Non-commutative maximal inequality on amenable groups

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based on joint work with S. Wang (Harbin)

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(Ω, μ) a measure space
$T : \Omega \to \Omega$ measure preserving

Theorem (Birkhoff)

Let $p \in [1, \infty)$ and $f \in L_p(\Omega)$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(T^n \omega) = F(\omega) \quad \text{a.e.}$$

where F is a T-invariant function ($T(f) := f(T\cdot)$).

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Goal

- Replace T by an amenable group of transformation G
- Replace (Ω, μ) by a noncommutative measure space $(\mathcal{M}, \tau)$

Ergodic theory of amenable group actions

- Birkhoff ('31), Calderón ('53), Templeman ('67), Emerson-Greenleaf ('74)
- Ornstein-Weiss ('87)): Rohklin lemma, leap in the understanding of amenable group actions
- Lindenstrauss (2001): general amenable groups

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Early days of noncommutative ergodic theory (70's)

- Lance (1975): $\mathcal{M}$ finite and $x \in \mathcal{M}$, single transformation
- followed by Kümmerer (1978) and Conze, Dang Ngoc (1979)
- Yeadon(1977): $\mathcal{M}$ tracial and $x \in L_1(\mathcal{M})$

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After developments in nc harmonic analysis

- Junge, Q.Xu (2006): $x \in L_p(\mathcal{M})$, $p \in [1, \infty)$, single transformation
- Hong, Liao, Wang (2020): G of polynomial growth

Noncommutative integration

- Origins in Murray and von Neumann's work on operator algebras in the 40's, then mathematical physics and Banach space geometry in the 70's.

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Classical integration	**Noncommutative integration**
$(\Omega, \mathcal{A}, \mu)$	$\dots$
$L_\infty(\Omega)$	$\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ une algèbre de von Neumann
f a function	x an operator
$\int$	τ a trace with nice properties
$\|f\|_p$, $L_p(\Omega)$	$\tau(\|x\|^p)^{1/p}$, $L_p(\mathcal{M})$
χ_E a characteristic function	e a projection

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- we can also define other noncommutative function spaces and in particular **weak L_p-spaces**

$$\|x\|_{p,\infty} = \sup_{\lambda>0} \lambda\tau\left(\{|x| \geq \lambda\}\right)^{1/p}.$$

Example of nc measure spaces

- $\mathcal{M} = \mathbb{M}_n(\mathbb{C})$ and $\tau = \mathrm{tr}$. In this case, if $x \in \mathcal{M}$ is normal, and $\lambda_1, \ldots, \lambda_n$ are its eigenvalues,

$$\|x\|_p = \left(\sum_{i=1}^{n} |\lambda_i|^p \right)^{1/p}.$$

  For $n = \infty$, $\mathcal{M} = \mathcal{B}(\ell^2)$.

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- *Semicommutative spaces:* $\mathcal{N} = \mathcal{M} \overline{\otimes} L_\infty(\Omega)$ the algebra of bounded measurable $\mathcal{M}$-valued functions on Ω with a natural trace φ.

$$\varphi(f) = \int_\Omega \tau(f(\omega)) d\omega = \tau\left(\int_\Omega f(\omega) d\omega \right).$$

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- von Neumann algebras can also be constructed from groups and dynamical systems

Noncommutative harmonic analysis

- '78 Haagerup shows that $C^*(\mathbb{F}_n)$ has AP using Fourier multipliers
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Nc Khinchin inequalities

Let $2 \leq p < \infty$ and $(x_i)_{i \leq N}$ be elements in $L_p(\mathcal{M})$, then

$$\mathbb{E}_\varepsilon \left[|| \sum_{i \leq n} \varepsilon_i x_i ||_p \right] \approx_p \max \left(||(\sum_{i \leq n} |x_i|^2)^{1/2}||_p, ||(\sum_{i \leq n} |x_i^*|^2)^{1/2}||_p \right)$$

where ε_i are independent Rademacher variables.

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- A dual formula for $p \leq 2$, extended to $p < 1$ and other function spaces
- Gives a first understanding of nc square functions

Maximal norms in $L_p(\mathcal{M})$

$L_p(\mathcal{M}, \ell^\infty)$ (Pisier, Junge)

Let (x_n) be a sequence in $L_p(\mathcal{M})$. Set

$$\left\| \sup_{n\geq 1}^{+} x_n \right\|_p := \inf_{x_n = a y_n b} \|a\|_{2p} \, \|b\|_{2p} \sup_{n\geq 1} \|y_n\|_\infty .$$

If each x_n is positive then

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- We say that $\left\| \sup_{n\geq 1}^{+} x_n \right\|_{1,\infty} \leq C$ if for any $\lambda > 0$ there exists a projection e $(e = \sup_{n\geq 1}^{+} x_n \leq \lambda)$ such that

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  - $\lambda \tau(1-e) \leq C$,
  - $\forall n \geq 1 \ \ \|e x_n e\|_\infty \leq \lambda$.

A nc analog of Birkhoff's theorem

Theorem (Junge, Xu '07)

Let $p \in (1, \infty)$ and T be a trace preserving automorphism of $\mathcal{M}$. Then for any $x \in L_p(\mathcal{M})$

$$\left\| \sup_{n \geq 1}^{+} \mathbb{A}_n(x) \right\|_p \lesssim \|x\|_p .$$

As a consequence, $(\mathbb{A}_n(x))_{n \geq 1}$ converges a.u.

$\mathbb{A}_n(x) = \frac{1}{n} \sum_{k=1}^{n} T^k x.$

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What next?

- Ergodic theorem for a positive contraction on a single L_p-space
- Trace replaced by a weight, general notion of nc measure
- Multiple transformations

Set up

$\mathcal{M}$ a von Neumann algebra and τ a n.s.f. trace on $\mathcal{M}$
G a second countable amenable locally compact group, m a Haar measure on G (also denoted by $|\cdot|$)
α an action of G on $\mathcal{M}$ by $*$-automorphisms
(F_n) a sequence of compact subsets of G

Notation (ergodic averages)

Denote by A_n^α the operator defined for any measurable function $x \in L_0(\mathcal{M})$ by

$$A_n^\alpha(x) := \frac{1}{|F_n|} \int_{g \in F_n} \alpha_g(x) dm(g).$$

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Question

Can we find a sequence $(F_n)_{n \geq 0}$ such that for any $x \in L_p(\mathcal{M})$, $p \in [1, \infty)$

$$A_n^\alpha(x) \to P(x) \quad \text{“pointwise”}$$

where $P(x)$ is G-invariant?

Asymptotic invariance

$G = \mathbb{Z}$, $F_n = \{1, \ldots, n\}$

Key observation

For any $f \in L_p(\Omega)$,

$$\| T(A_n^\alpha(f)) - A_n^\alpha(f) \|_p \leq \frac{2}{n} \| f \|_p .$$

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Proof:

$$\begin{aligned} \| T(A_n^\alpha(f)) - A_n^\alpha(f) \|_p &= \left\| \frac{1}{n} \sum_{k \in F_n + 1} T^k(f) - \frac{1}{n} \sum_{k \in F_n} T^k(f) \right\|_p \\ &\leq \frac{|F_n \Delta (F_n + 1)|}{|F_n|} \| f \|_p . \end{aligned}$$

Precising the question: amenable groups

Definition (Følner sequence)

A sequence $(F_n)_{n\geq 1}$ of subsets of G is said to be a *Følner sequence* if for any $K \subset G$ compact,

$$\lim_{n\to\infty} \frac{|K.F_n \setminus F_n|}{|F_n|} = 0.$$

G admits a Følner sequence if and only if G is *amenable*.

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Mean ergodic theorem for Følner sequences

Assume that $(F_n)_{n\geq 1}$ is a Følner sequence. Let $f \in L_p(\Omega)$, $p \in (1,\infty)$ then

$$\|A_n^\alpha(f) - P(f)\|_p \to 0$$

where P is a projection onto G-invariant functions.

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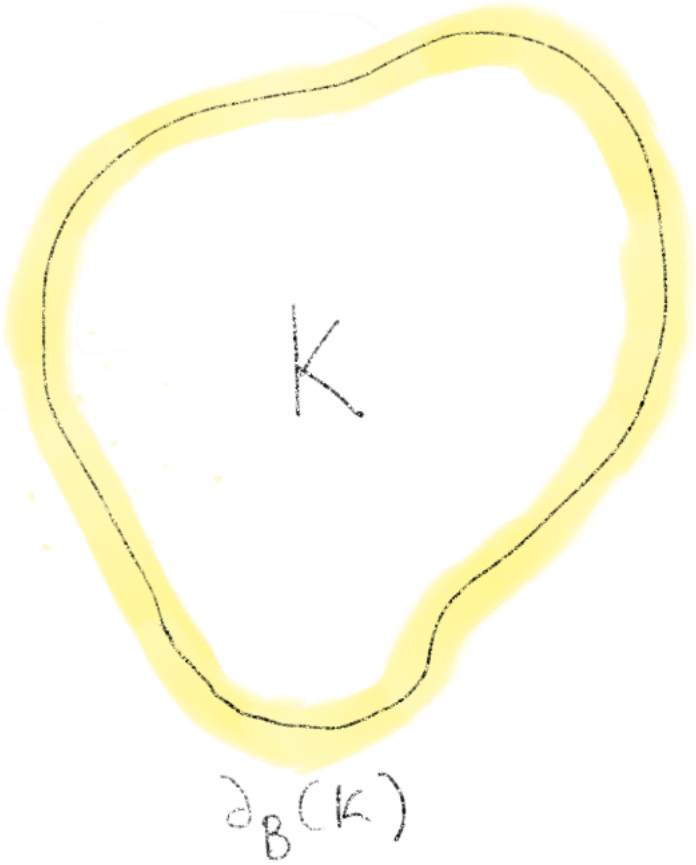
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Not enough for pointwise convergence!

