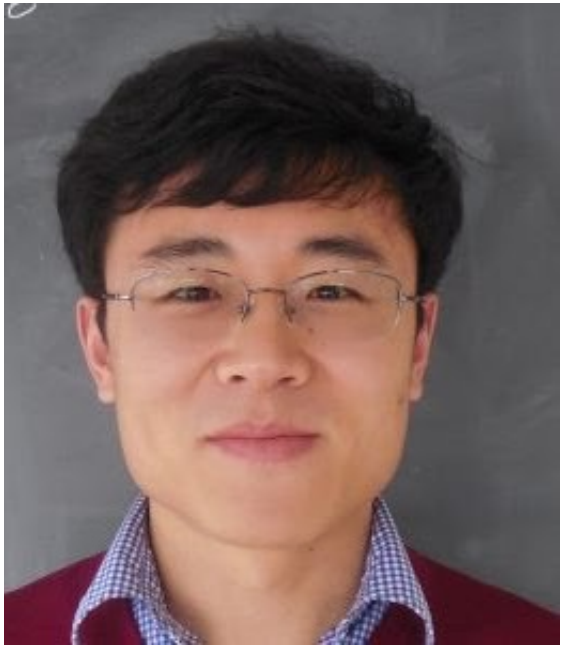
MAXIMAL CUBIC MODULATIONS OF THE HILBERT TRANSFORM ALONG THE PARABOLA

EL ESCORIAL

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JOINT WORK WITH



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Carleson operator

Hilbert transform $$Hf(x) := \text{p.v.} \int_{\mathbb{R}} f(x-y) \frac{dy}{y}$$

Carleson operator $$Cf(x) := \sup_{\lambda \in \mathbb{R}} \left| \text{p.v.} \int_{\mathbb{R}} f(x-y) e^{i\lambda y} \frac{dy}{y} \right|$$

Theorem (Carleson 1966, Hunt 1968)

Let $1 < p < \infty$. Then $\| Cf \|_{L^p(\mathbb{R})} \lesssim \| f \|_{L^p(\mathbb{R})}$

Also Fefferman (1973), Lacey-Thiele (2000)

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Hilbert transform $$Hf(x) := \text{p.v.} \int_{\mathbb{R}} f(x-y) \frac{dy}{y}$$

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modulation invariant

$f(x)$ replaced by $f(x)e^{iax}$

Theorem (Carleson 1966, Hunt 1968)

Let $1<p<\infty$. Then $\| Cf \|_{L^p(\mathbb{R})} \lesssim \| f \|_{L^p(\mathbb{R})}$

Also Fefferman (1973), Lacey-Thiele (2000)

Maximally modulated CZ operators

Let K be a suitable Calderón–Zygmund convolution kernel in $\mathbb{R}^d$:

(i) $K \in \mathcal{S}'(\mathbb{R}^d)$ s.t. $K \in C^1(\mathbb{R}^d \setminus \{0\})$

(ii) $\|\hat{K}\|_\infty \leq C$

(iii) $|K(y)| \leq \frac{C}{|y|^d}$

(iv) $|\partial_{y_i} K(y)| \leq \frac{C}{|y|^{d+1}}$

$$T^* f(x) := \sup_{\lambda \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)\, e^{i\lambda \cdot y} K(y)\, dy \right|$$

Theorem (Sjölin, 1971)

Let $1 < p < \infty$. Then $\|T^* f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$

Polynomial modulations

$$T^* f(x) := \sup_{\lambda \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)\, e^{i\lambda \cdot y}\, K(y)\, dy \right|$$

Stein : Do L^p- bounds still hold if $\lambda \cdot y$ is replaced by a polynomial $P_\lambda(y) := \sum_{1 \le |\alpha| \le N} \lambda_\alpha y^\alpha$ of fixed degree N and supremum is taken over all coefficients $\lambda = (\lambda_\alpha)_{1 \le |\alpha| \le N}$?

