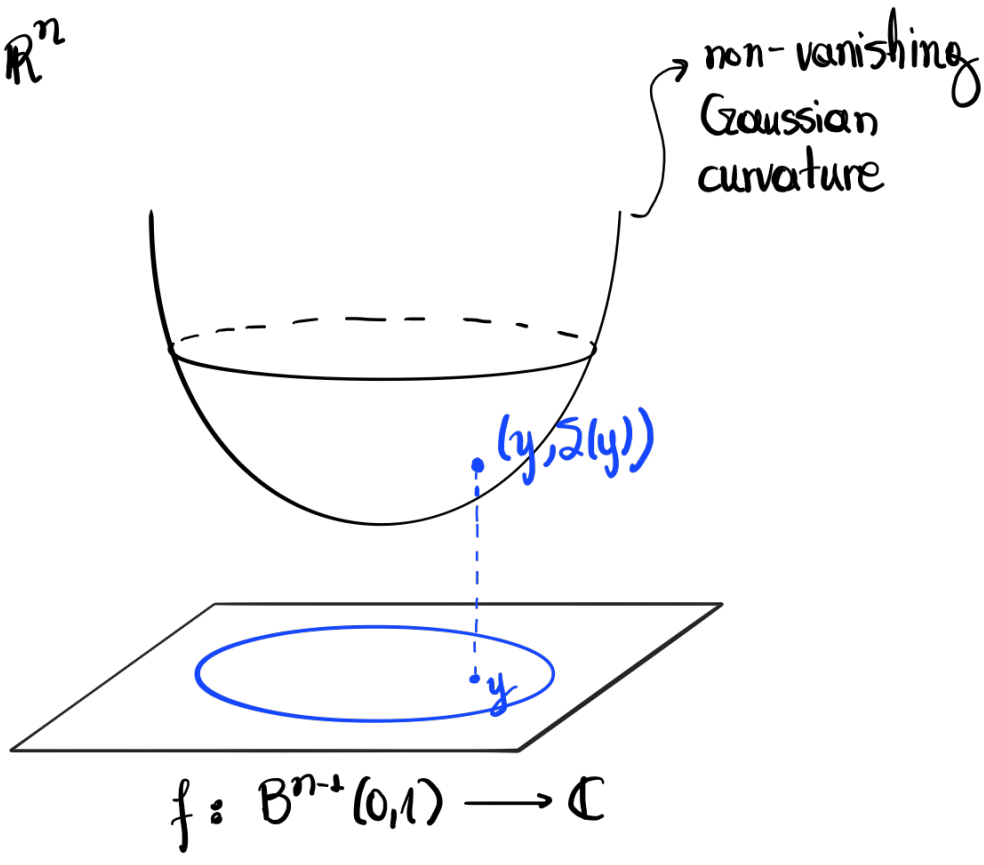
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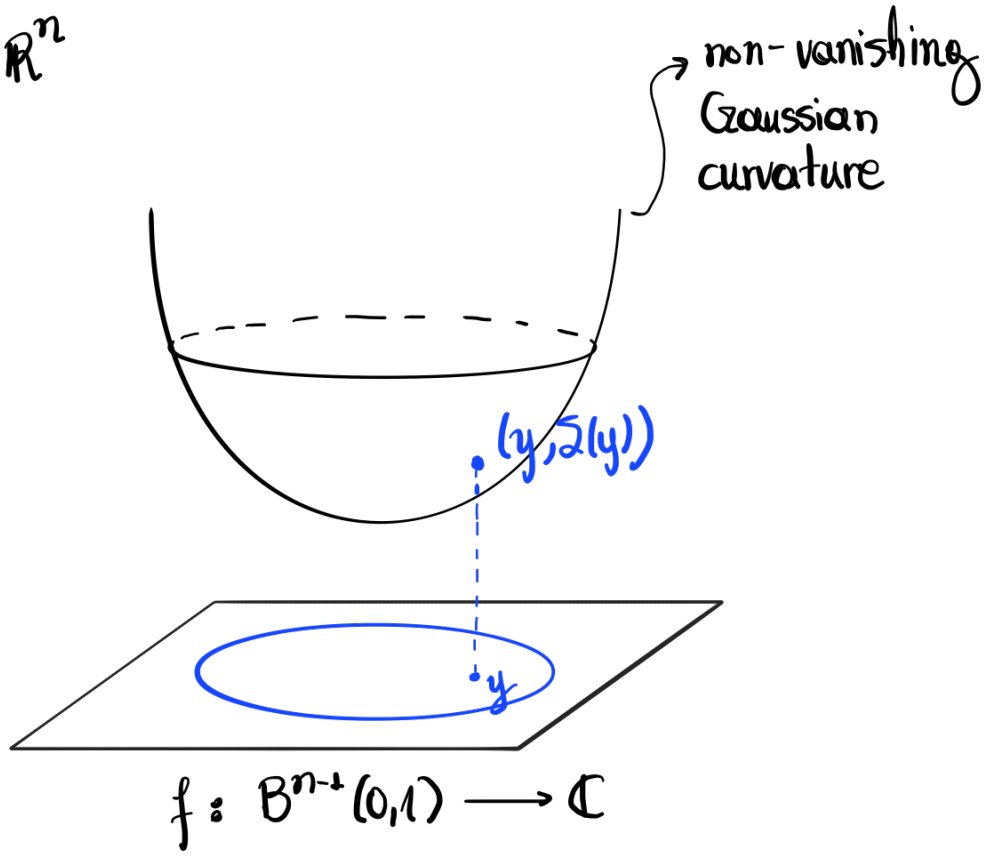
for some $q = q(\phi) < \frac{2(n+1)}{n-1}$.

The extension operator:



$$Ef(x,t) := \int_{y \in B^{n-1}(0,1)} e^{i\langle (x,t),\,(y,\Sigma(y))\rangle} f(y)\,dy$$