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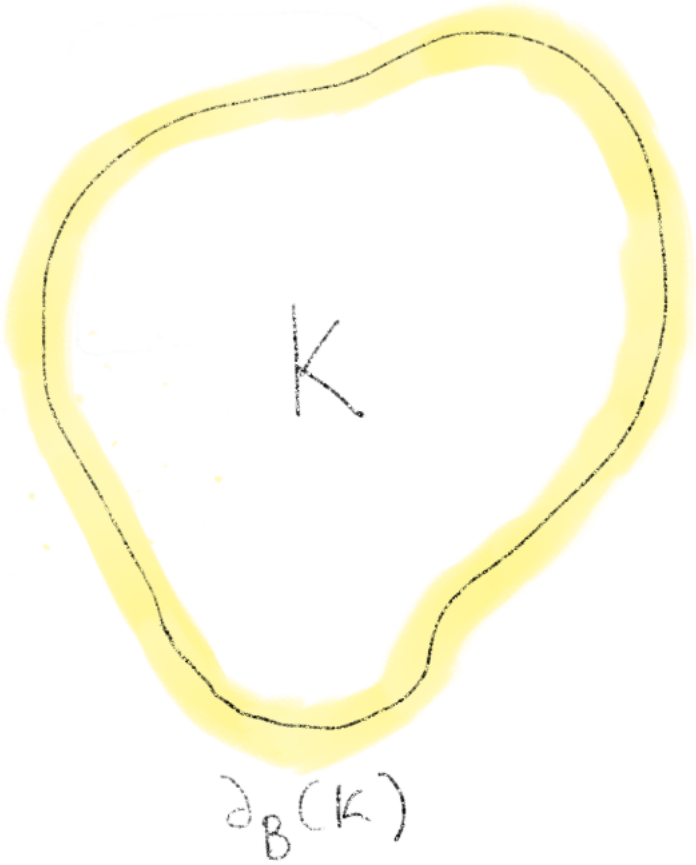
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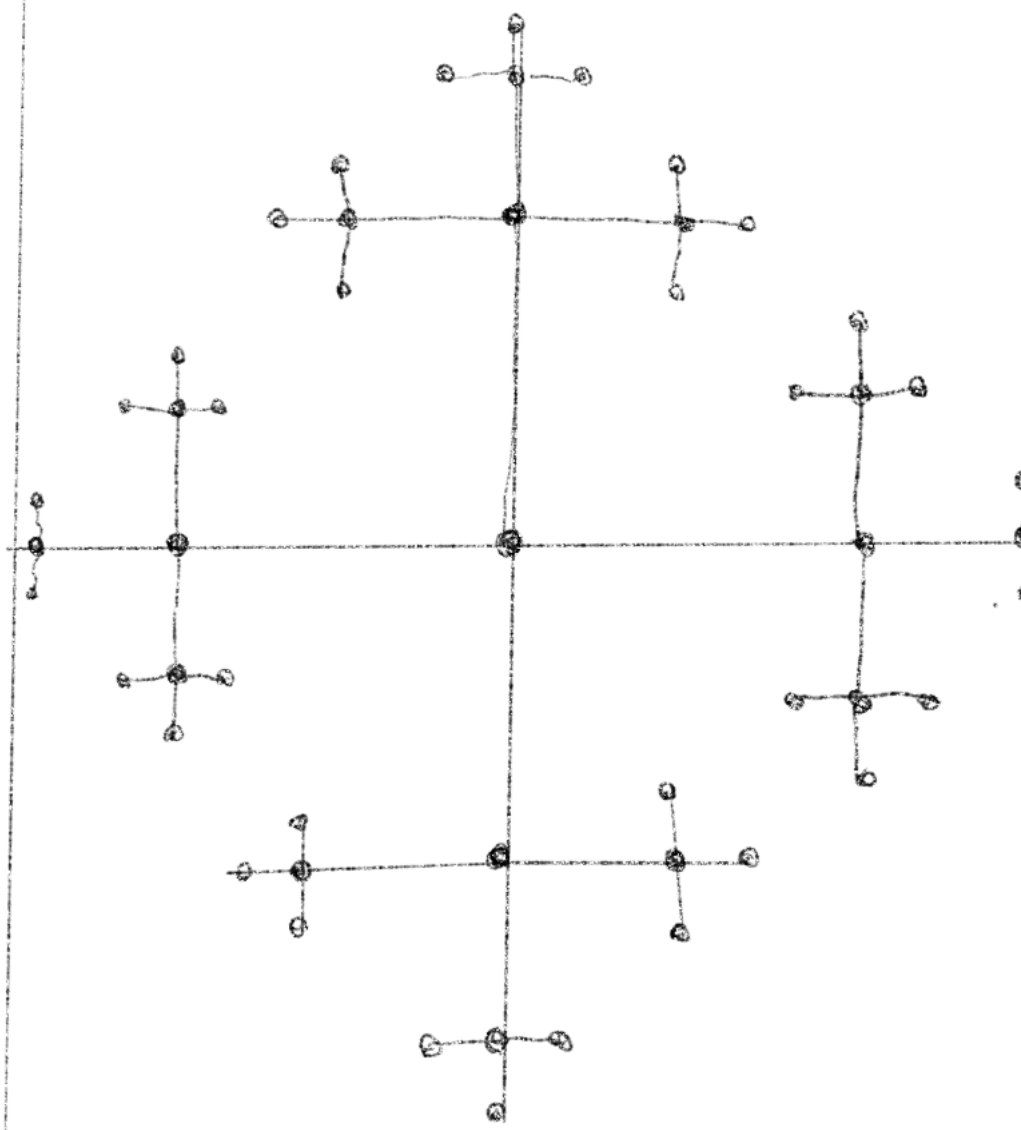