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- **Ricci - Stein** (1987): $T_p f(x) = \int_{\mathbb{R}^d} f(x-y)\, e^{iP(x,y)}\, K(y)\, dy$,

  where $P(x,y)$ real polynomial in $\mathbb{R}^d \times \mathbb{R}^d$

  and proved L^p - bounds only depending on the degree

Polynomial modulations

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  where $P(x,y)$ real polynomial in $\mathbb{R}^d \times \mathbb{R}^d$

  and proved L^p - bounds only depending on the degree

  Above, $\sum_{1 \le |\alpha| \le N} \lambda_\alpha(x)\, y^\alpha$, $\lambda_\alpha(x)$ any measurable function

Polynomial modulations

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- **Stein (1995)** : $L^p(\mathbb{R})$ bounds hold if $P_\lambda(y) = \lambda y^2$

$$T^* f(x) = \sup_{\lambda \in \mathbb{R}} \left| \text{p.v.} \int_{\mathbb{R}} f(x-y)\, e^{i\lambda y^2}\, \frac{dy}{y} \right|$$

using assymptotics for FT of $\dfrac{e^{i\lambda y^2}}{y}$

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no linear term! modulation invariance is broken

Polynomial modulations

Theorem (Stein-Wainger, 2001)

Let $1 < p < \infty$, $N \in \mathbb{N}$. Then

$$\Big\| \sup_{\lambda} |T_\lambda f| \Big\|_{L^p(\mathbb{R}^d)} \lesssim_N \|f\|_{L^p(\mathbb{R}^d)}$$

for all $P_\lambda(y) = \sum_{2 \le |\alpha| \le N} \lambda_\alpha y^\alpha$.

misses linear terms

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Theorem (Lie, 2020)

The above holds for all $P_\lambda(y) = \sum_{1 \le |\alpha| \le N} \lambda_\alpha y^\alpha$.

includes linear terms

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Further contributions by Zorin-Kranich (2021)

Stein - Wainger's approach

model case: $$C f(x) = \sup_{j \in \mathbb{Z}} \sup_{|\lambda| \approx 2^{2j}} \left| \text{p.v.} \int_{\mathbb{R}} f(x-y)\, e^{i\lambda y^2} \frac{dy}{y} \right|$$

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$$|y| \leq 2^{-j} + \sum_{\ell > 0} |y| \approx 2^{-j+\ell}$$

$\downarrow$

no oscillation

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Hardy-Littlewood and maximal truncations Hilbert transform

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uniform estimate: $$\| C_\ell f \|_{L^p(\mathbb{R})} \lesssim \| f \|_{L^p(\mathbb{R})}, \quad 1 < p \leq \infty, \quad \ell \geq 1$$

$$C_\ell f(x) = \sup_{j \in \mathbb{Z}} \sup_{|\lambda| \approx 2^{2j}} \left| \int_{|y| \approx 2^{-j+\ell}} f(x-y)\, e^{i\lambda y^2}\, \beta(2^{j-\ell} y) \frac{dy}{y} \right|$$

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$$\lesssim \sup_{j \in \mathbb{Z}} \sup_{|\lambda| \approx 2^{2j}} \frac{1}{2^{-j+\ell}} \int_{|y| \leq 2^{-j+\ell}} |f(x-y)|\, dy$$

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$$\lesssim \sup_{j \in \mathbb{Z}} \sup_{|\lambda| \approx 2^{2j}} \frac{1}{2^{-j+\ell}} \int_{|y| \leq 2^{-j+\ell}} |f(x-y)|\, dy$$

$$\lesssim M_{HL} f(x)$$

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Key estimate: $\exists\, \varepsilon > 0$ s.t. $$\| C_\ell f \|_{L^2(\mathbb{R})} \lesssim 2^{-\ell\varepsilon} \| f \|_{L^2(\mathbb{R})}$$