The solution to the Schrödinger equation



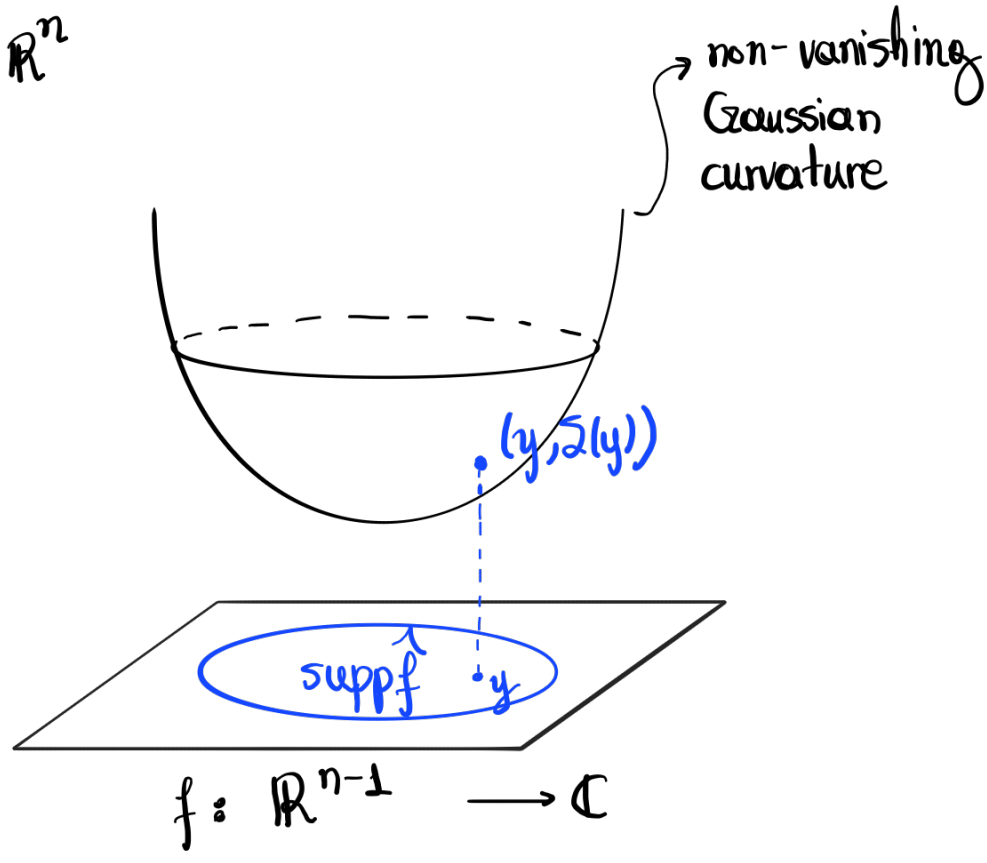
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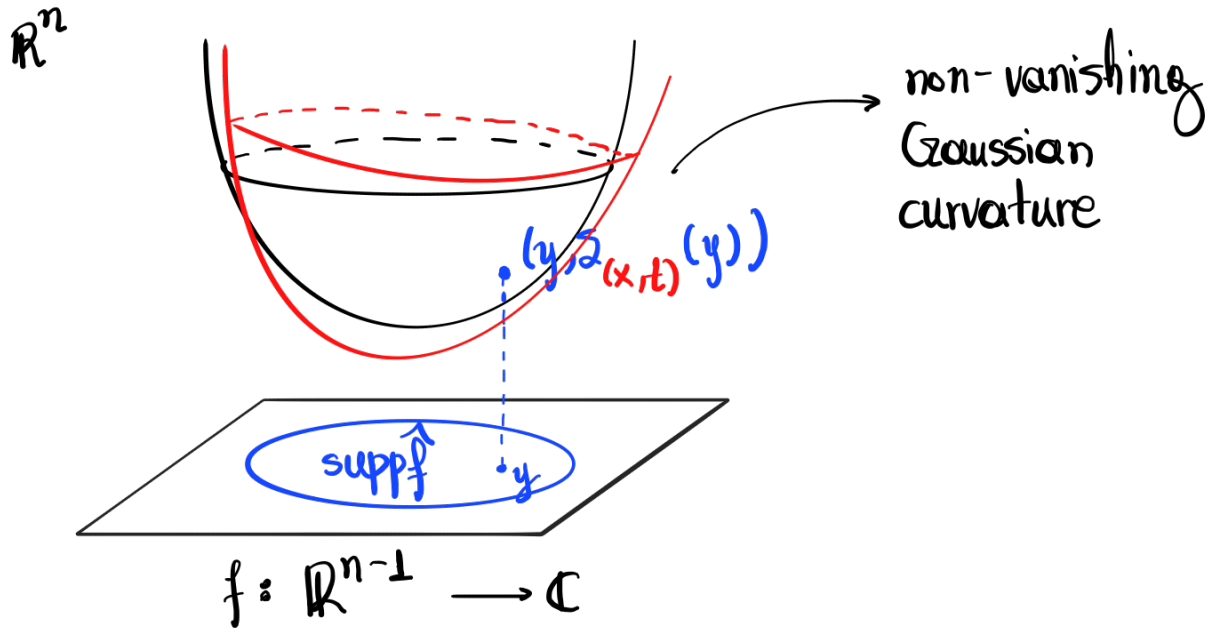
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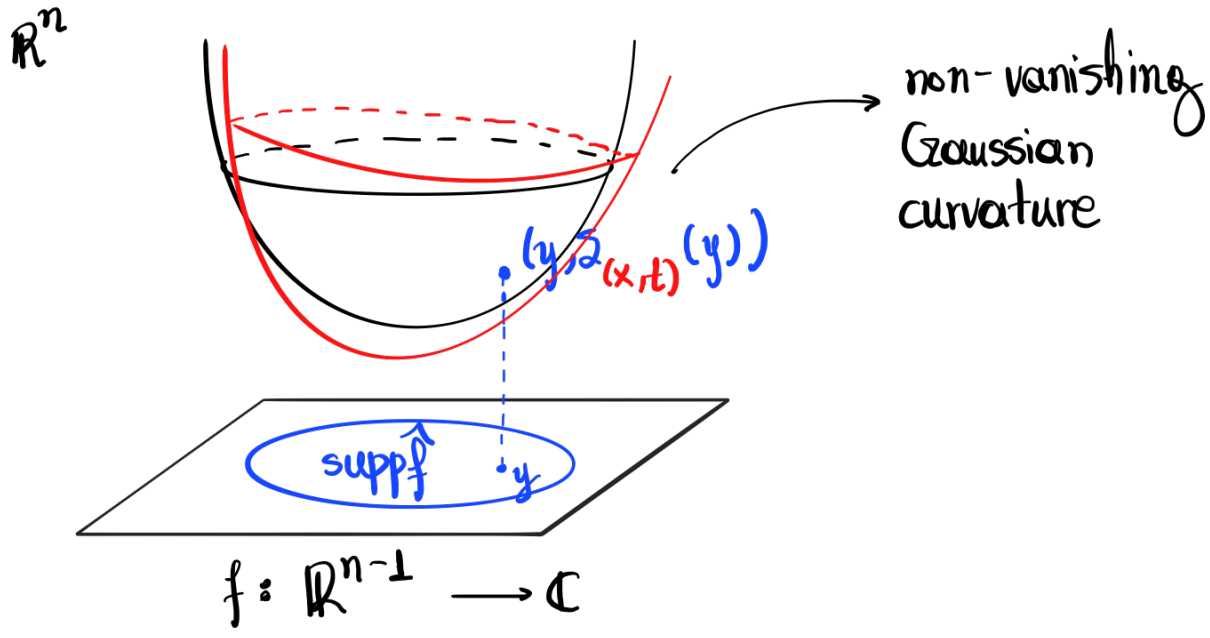


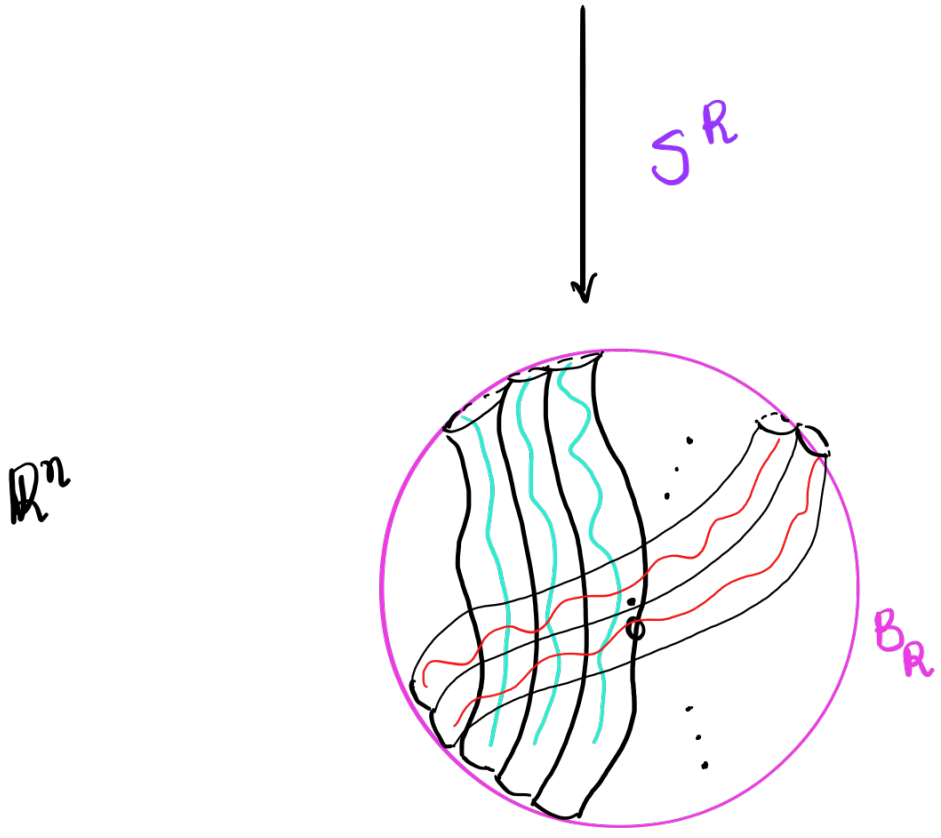
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$\Big\{$