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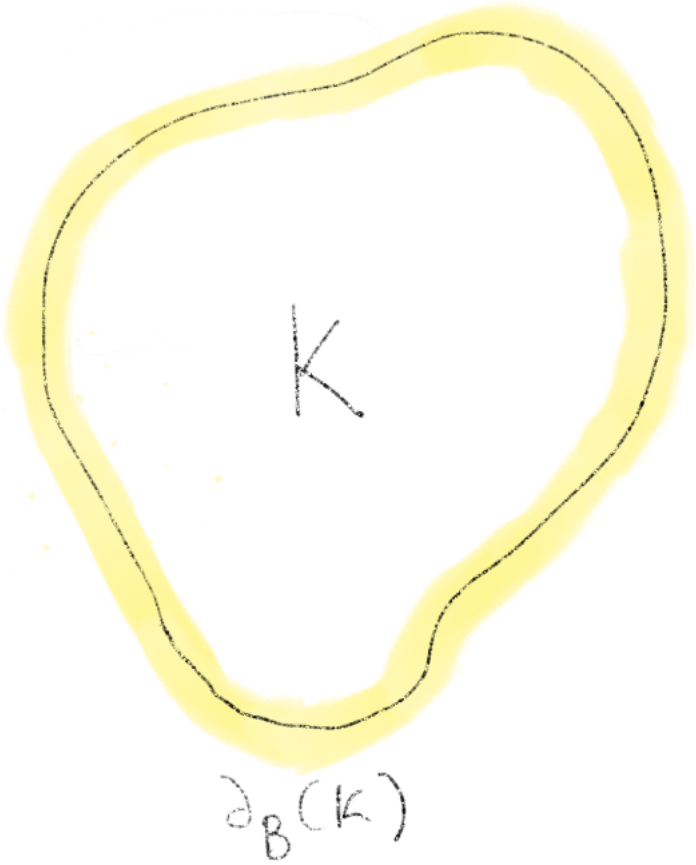
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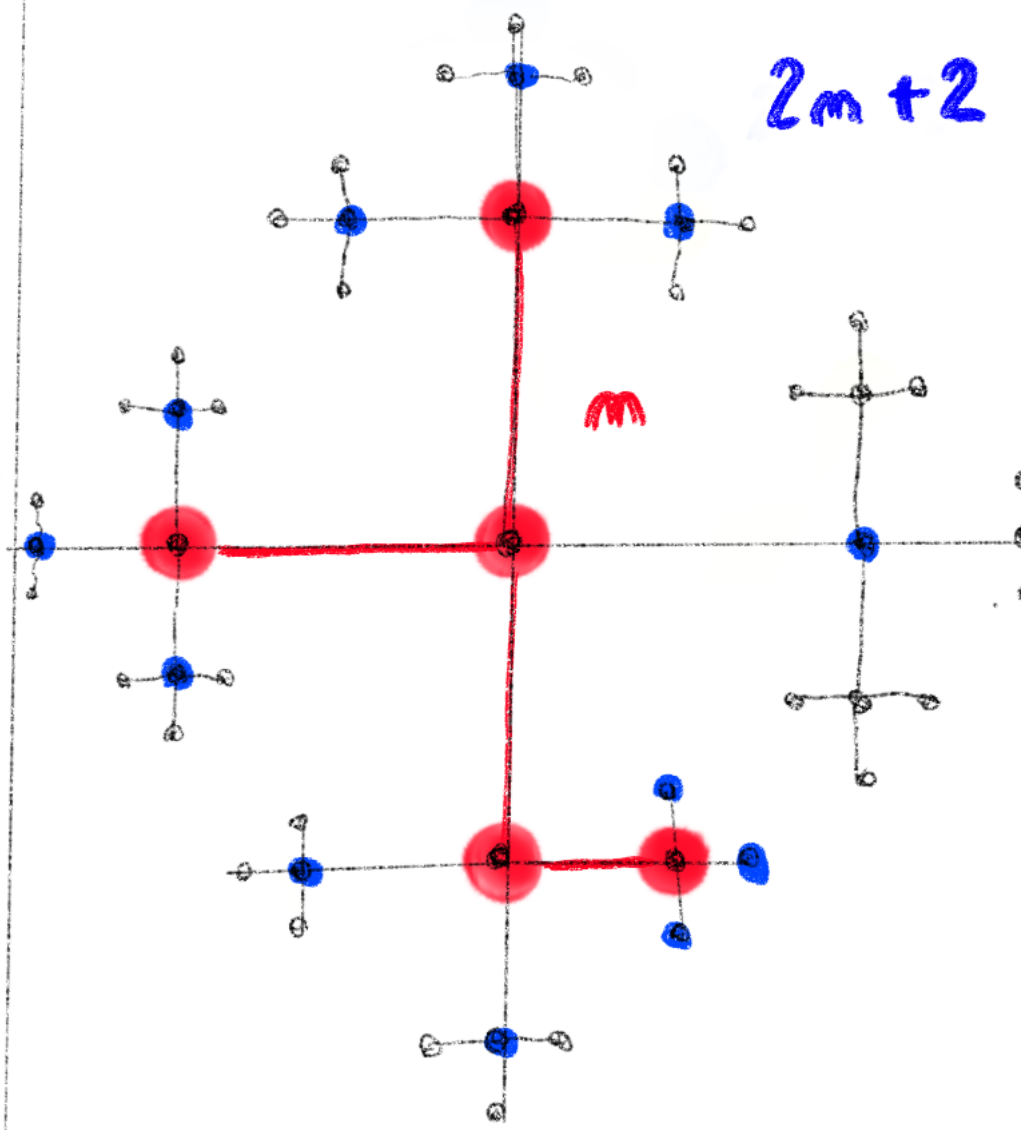
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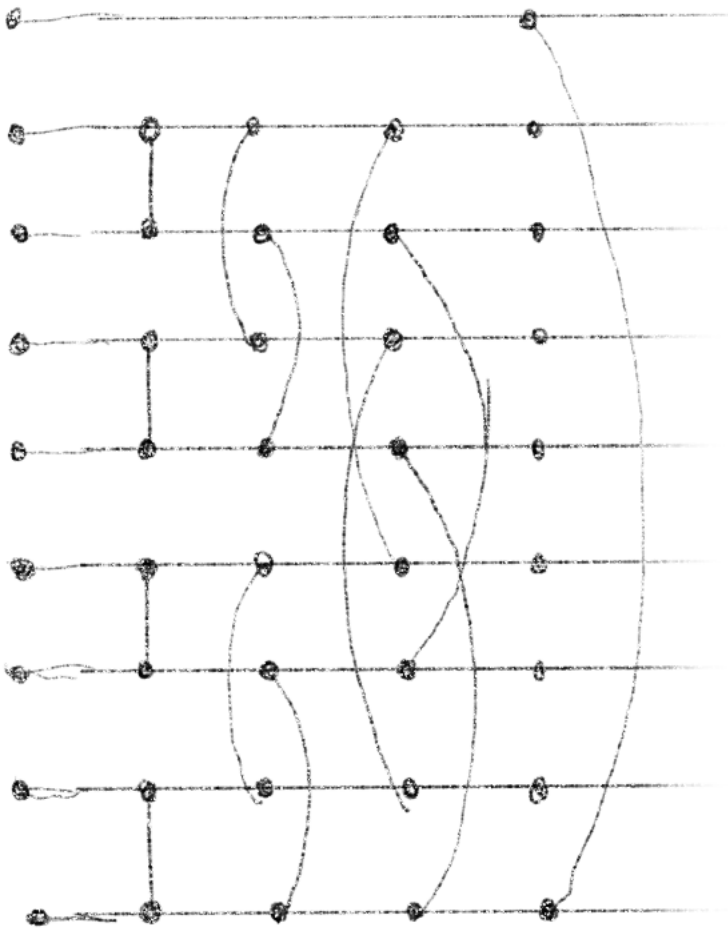
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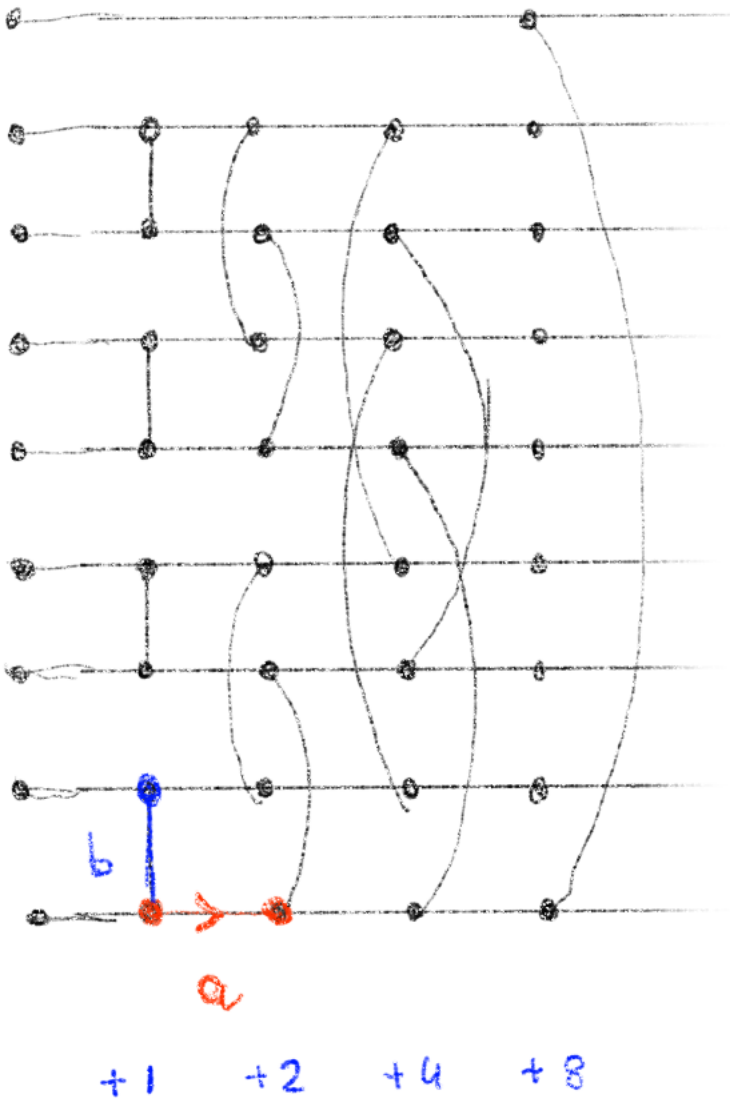


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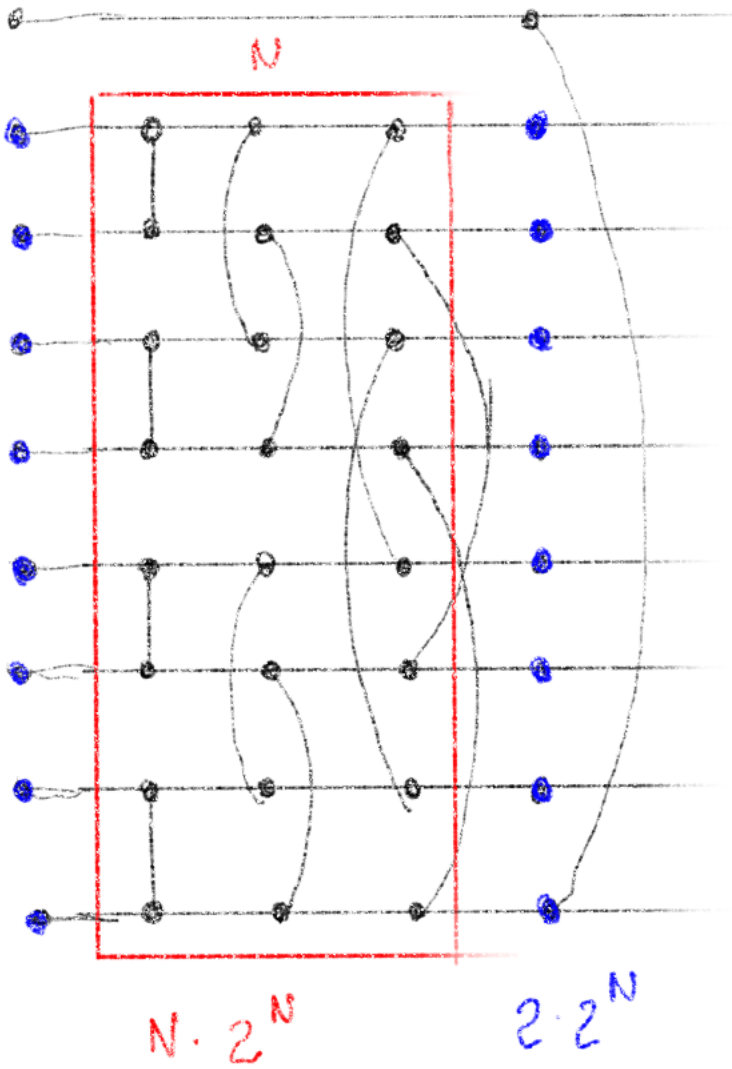