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↓

propagates to all $1 < p < \infty$

Stein - Wainger's approach

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- **Linearisation** : $x \longmapsto j(x)$ and $\lambda(x)$, with $|\lambda(x)| \approx 2^{2j(x)}$

$$C_\ell f(x) = \int_{|y| \approx 2^{-j(x)+\ell}} f(x-y)\, e^{i\lambda(x) y^2}\, \beta(2^{j(x)-\ell} y)\, \frac{dy}{y}$$

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$\Phi_\ell^{\lambda,j}(y)$

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- **TT^* argument** : $\| C_\ell \|_{2\to 2} = \| C_\ell C_\ell^* \|_{2\to 2}^{1/2}$, where

$$C_\ell C_\ell^* f(x) = \int \Phi_\ell^{\lambda(x),j(x)} * \tilde{\Phi}_\ell^{\lambda(z),j(z)}(x-z)\, f(z)\, dz$$

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$\beta(2^{j-\ell} y)$: $\Phi_\ell^{\lambda, j}(y)$

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Van der Corput arguments to estimate Kernel $\Rightarrow$ geometric maximal function bounds

Introducing curvature

Hilbert transform along the parabola

$$H_{par} f(x,t) = \text{p.v.} \int_{\mathbb{R}} f(x-y, t-y^2) \frac{dy}{y}$$

which arised in parabolic PDE (Fabes – Rivière)

Theorem (Nagel – Rivière – Wainger, 1974-1976)

Let $1<p<\infty$. Then $\| H_{par} f \|_{L^p(\mathbb{R}^2)} \lesssim \| f \|_{L^p(\mathbb{R}^2)}$

More general theory in $\mathbb{R}^d$ over submanifolds.

Carleson-Radon like operators

Pierce-Yung (2015) : let K be a CZ kernel in $\mathbb{R}^{d-1}$ and

$$T_\lambda f(x,t) = \int_{\mathbb{R}^{d-1}} f(x-y, t-|y|^2)\, e^{iP_\lambda(y)} K(y)\, dy \,, \quad (x,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$$

where $P_\lambda(y) = \sum_{1 \le |\alpha| \le N} \lambda_\alpha y^\alpha$, and $f \longmapsto \sup_\lambda |T_\lambda f|$

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To **avoid modulation invariance** (f replaced by $f(x,t)\, e^{i a\cdot(x,t)}$) one needs to **remove linear terms** and multiples of $|y|^2$ in $P_\lambda(y)$.

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To **avoid modulation invariance** (f replaced by $f(x,t)\, e^{i a\cdot(x,t)}$)
one needs to **remove linear terms** and multiples of $|y|^2$ in $P_\lambda(y)$.

positive answer : in the absence of such modulations and $\boxed{d \geq 3}$

$(x,t) \in \mathbb{R}^2 \times \mathbb{R}$

Carleson-Radon like operators

Theorem (Pierce-Yung, 2015)

Let $d \geq 3$ and $N \in \mathbb{N}$, $N \geq 2$. Let

$$\mathcal{P} := \{ P_2(y), P_3(y), \dots, P_N(y) \}$$

where $P_j(y)$ is homogeneous of degree j, $P_2(y) \neq C|y|^2$.

For each $\lambda = (\lambda_2, \dots, \lambda_N)$, let $P_\lambda(y) = \sum_{j=2}^{N} \lambda_j P_j(y)$

Then

$$\Big\| \sup_{\lambda = (\lambda_2, \dots, \lambda_N) \in \mathbb{R}^{N-1}} \Big| \int_{\mathbb{R}^{d-1}} f(x-y, t-|y|^2)\, e^{iP_\lambda(y)} K(y)\, dy \Big| \Big\|_{L^p(\mathbb{R}^d)}$$

$$\lesssim_N \|f\|_{L^p(\mathbb{R}^d)}$$

Carleson-Radon like operators

Theorem (Anderson - Maldague - Pierce - Yung, 2024)

Generalisation to quadratic hypersurfaces.

Carleson-Radon like operators

Theorem (Anderson - Maldague - Pierce - Yung, 2024)

Generalisation to quadratic hypersurfaces.