supp in $\left\{ \partial_y \left(\phi(\cdot; y(\theta)) - \langle y(\theta); w\rangle \right) = 0 \right\}$

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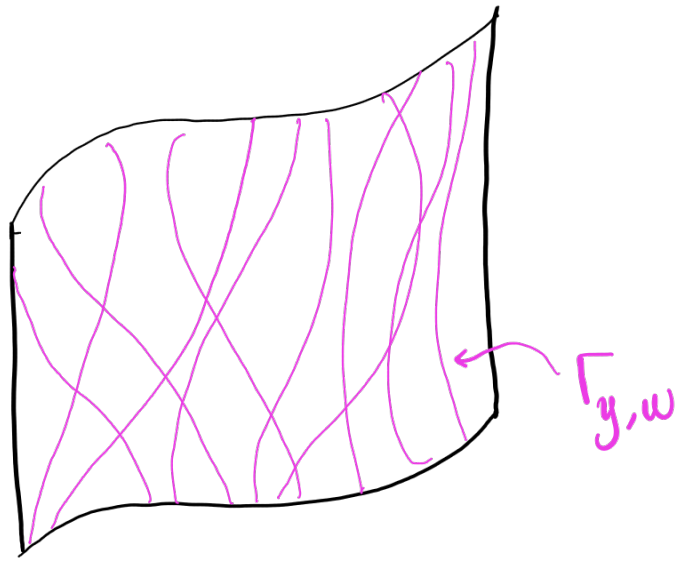
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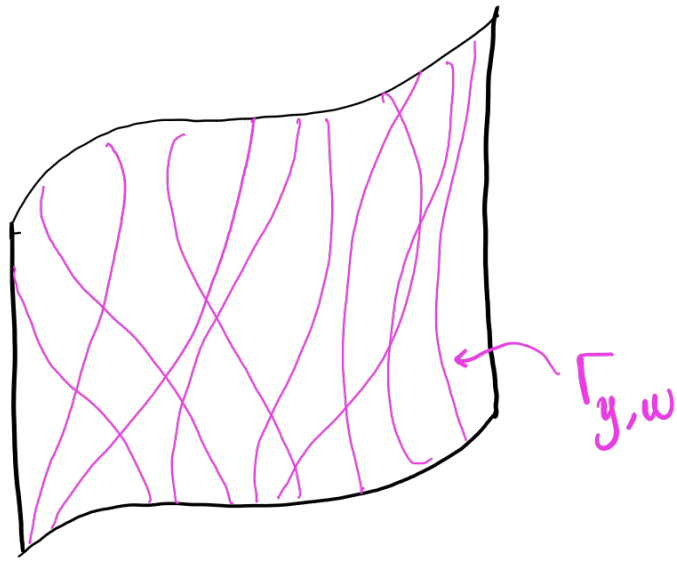
(tube in "direction" θ, with "base pt" v)

So, for the same curves $\Gamma_{y,w}$, we want to avoid bad Nikodym phenomena

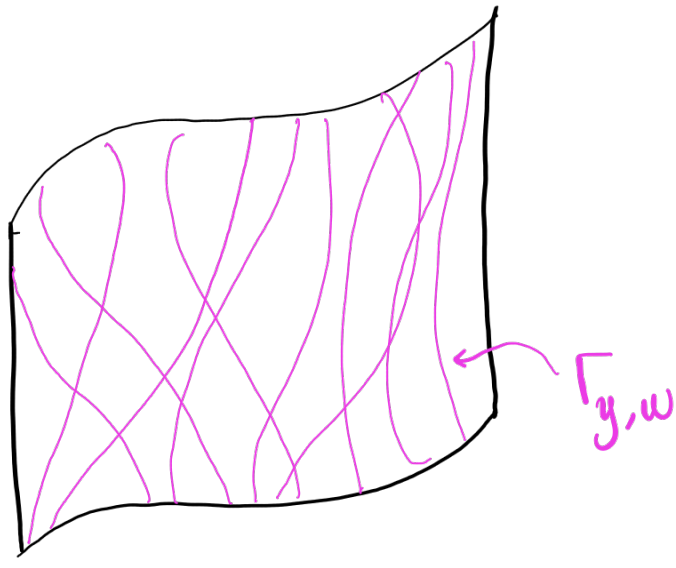
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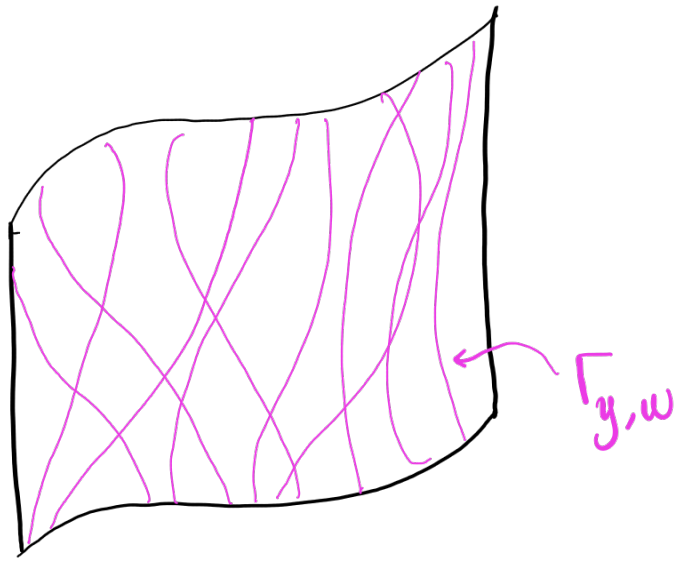
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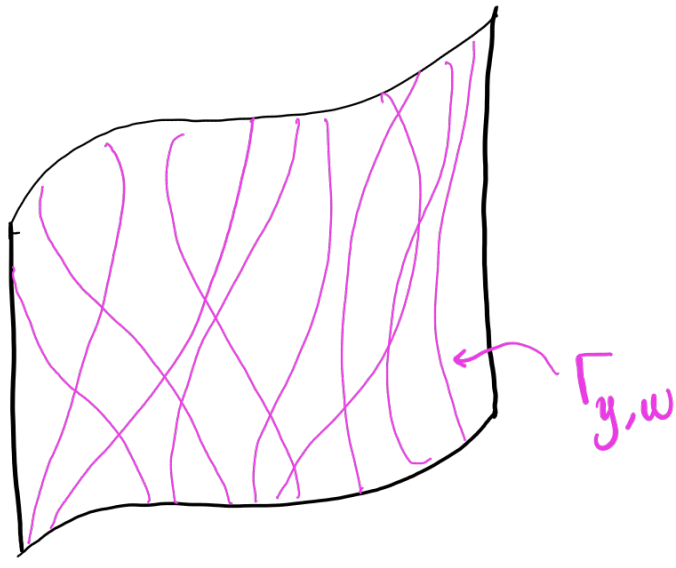
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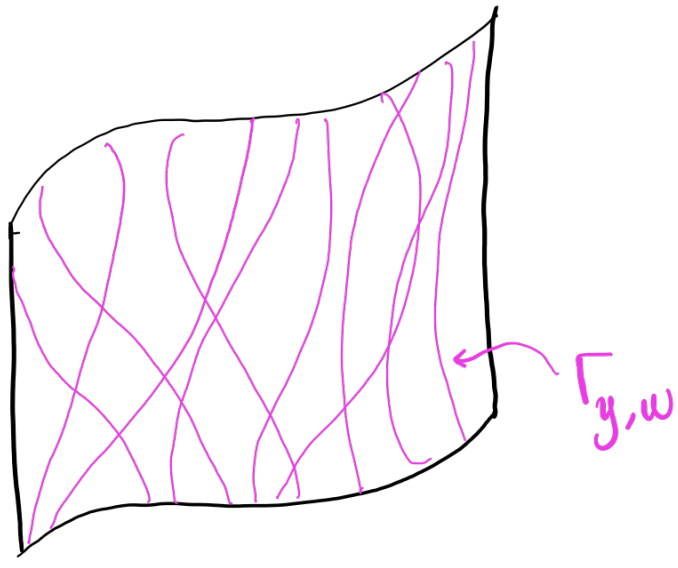