$b = b^{-1}$

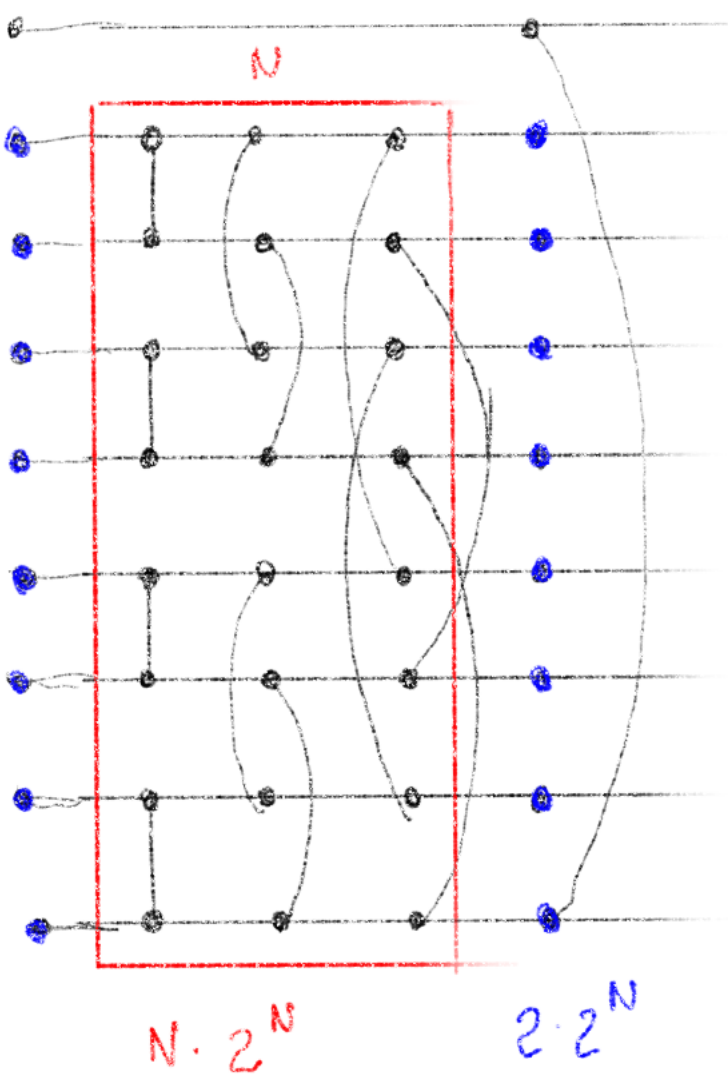
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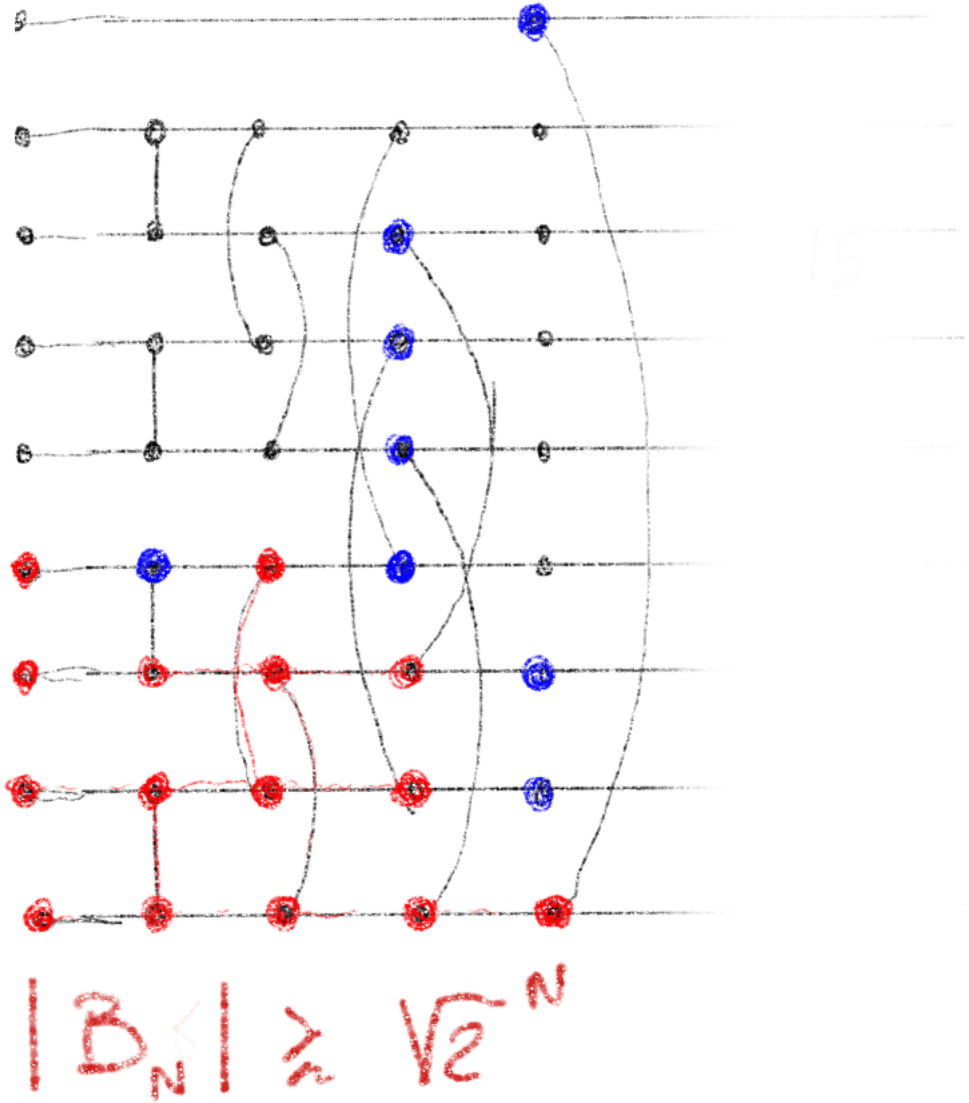


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For groups of polynomial growth

G group of polynomial growth
α an action of G on $\mathcal{M}$ by trace preserving automorphisms
S a finite symmetric generating set of G, $F_n = S^n$.

Theorem (Hong, Liao, Wang '21)

Let $p \in (1, \infty)$ and $x \in L_p(\mathcal{M})$ then $A_n^\alpha(x)$ converges a.u. to Px its projection onto fixed points of α.

- Maximal inequality $\Rightarrow$ a.u. convergence,
- **Transference:** it suffices to consider the action π of G onto $\ell^\infty(G)\overline{\otimes}\mathcal{M}$,
- Limiting factor: maximal inequality for π (from now on we omit π).

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Theorem (C., Wang)

There exists $(F_n)_{n\geq 0}$ and a constant $c > 0$ independent of f (and $\mathcal{M}$) such that

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1. The Følner condition guarantees convergence for a dense subset and so the missing ingredient is the maximal inequality.
2. The weak type $(1,1)$ inequality implies maximal inequalities in L_p, $p > 1$ by an interpolation theorem of Junge and Q. Xu.
3. The method also provides jump and variational inequalities, completing Lindenstrauss's result

Tools

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- Their non-commutative analogs (Hong-Xu, C., Conde-ALonso-Parcet)

Euclidean setting: dyadic filtration

- Set $\mathcal{Q}_n = \left\{[0, 2^n)^d + k2^n : k \in \mathbb{Z}^d\right\}$. $\mathcal{Q}_n$ is a partition of $\mathbb{R}^d$ into dyadic cubes of edge-length 2^n.

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- For $t \in \mathbb{R}^d$, set $Q_{t,n}$ to be the element of $\mathcal{Q}_n$ containing t.
- For $f \in L_1(\mathcal{N})$, we define

$$\mathbb{E}_n[f](t) = \frac{1}{|Q_{t,n}|} \int_{Q_{t,n}} f(u)du.$$

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$$\mathbb{E}_n[f](t) = \frac{1}{|Q_{t,n}|} \int_{Q_{t,n}} f(u)du.$$

- $\mathbb{E}_n$ is the conditional expectation from $L_\infty(\mathbb{R}^d)$ to the algebra of bounded functions that are constant on every cube of $\mathcal{Q}_n$.

Doob/Cuculescu's maximal inequality

For $f \in L_1(\mathbb{R}^d)$, we have

$$\left\|\sup\nolimits_{n\geq 0}^{+} \mathbb{E}_n[f]\right\|_{1,\infty} \leq c \left\|f\right\|_1 .$$

Avoiding covering lemmas: the euclidean case

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The ball/grid difference operator

Set for any $n \geq 0$, $\mathbb{D}_n = \mathbb{A}_n - \mathbb{E}_n$ and

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Theorem (Jones et al., Hong-Xu)

For any f in $L_1(\mathbb{R}^d)$ then
$$\|D[f]\|_{1,\infty} \leq c \|f\|_1 .$$

Steps of the proof

Recall that $\mathbb{D}_k := \mathbb{A}_k - \mathbb{E}_k$.