Approach: follows Stein-Wainger TT^*, but **not directly**

- Replace by smoother variant, run (TT^*)
- Control square function for the error of the approximation

Carleson-Radon like operators

- **Pierce-Yung**: one-parameter "model" cubic case $d \geq 3$

$$C_d f(x,y) = \sup_{v \in \mathbb{R}} \left| \int_{\mathbb{R}^{d-1}} f(x-t, y-|t|^2)\, e^{iv(t_1^3 + \cdots + t_{d-1}^3)} K(t)\, dt \right|$$

- What about $d=2$?

$$C f(x,y) := \sup_{v \in \mathbb{R}} \left| p.v. \int_{\mathbb{R}} f(x-t, y-t^2)\, e^{ivt^3} \frac{dt}{t} \right|$$

Carleson-Radon like operators

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Thm (B.-Guo-Hickman, 2024)

For all $1 < p < \infty$, $\|Cf\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$

Carleson-Radon like operators

- **Pierce-Yung:** one-parameter "model" cubic case — $d \geq 3$

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Thm (B.-Guo-Hickman, 2024) Hsu-Lie (2024)

For all $1 < p < \infty$, $\| Cf \|_{L^p(\mathbb{R}^2)} \lesssim \| f \|_{L^p(\mathbb{R}^2)}$

Very different approaches

Earlier related results

Our result says that for $1<p<\infty$

$$\Big\| \text{p.v.} \int_{\mathbb{R}} f(x-t, y-t^2)\, e^{i r(x,y) t^3} \frac{dt}{t} \Big\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for all measurable functions $r(x,y)$.

- **Guo - Pierce - Yung - Roos (2017)** proved it for $r(x)$

  This allows to take Fourier transform in the y - variable

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  This allows to take Fourier transform in the y - variable

- Roos (2019)
- Ramos (2021)
- Becker (2023, 2024)

} partial results on modulation invariant cases $e^{iNt+iMt^2}$

Why dimension matters ?

- Under **no curvature**, $C_\ell f(x) := \int_{\mathbb{R}^d} f(x-t)\, e^{i\lambda(x)|t|^2}\, \beta(2^{-j(x)+\ell} t) K(t)\, dt$

TT* argument: $\| C_\ell \|_{2\to 2} = \| C_\ell C_\ell^* \|_{2\to 2}^{1/2}$

$C_\ell C_\ell^* f(x) = \int_{\mathbb{R}^d} K(x-z) f(z)\, dz$

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L_ℓ was a convolution : double integral in $\mathbb{R}^d$ - integral in $\mathbb{R}^d$
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L_b was a convolution : double integral in $\mathbb{R}^d$ - integral in $\mathbb{R}^d$ = integral in $\mathbb{R}^d$

- **With curvature**

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$$C_\ell C_\ell^* f(x,y) = \int_{\mathbb{R}^d} K((x,y)-(z,u))\, f(z,u)\, dz\, du$$

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↳ was a convolution : double integral in $\mathbb{R}^d$ - integral in $\mathbb{R}^d$ = integral in $\mathbb{R}^d$

- **With curvature**

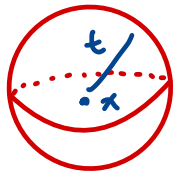
$$C_\ell f(x,y) := \int_{\mathbb{R}^{d-1}} f(x-t, y-|t|^2)\, e^{iv(x,y)|t|^3}\, \beta(2^{-j(x,t)+\ell} t)\, K(t)\, dt$$

$$C_\ell C_\ell^* f(x,y) = \int_{\mathbb{R}^d} K((x,y)-(z,u))\, f(z,u)\, dz\, du$$

↳ double integral in $\mathbb{R}^{d-1}$ - integral in $\mathbb{R}^d$ = integral in $2(d-1)-d = d-2$ variables

Related phenomenon

spherical averages : $A_t f(x) = \int_{S^{d-1}} f(x - ty)\, d\sigma(y)$



$\partial\{x\} + tS^{d-1}$

spherical maximal function $Sf(x) = \sup_{t>0} |A_t f(x)|$

Theorem (Stein, $d \geq 3$ (1976); Bourgain, $d = 2$ (1986))