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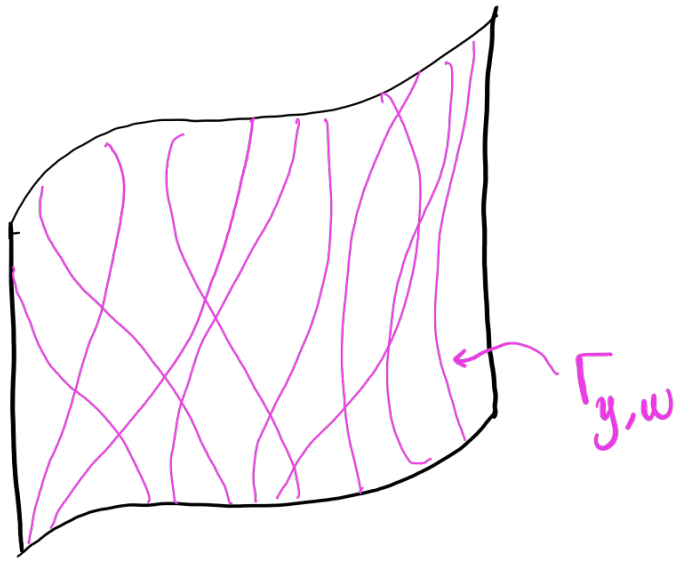


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Plan :

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Katz-Rogers

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Get improved multilinear Kakeya / Nikodym estimates for ϕ

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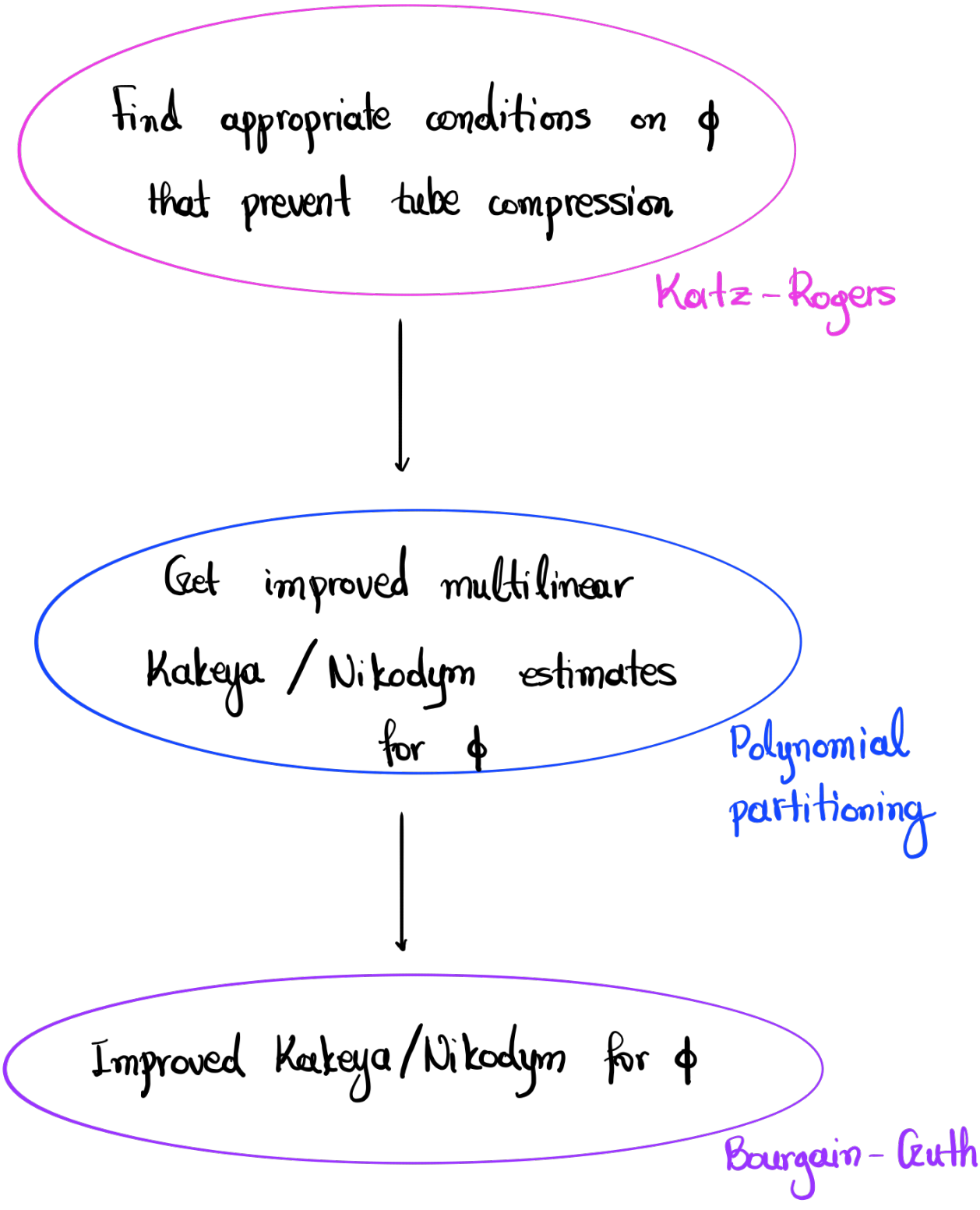
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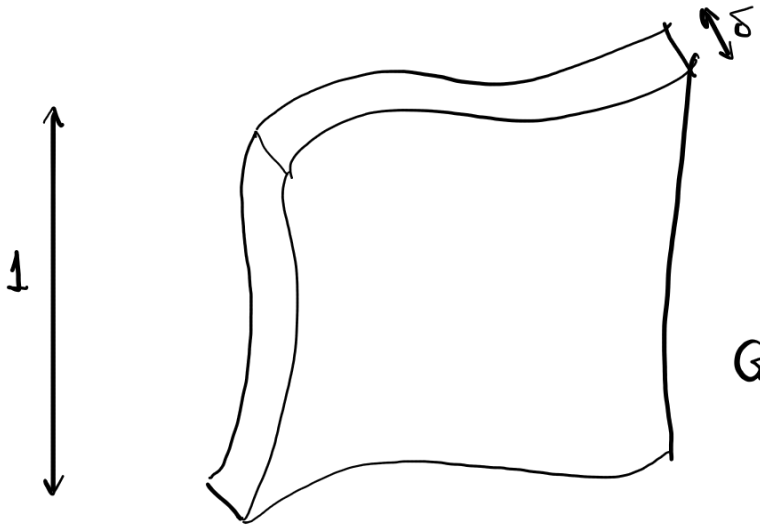
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↓