- Local estimates:
  - if $\mathbb{E}_n(f) = 0$, for $p \in [1, \infty]$,

$$\|\mathbb{D}_k(f)\|_p \leq 2^{n-k} \|f\|_p ,$$

  - if $f \in \mathcal{N}_n$, for $p \in [1, \infty)$,

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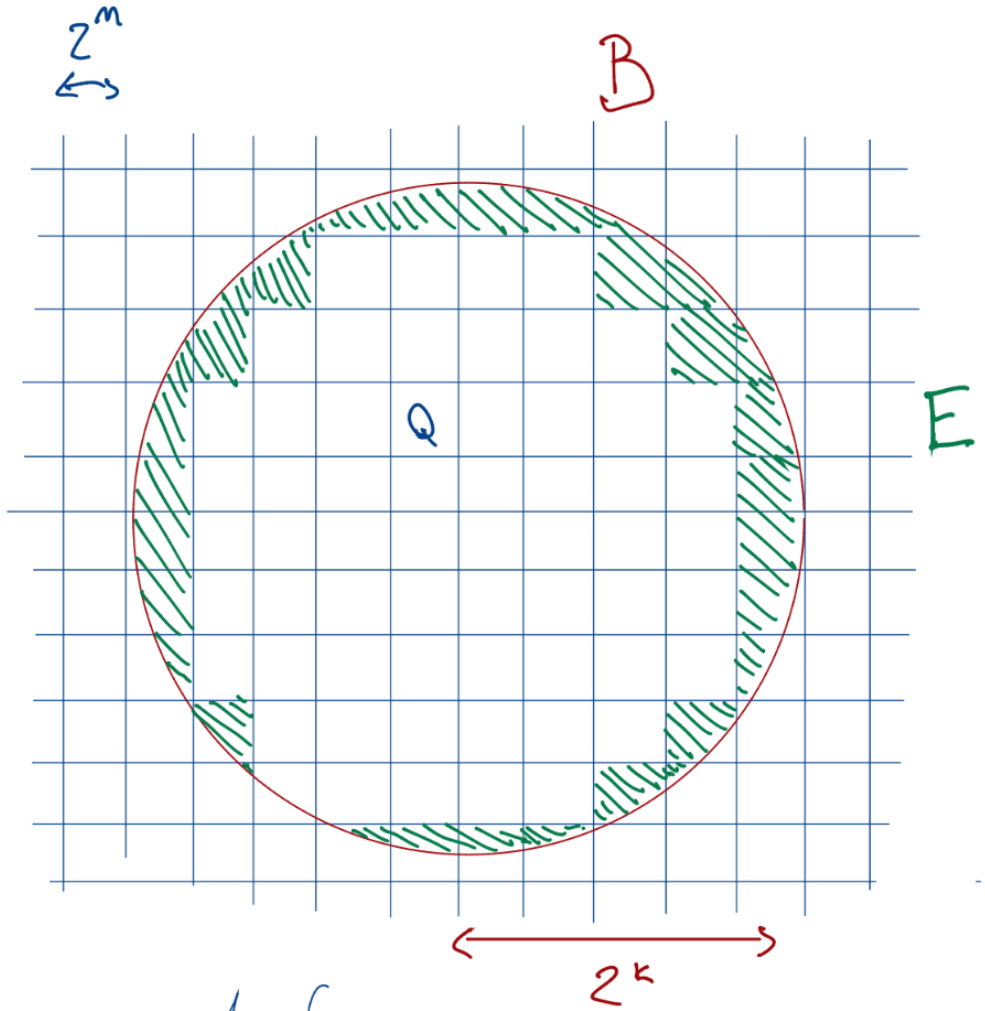
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$$f = g + h + b.$$
  - $g = \sum_n g_n$ with g_n adapted to the n-th generation of the filtration
  - same for h and b.

Geometry of local estimates



- $\int_Q f = 0$
- $\frac{|E|}{|B|} \simeq \frac{2^{(d-1)k}\, 2^m}{2^{dm}}$

$\simeq 2^{k-m}$

$$\mathbb{D}_k(f) = \mathbb{A}_k(f) = \frac{1}{|B|} \int_E f(v)\, dv$$

A measure of invariance

Let G be a l.c. group

Relative boundary

Let $E, K \subset G$ and define

$$\partial_K(E) := \cup \{\gamma K : \gamma \in G, \gamma K \cap E \neq \emptyset, \gamma K \cap E^c \neq \emptyset\}.$$

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The local estimates of the previous slides rely on two quantities:

- when $k \geq n$, a measure of the invariance of F_k with respect to $Q_n := [0, 2^n)^d$

$$\frac{|\partial_{Q_k}(F_n)|}{|F_n|} \approx 2^{n-k}$$

  where Q_n is the "shape" of the elements of $\mathcal{Q}_n$.
- when $n > k$, a measure of the invariance of the elements A of $\mathcal{Q}_n$ with respect to F_k

$$\frac{|A \cdot F_n \setminus A|}{|A|} \approx 2^{k-n}.$$

The grid/ball scheme in amenable groups

G an amenable group. We need:

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Ergodic and martingale averages

$$\mathbb{A}_n[f](g) = \frac{1}{|gF_n|}\int_{gF_n} f(h)dh \quad \text{and} \quad \mathbb{E}_n[f](g) = \frac{1}{|Q_{n,g}|}\int_{Q_{n,g}} f(h)dh$$

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Theorem (Wang, C.)

Assuming suitable invariance conditions the resulting grid/ball difference operator $D = \left(\sum_{n\in\mathbb{N}} |\mathbb{A}_n - \mathbb{E}_n|^2\right)^{1/2}$ is of weak type $(1,1)$.

Invariance conditions and tilability

- For any $k \geq n$,
$$|\partial_{Q_k}(F_n)| \leq 2^{n-k} |F_n| .$$
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Definition (Tilability)

A group is *Følner tilable* if for any compact $K \subset G$ and any $\delta > 0$ there exists a partition $\mathcal{Q}$ of G such that:

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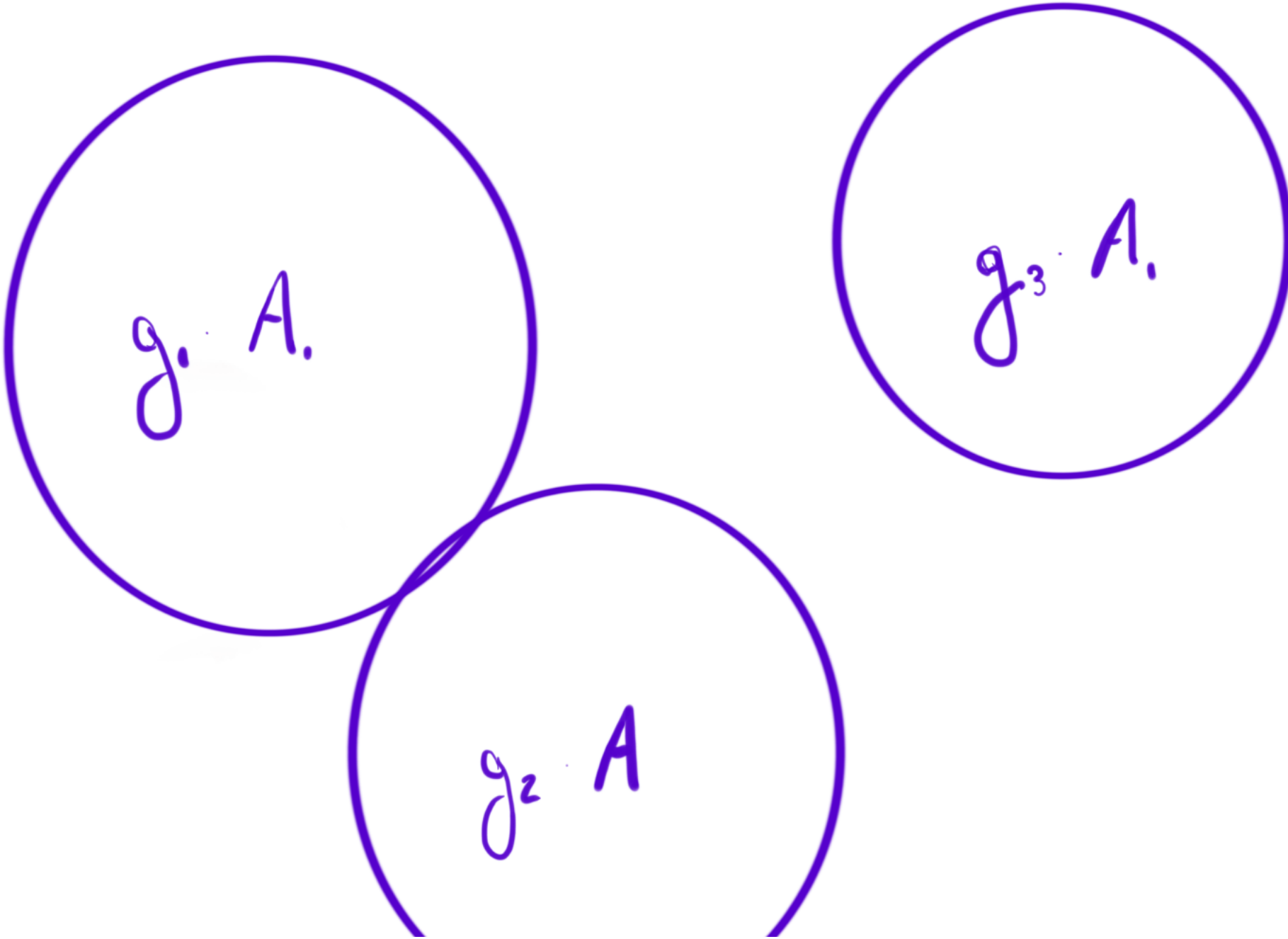
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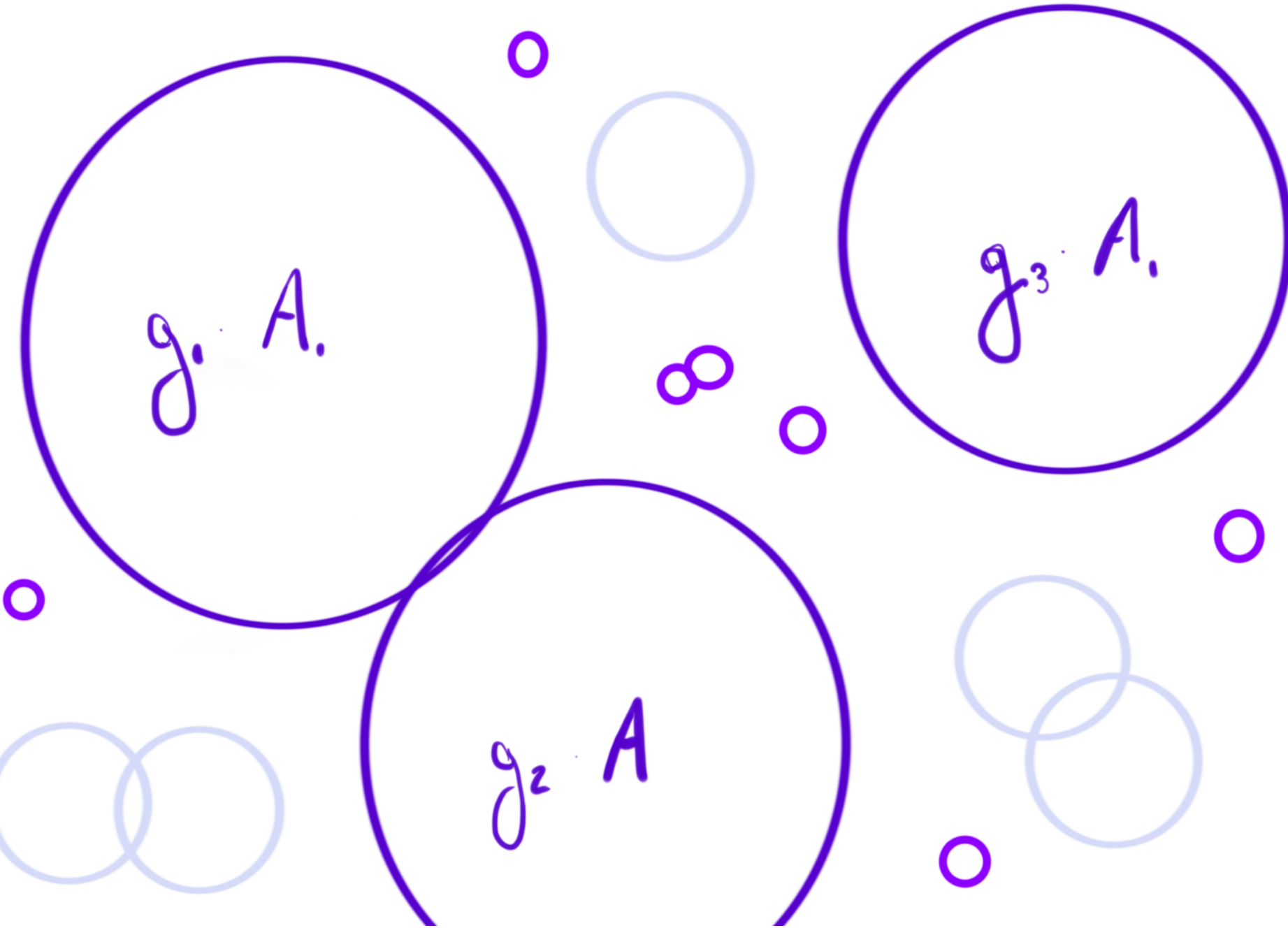
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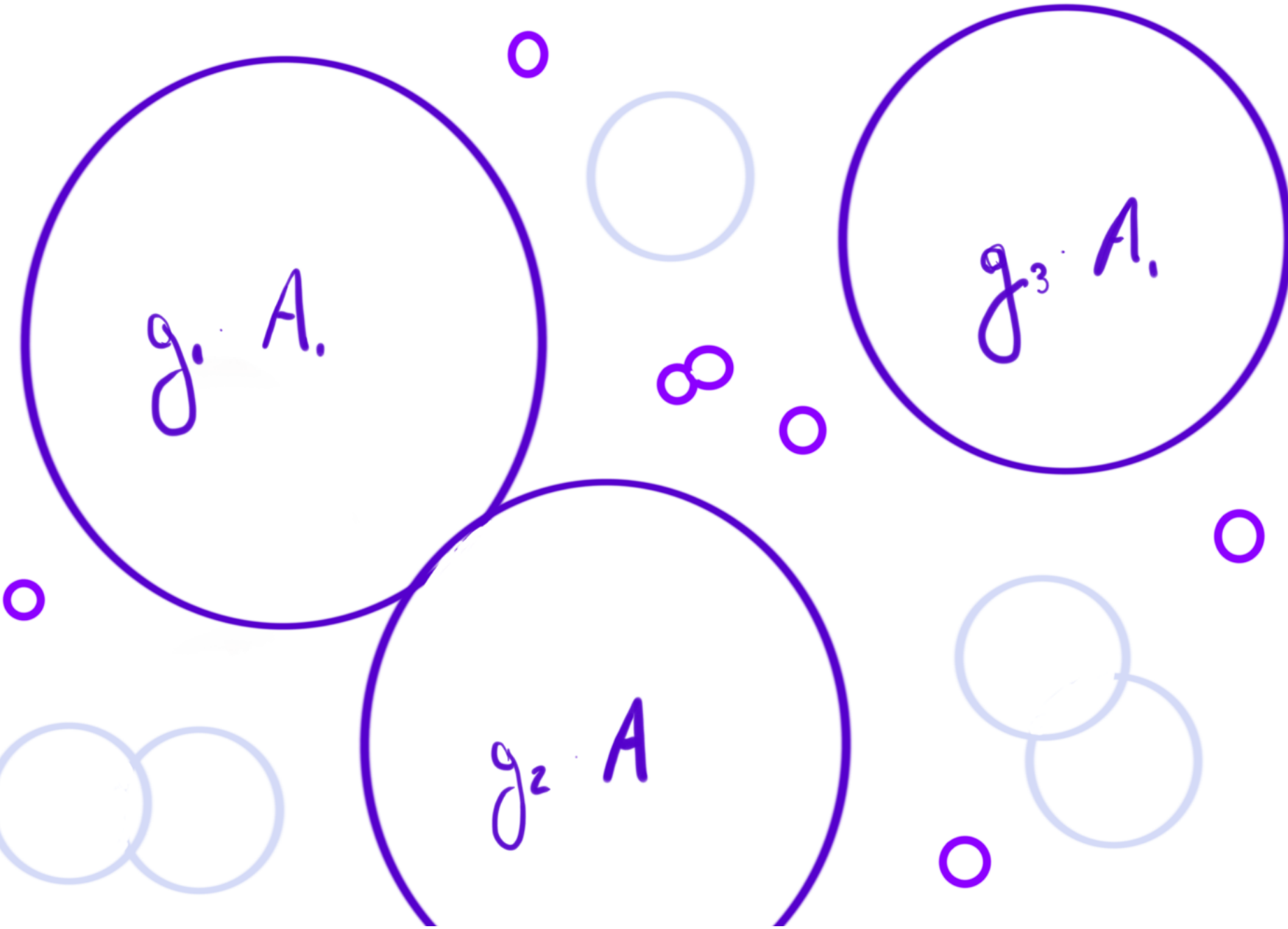
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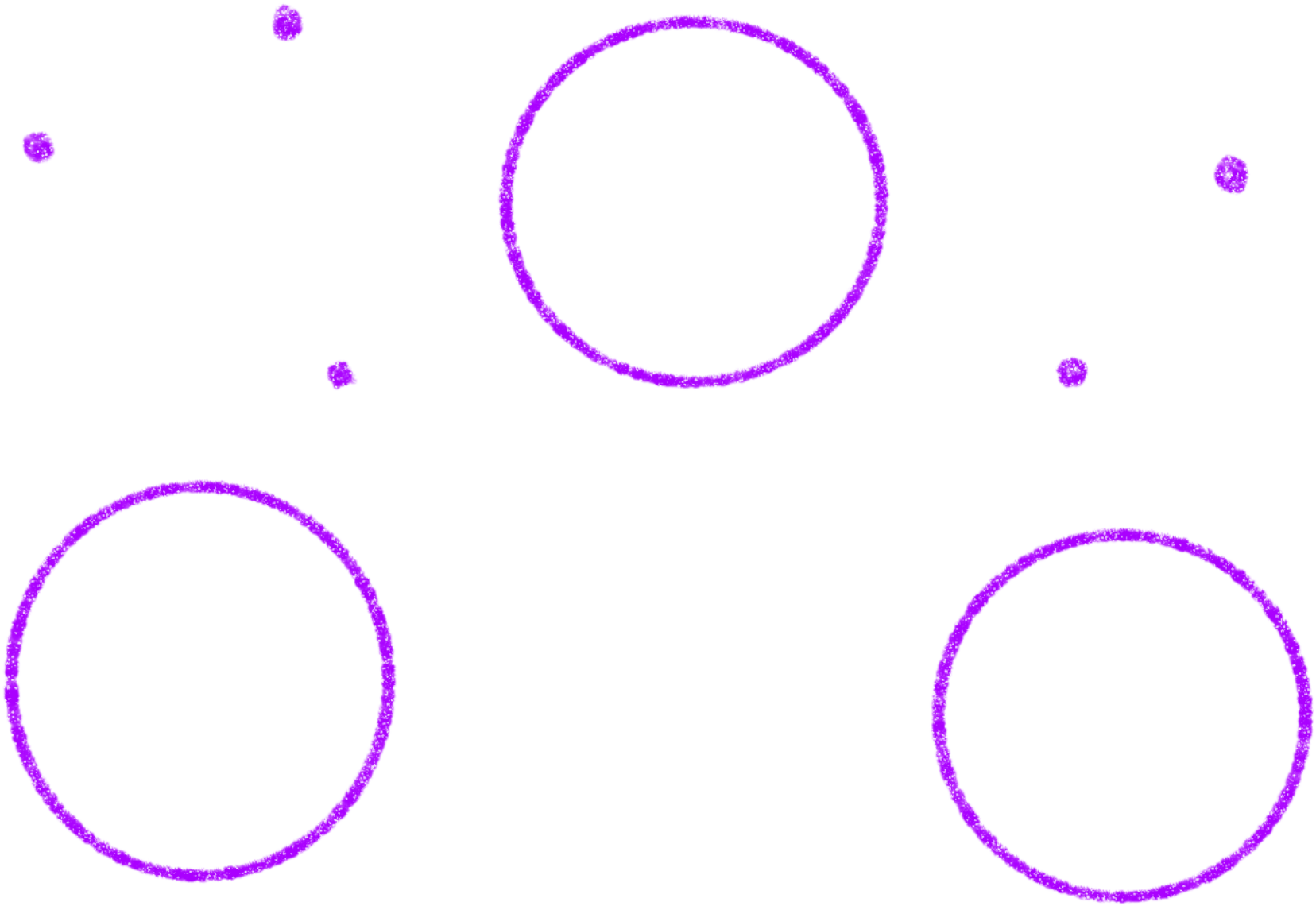
Theorem (Downarowicz/Huczek/Zhang, Wang/C.)