$\| Sf \|_{L^p(\mathbb{R}^d)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}$ if and only if $p > \frac{d}{d-1}$

Fourier transform (L^2- bounds)

- $\widehat{d\sigma_t}(\xi) = \sum_{\pm} e^{\pm it|\xi|} (1+t|\xi|)^{-(d-1)/2} \, a_{\pm}(t,\xi)$,

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for $\operatorname{supp}(\widehat{f_k}) \subseteq \{ |\xi| \approx 2^k \}$

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- Trade $\sup_{1 \le t \le 2}$ by $\frac{1}{p}+$ derivatives in L^p (FTC + Hölder)

$$\| \sup_{1 \le t \le 2} |A_t f_k| \|_p \lesssim 2^{k/p} \| \left(\int_1^2 |A_t f_k|^p \, dt \right)^{1/p} \|_p$$

Fourier transform (L^2- bounds)

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- By Littlewood-Paley theory $\sup_{1<t<2} \rightsquigarrow \sup_{t>0}$

Local smoothing estimates

For $d=2$, one aims to prove

$$\left(\int_1^2 \| A_t f_k \|_{L^p(\mathbb{R}^2)}^p \, dt \right)^{1/p} \lesssim 2^{-k/p} \, 2^{-k\sigma} \, \| f \|_{L^p(\mathbb{R}^2)}$$

for some $\sigma > 0$ and some $2 < p < \infty$.

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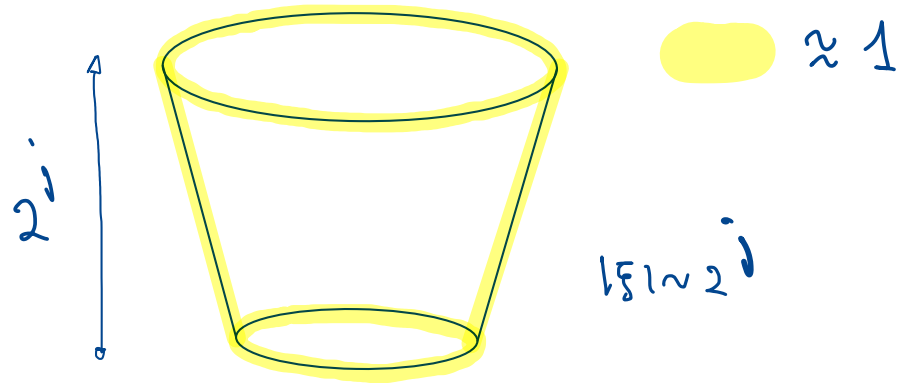
- $\widehat{A_t f_K}(\xi) = 2^{-K(d-1)/2} \sum_{\pm} e^{\pm it|\xi|} a_\pm(t,\xi) \hat{f}(\xi)$

- Essentially equivalent to Sobolev space-time estimate for

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi + it|\xi|} \hat{f}(\xi) \, d\xi$$