Improved oscillatory integral estimates

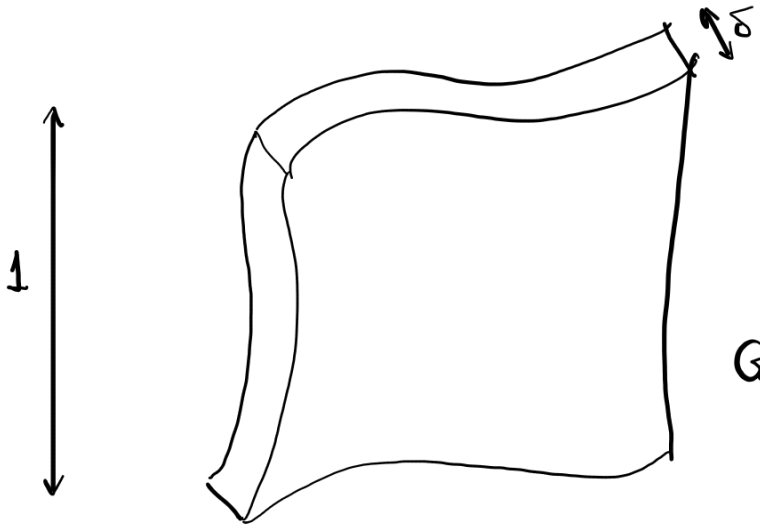
Bourgain

Thoughts - Kakeya case:



G: grain of dimension $\frac{n+1}{2}$

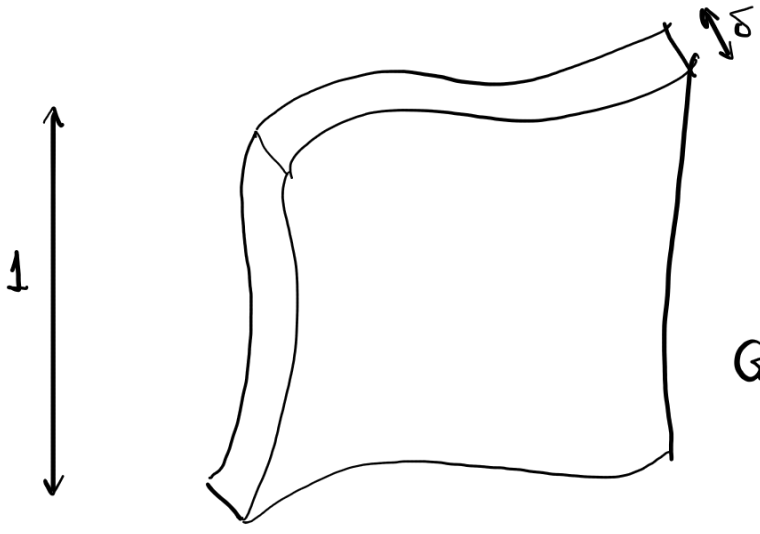
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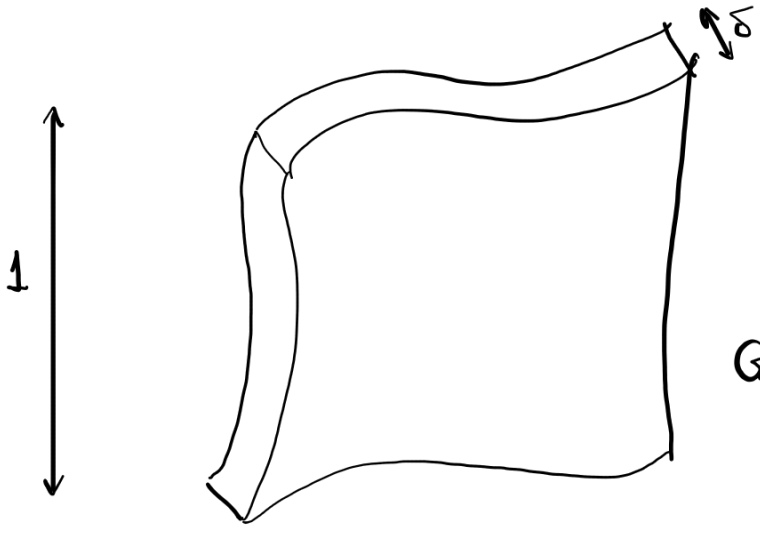
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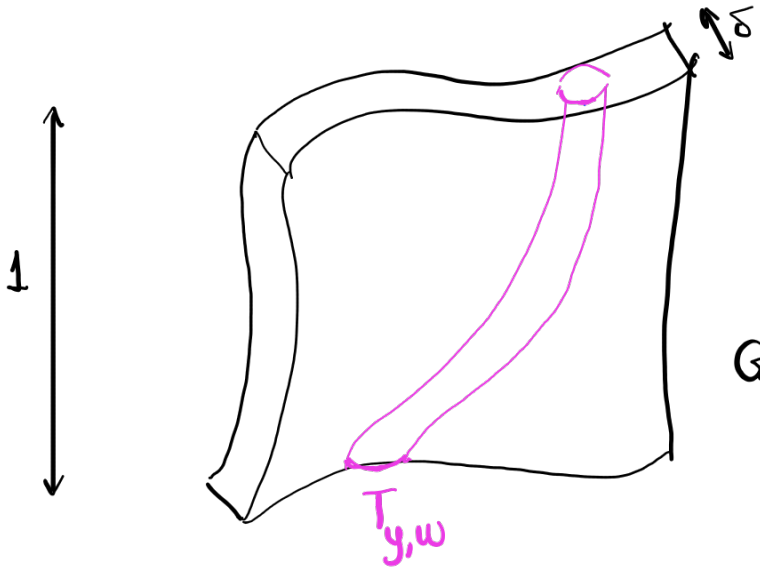
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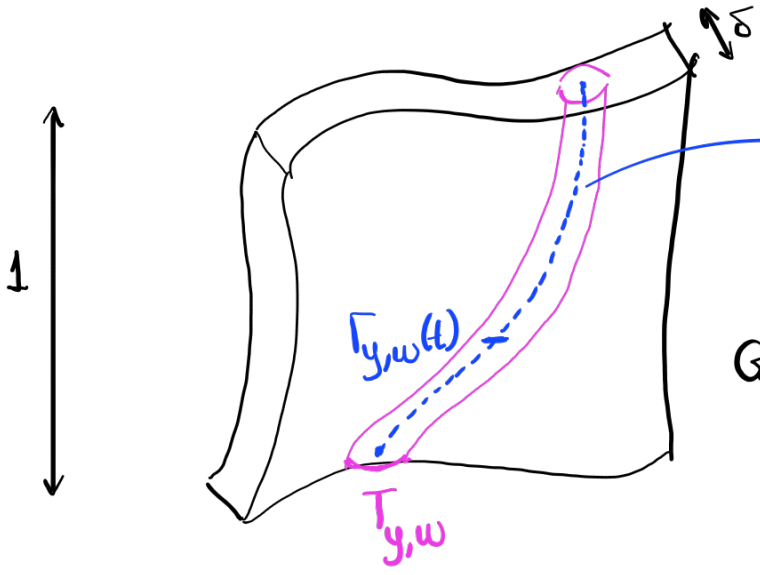
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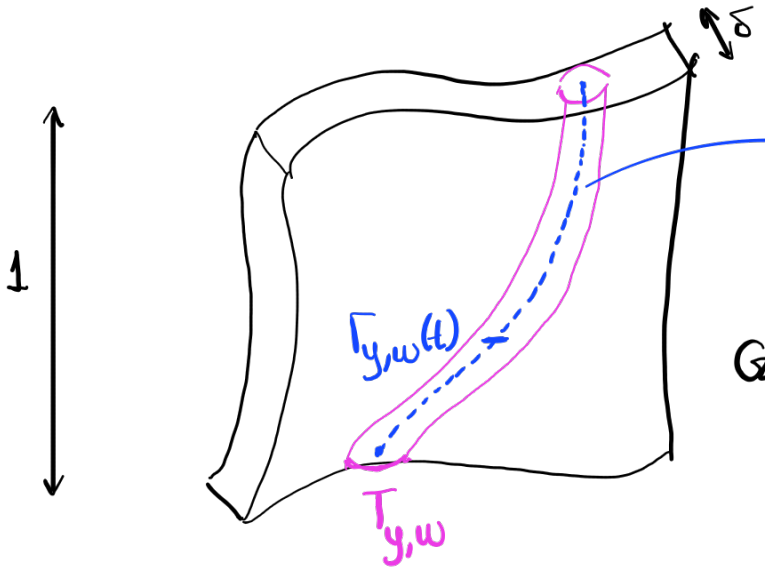