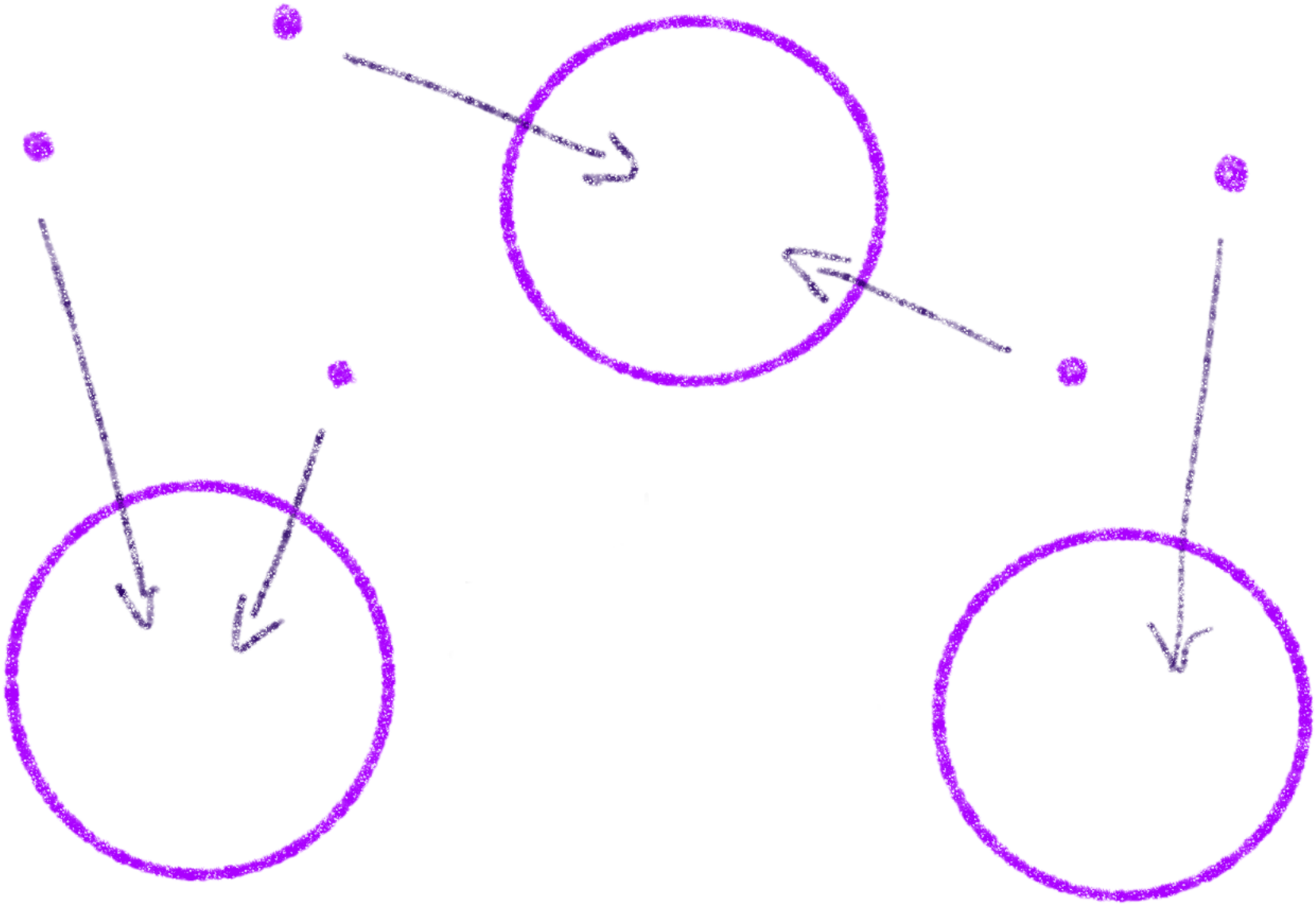
Every l.c.s.c. unimodular amenable group is Følner tilable.