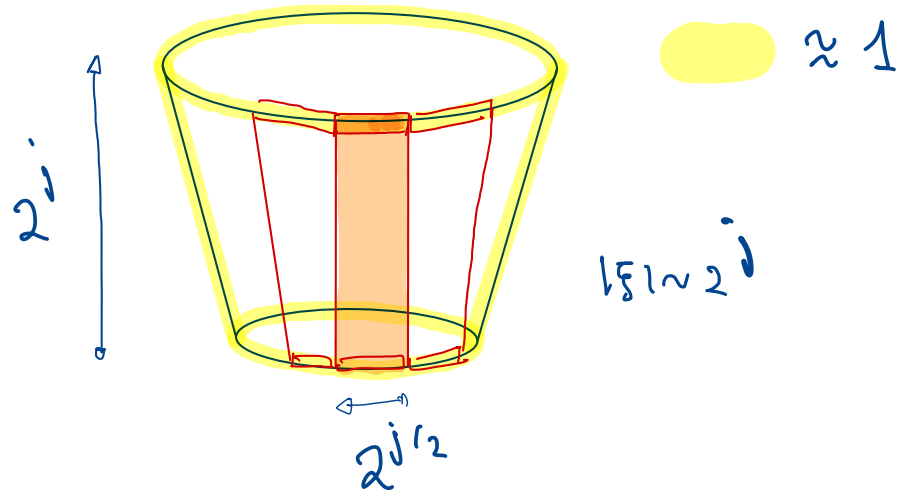
Local smoothing estimates

Space - time Fourier transform supported in



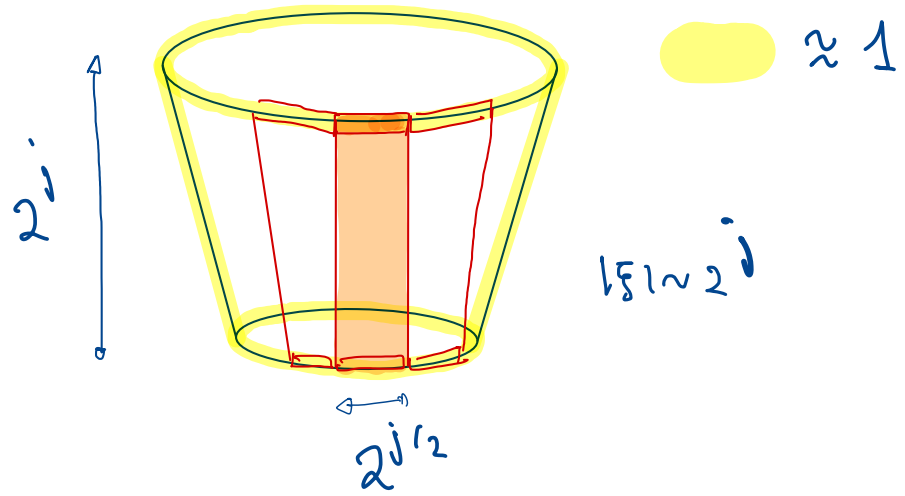
Local smoothing estimates

Space - time Fourier transform supported in



Local smoothing estimates

Space - time Fourier transform supported in



A useful technique (but not the only one) has been **decoupling**

Theorem (Bourgain-Demeter, 2015) For $2 \le p \le \frac{2(d+1)}{d-1}$

$$\| \sum_\nu e^{it\sqrt{-\Delta}} f_\nu \|_{L^p(\mathbb{R}^d \times [1,2])} \le 2^{j\varepsilon} \left(\sum_\nu \| e^{it\sqrt{-\Delta}} f_\nu \|^2_{L^p(\mathbb{R}^d \times [1,2])} \right)^{1/2}$$

L^p- approach to cubic maximal modulations for H_{par}

$$C_\ell f(x,y) := \sup_{j\in\mathbb{Z}} \sup_{|v|\approx 2^{3j}} \int_{\mathbb{R}} f(x-t, y-t^2)\, e^{ivt^3}\, \beta(2^{j-\ell}t)\, \frac{dt}{t}$$

TT^* method of **Pierce-Yung / Stein-Wainger** does not seem to work

Theorem (B.-Guo-Hickman)

There exists $p_0 > 2$ and $\varepsilon_0 > 0$ s.t.

$$\| C_\ell f \|_{L^{p_0}(\mathbb{R}^2)} \lesssim 2^{-\ell\varepsilon_0} \| f \|_{L^{p_0}(\mathbb{R}^2)}.$$

L^p- approach to cubic maximal modulations for H_{par}

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- Interpolation with $\| C_\ell f \|_1 \lesssim \| f \|_1$ $\Rightarrow$ also holds for $\boxed{p_0 = 2}$
- **Don't Know** how to prove L^2- bounds directly.