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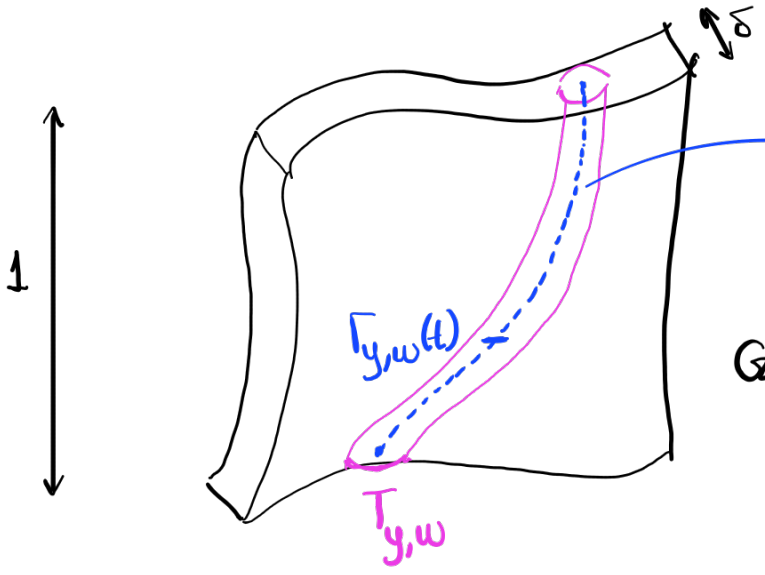
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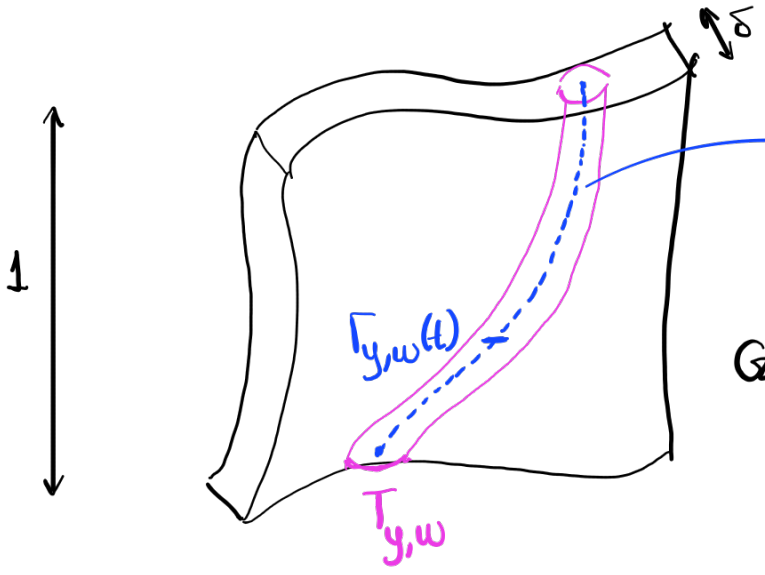
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$$\Downarrow$$