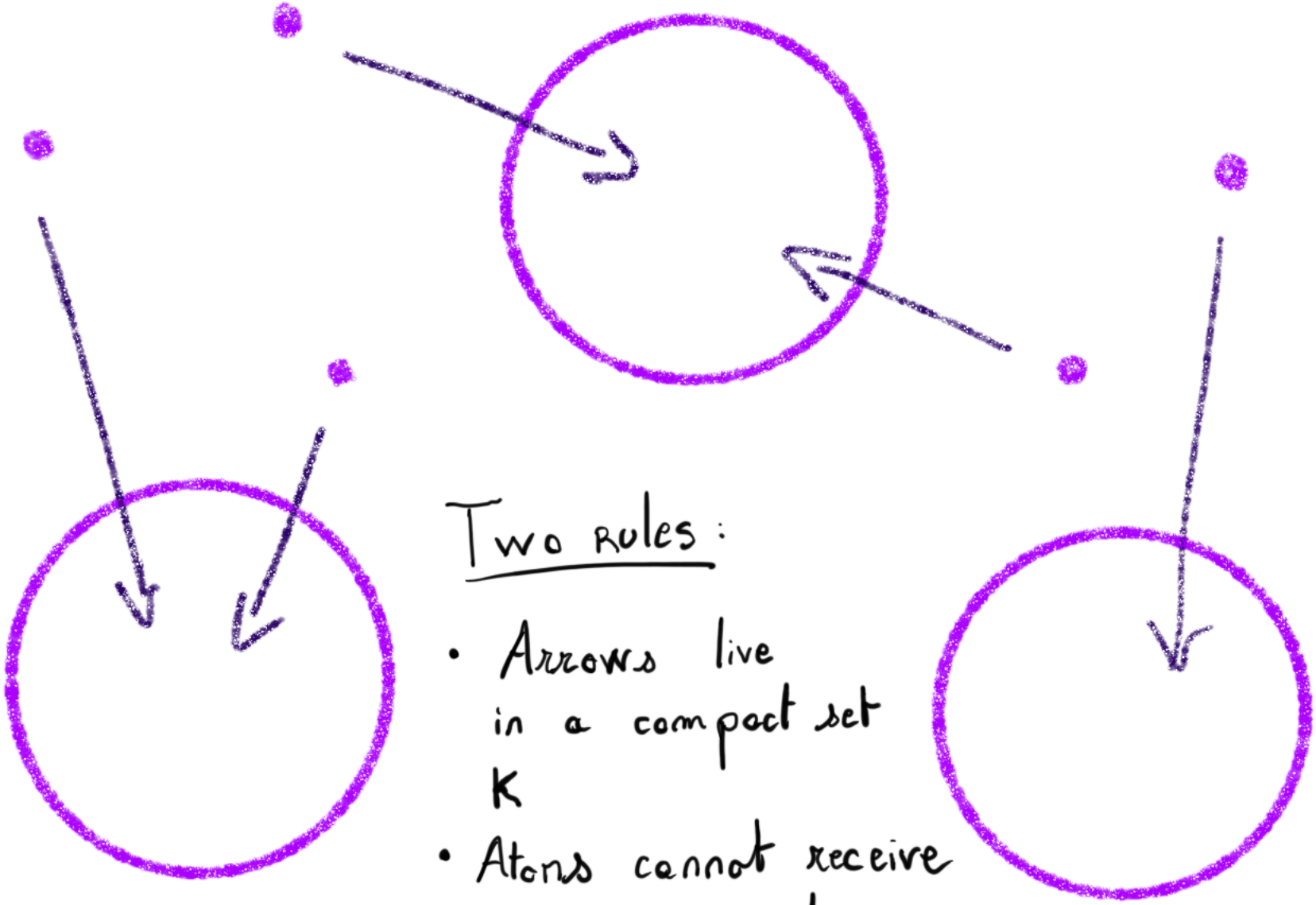

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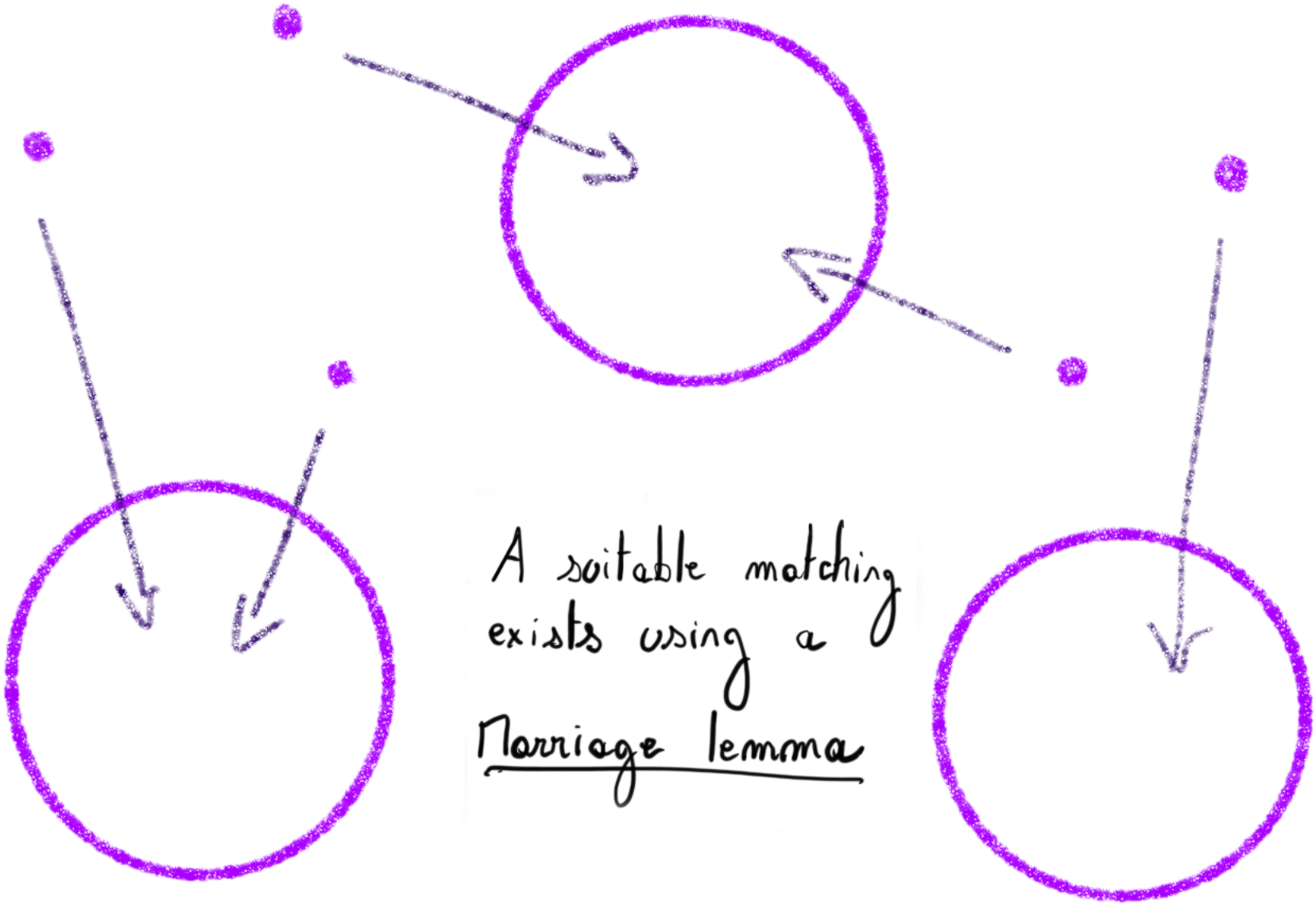

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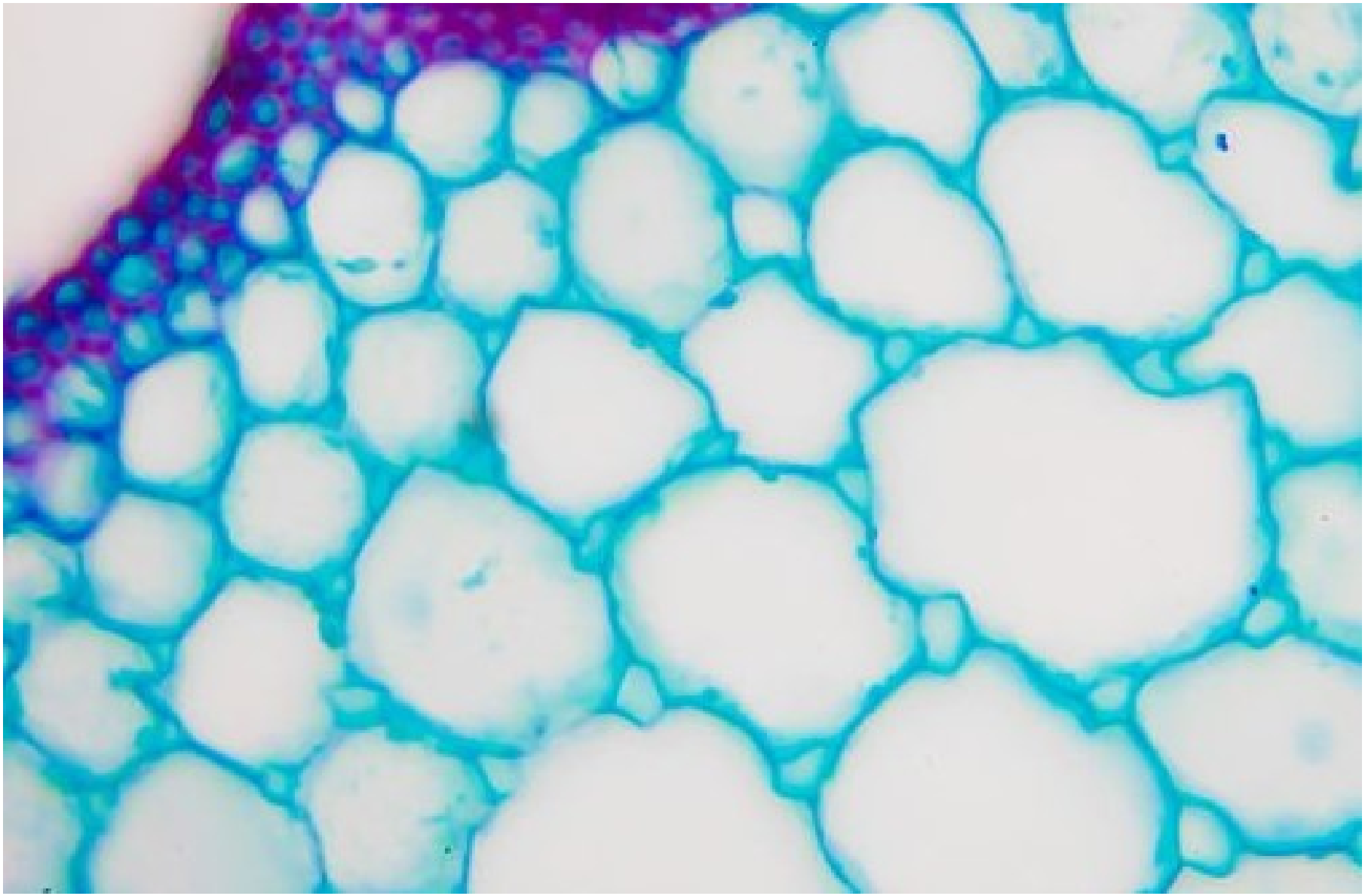

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Two rules:
• Arrows live in a compact set K
• Atoms cannot receive too many points


A suitable matching
exists using a
Marriage lemma



Thank you!