Fourier analytic approach

$$C_\ell f(x,y) := \sup_{j\in\mathbb{Z}} \sup_{|v|\approx 2^{3j}} \int_{\mathbb{R}} f(x-t, y-t^2)\, e^{ivt^3}\, \beta(2^{j-\ell}t)\, \frac{dt}{t}$$

Fourier multiplier $$m(\xi,\eta) = \int_{\mathbb{R}} e^{-i(t\xi + t^2\eta - t^3 v)}\, \beta(2^{j-\ell}t)\, \frac{dt}{t}$$

Since $|t| \approx 2^{-j+\ell}$ and $|t^3 v| \approx 2^{3\ell}$

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low freq part

- $|\xi| \lesssim 2^j 2^{2\ell - C_1}$
- $|\eta| \lesssim 2^{2j+\ell - C_2}$

$\Rightarrow$ decay in 2^ℓ in Kernel estimates + strong max fn bounds

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LP dec in ξ, η

high $|\xi|$, low $|\eta|$

low $|\xi|$, high $|\eta|$

high $|\xi|$, high $|\eta|$

Fourier analytic approach

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- **Littlewood-Paley theory** + rescaling : $\sup_{j\in\mathbb{Z}} \longrightarrow j=0$
- **Sobolev embedding** (FTC + Hölder) : $\sup_{|v|\sim 1} \longrightarrow 2^{3\ell/p} \left(\int_1^8 |\,|^p dv \right)^{\frac{1}{p}}$

Local smoothing estimates

$$H^{\nu}_{\ell} f(x,y) := \int_{\mathbb{R}^2} e^{i(x\xi+y\eta)} \int_{\mathbb{R}} e^{-i 2^{3\ell}(t\xi+t^2\eta-t^3\nu)} \beta(t) \frac{dt}{t} \hat{f}(\xi,\eta)\, d\xi d\eta$$

Main estimates : $\exists\ \varepsilon_0, \delta_0 > 0$ s.t.

$$(HH) \quad \left(\int_1^8 \| H^{\nu}_{\ell,\kappa_1,\kappa_2} f \|^{p_0}_{L^{p_0}(\mathbb{R}^2)} d\nu \right)^{1/p_0} \lesssim 2^{-3\ell/p_0} 2^{-\ell\varepsilon_0} 2^{-(\kappa_1+\kappa_2)\delta_0} \| f \|_{L^{p_0}(\mathbb{R}^2)}$$

for all $\kappa_1 > -C_1$, $\kappa_2 > -C_2$

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for all $\kappa_1 > -C_1$, $\kappa_2 > -C_2$

Multiplier behaviour

$$m_\ell^\nu(\xi,\eta) := \int e^{i 2^{3\ell} \phi^\nu_{\xi,\eta}(t)} \frac{\beta(t)}{t}\, dt$$

where $\phi^\nu_{\xi,\eta}(t) := -t\xi - t^2\eta + t^3\nu = -\langle (t,t^2,t^3), (\xi,\eta,-\nu)\rangle$

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Van der Corput with 3rd derivatives:

$$|m_\ell^\nu(\xi,\eta)| \lesssim 2^{-3\ell/3}$$

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rapid decay in ℓ, k_1, k_2

(good L^2-bound)

unless

$$\begin{cases} 1 \lesssim |\xi| \approx |\eta| & HH \\ |\eta| \lesssim |\xi| \approx 1 & UL \\ |\xi| \lesssim |\eta| \approx 1 & LU \end{cases}$$

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Van der Corput : $\boxed{|m_\ell^\nu(\xi,\eta)| \lesssim |\eta|^{-1/2}\, 2^{-3\ell/2} \quad \text{if } |\eta| \gtrsim 1}$

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$2^{-\ell\varepsilon}$ – L^2-bound if $|\xi| \approx |\eta| \gtrsim 2^{\ell\varepsilon'}$

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If $|\eta| \approx 1$, can't obtain $2^{-\ell\varepsilon}$