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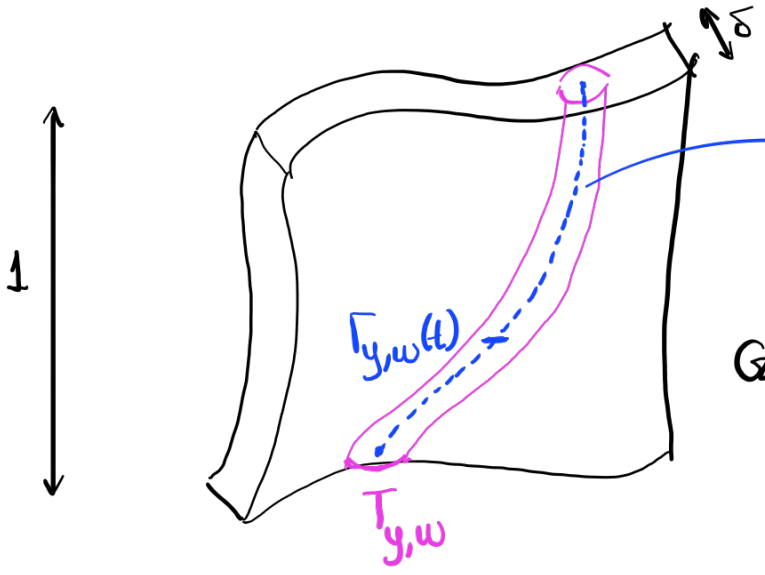
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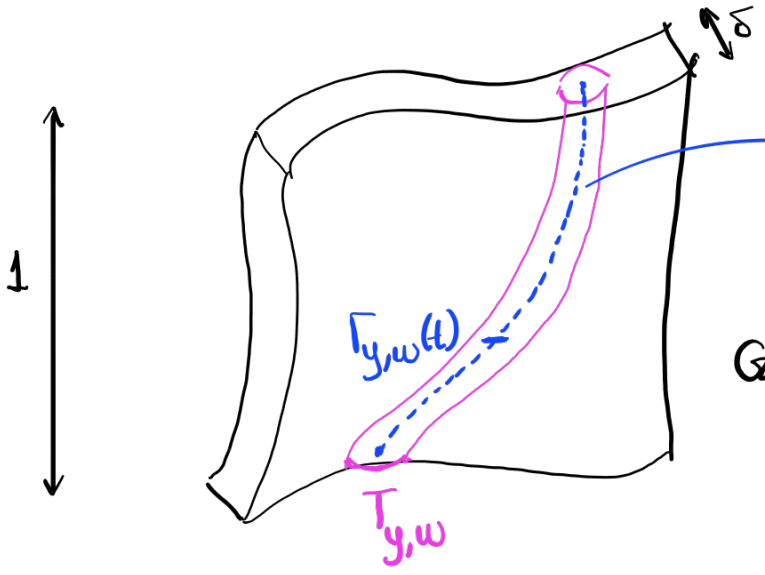
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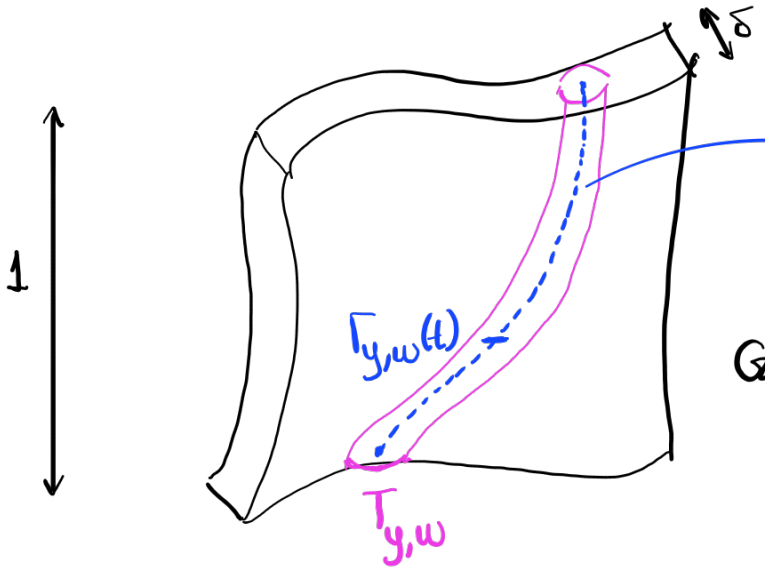
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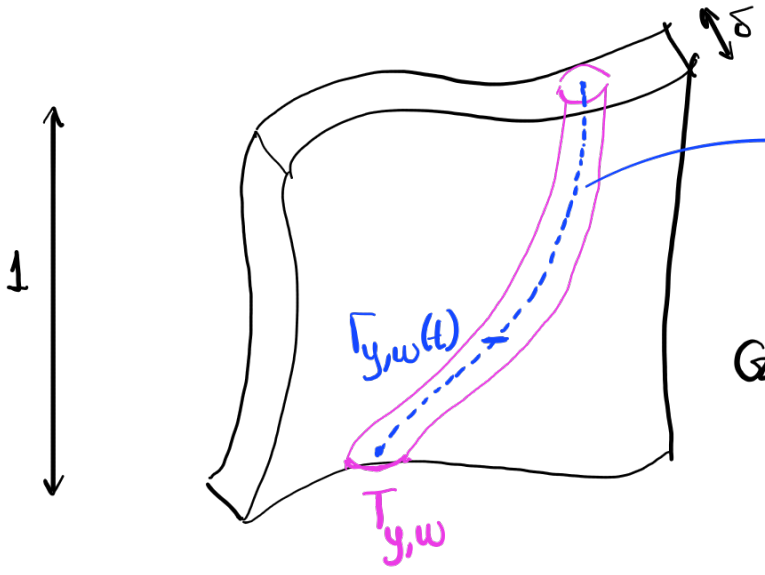
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$$\Phi : Y \times (-1,1) \longrightarrow G$$
$$(y,t) \longrightarrow \Gamma_{y,w(y)}(t)$$

Thoughts - Kakeya case:



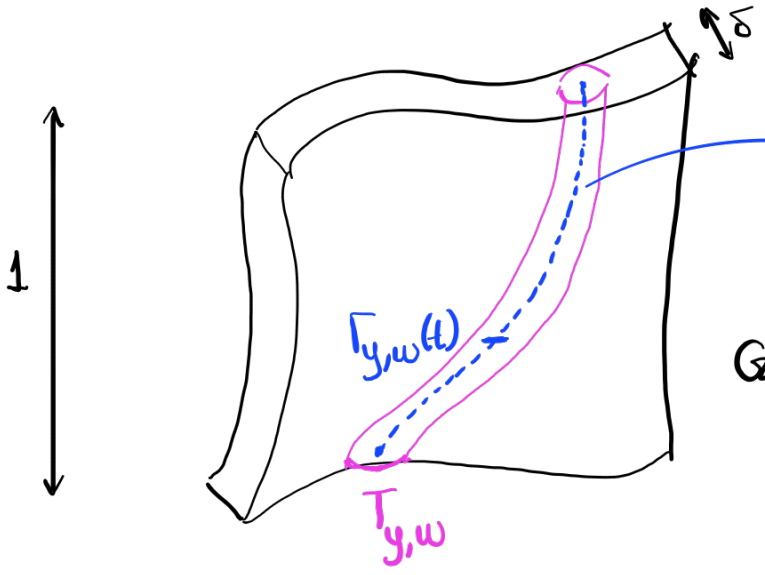
$\Gamma_{y,w} = \{(x,t) : \partial_y \phi(x,t;y) = w\}$

G: grain of dimension $\frac{n+1}{2}$

Need condition on ϕ s.t. a maximal set of direction δ-separated δ-tubes $T_{y,w}$ will not concentrate in G.

- $\left| \bigcup_{T \in \mathbb{T}} T \right| \gtrsim \left| \{ \Phi(y,t) : (y,t) \in Y \times (-1,1) \} \right|$

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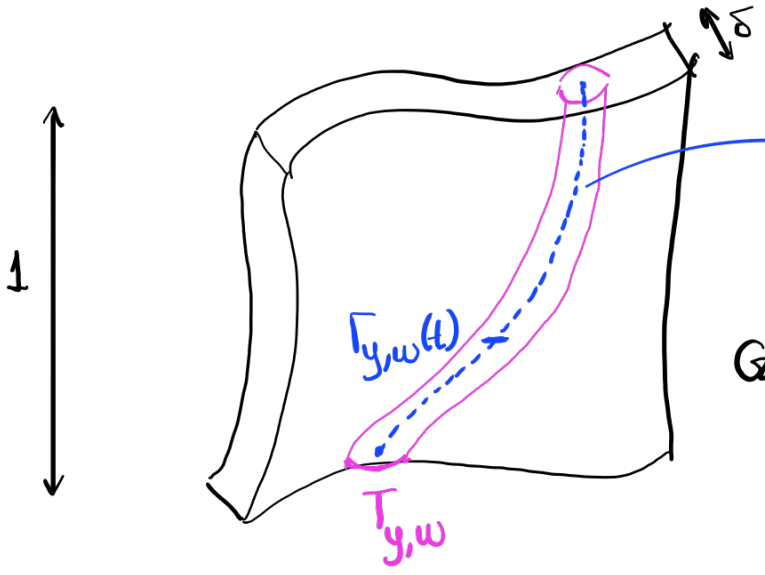
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$J\Phi(y,t)$: $\Gamma_{y, w(y)}(t)$

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(δ^{n-m})

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- Translates to:

$$\det\left(\partial^2_{yy}\phi\left(\Gamma_{y,w(y)}(t);y\right)-M(y)\right) \text{ large},$$

$\forall\ w$, M measurable.

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$$\det\left[\partial^2_{yy}\phi\left(\Gamma_{y,w(y)}(t);y\right)-M(y)\right]_{a,b} \text{ large},$$

$\forall\ w, M$ measurable.

an $\frac{n+1}{2}\times\frac{n+1}{2}$ minor

- Want: $\exists\ a, b$ s.t.: $\forall\ w$, M measurable,

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$\forall\ \gamma_1, \ldots, \gamma_m$,

$\left| f(t) - \sum_{i=1}^{m} \gamma_i g_i(t) \right|$ is small on small set.

ok if:

1.

AND

2.

OK if:

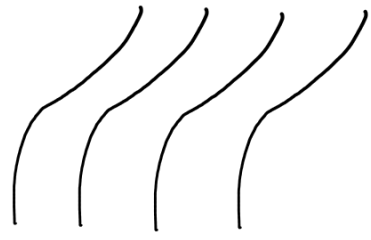
1. ϕ real analytic + "translation invariant":

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$\forall\ y \rightsquigarrow$ 

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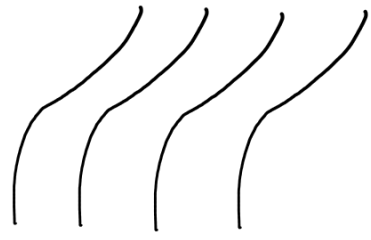
$\Gamma_{y,w}$ as w varies

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$$\phi(x,t;y) = \langle x,y\rangle + \psi(t,y), \text{ so:}$$

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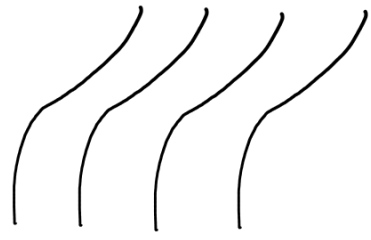
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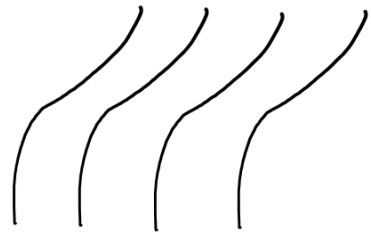
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2. $\exists\ a,b$ ($\frac{n+1}{2}$ rows, $\frac{n+1}{2}$ columns) s.t.:

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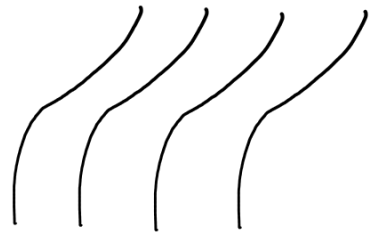
2. $\exists\ a,b$ ($\frac{n+1}{2}$ rows, $\frac{n+1}{2}$ columns) s.t.:

$$\text{for a.e. } y, \quad \det\left[\partial^2_{yy}\phi\ (0,\cdot\,;y)\right]_{a,b}$$

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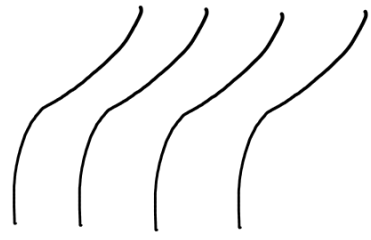
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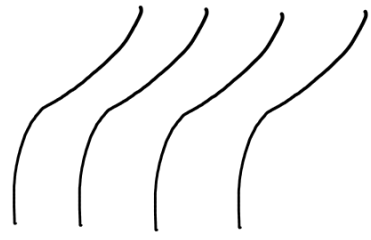
$$\text{for a.e. } y,\quad \det\Big[\overbrace{\partial^2_{yy}\phi\,(0,\cdot;y)}^{\text{a function of } t}\Big]_{a,b}$$

is <u>not</u> a linear combination

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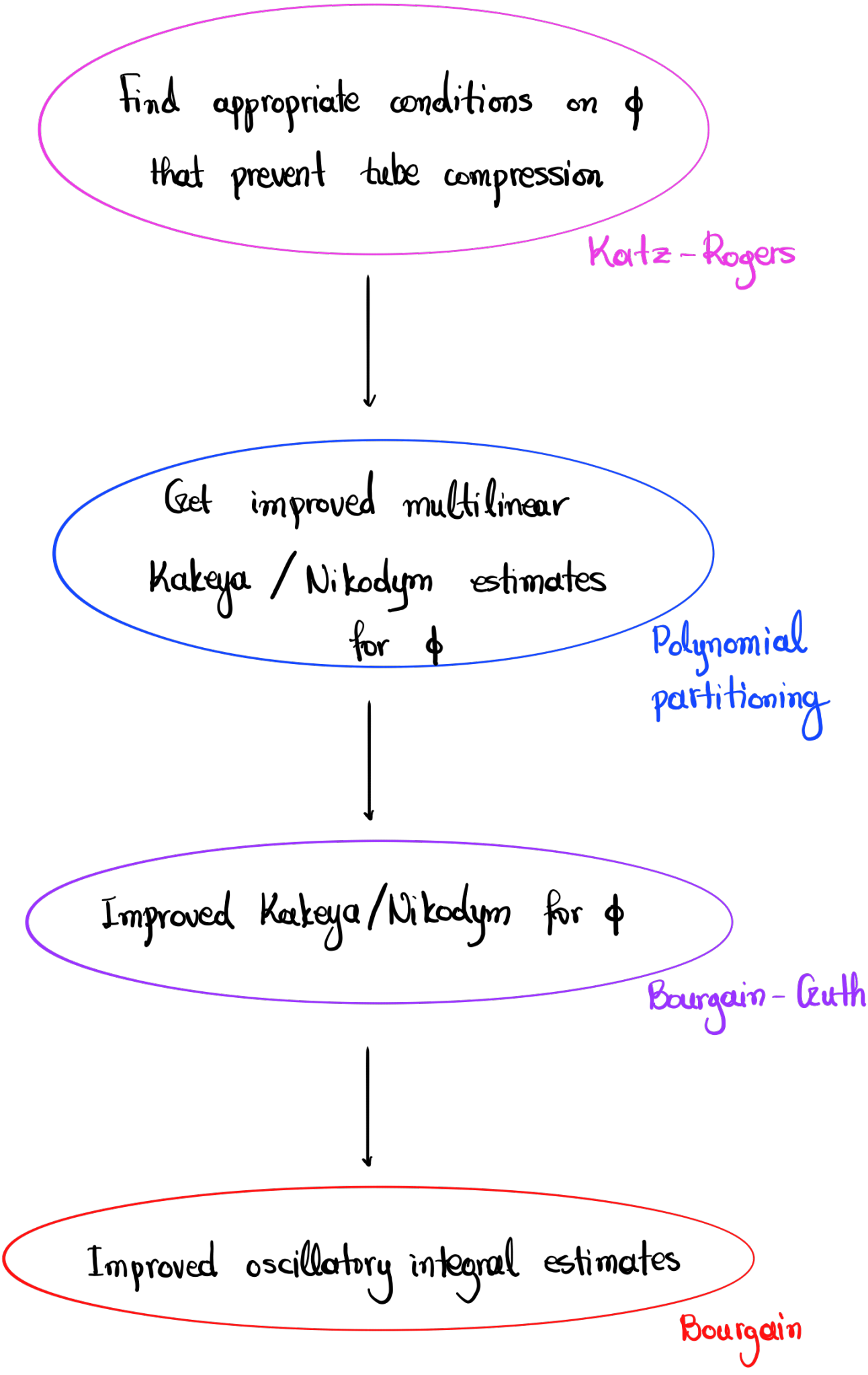
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for a.e. y, $\det\left[\partial^2_{yy}\phi\ (0,\cdot\,;y)\right]_{a,b}$ (a function of t)

is <u>not</u> a linear combination

of dets of minors of $\left[\partial^2_{yy}\phi\ (0,\cdot\,;y)\right]_{a,b}$.

Plan:


Find appropriate conditions on ϕ that prevent tube compression
Katz-Rogers
Get improved multilinear Kakeya / Nikodym estimates for ϕ
Polynomial partitioning
Improved Kakeya/Nikodym for ϕ
Bourgain-Guth
Improved oscillatory integral estimates
Bourgain

Thank you !