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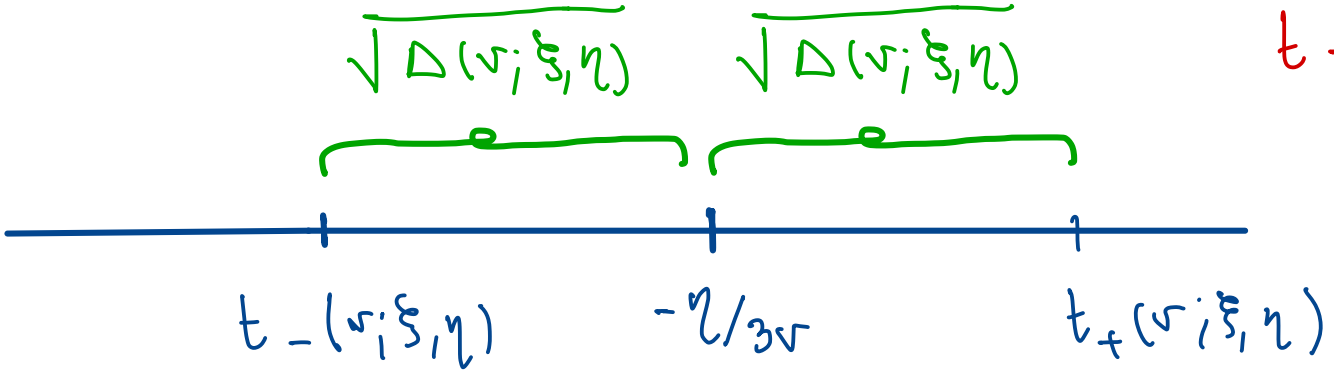
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$t = t_\pm(\nu;\xi,\eta)$



Multiplier behaviour

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Dyadically decompose $|\Delta(\nu;\xi,\eta)| \approx |\eta|^2\, 2^{-2n} \approx 2^{2K_2 - 2n}$

with $n = 0, \dots, \ell + K_2$ → matching VdC with 3rd der's

Bad L^2-decay but smaller r-support !

$$|\Delta(r;\xi,\eta)| \approx 2^{2k_2-2n} \Longrightarrow$$

$$|m_\ell^r(\xi,\eta)| \lesssim 2^{-3\ell/2}\, 2^{-k_2/2}\, 2^{n/2}$$

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$$\left.\begin{aligned} \Delta(r;\xi,\eta) &= \eta^2 + 3r\xi \\ |\Delta(r;\xi,\eta)| &\approx 2^{2k_2-2n} \end{aligned}\right\}$$

Bad L^2-decay but smaller ν-support !

$$|\Delta(\nu;\xi,\eta)| \approx 2^{2k_2-2n} \Rightarrow$$

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small support

Bad L^2-decay but smaller v-support !

$$|\Delta(v;\xi,\eta)| \approx 2^{2k_2-2n} \Rightarrow$$

$$|m_\ell^v(\xi,\eta)| \lesssim 2^{-3\ell/2}\, 2^{-k_2/2}\, 2^{n/2}$$

$$\left.\begin{aligned}\Delta(v;\xi,\eta) &= \eta^2 + 3v\xi \\ |\Delta(v;\xi,\eta)| &\approx 2^{2k_2-2n}\end{aligned}\right\} \Rightarrow \left|v + \frac{\eta^2}{3\xi}\right| \approx \frac{2^{2k_2-2n}}{|\xi|} \approx 2^{k_2-2n}$$

small support

$$\left(\int_1^2 |m_{\ell,n}^v(\xi,\eta)|^2\, dv\right)^{1/2} \lesssim 2^{-3\ell/2}\, 2^{-n/2}$$

Bad L^2-decay but smaller ν-support !

$$|\Delta(\nu;\xi,\eta)| \approx 2^{2k_2-2n} \Rightarrow$$

$$|m_\ell^\nu(\xi,\eta)| \lesssim 2^{-3\ell/2}\, 2^{-k_2/2}\, 2^{n/2}$$

$$\left.\begin{aligned}\Delta(\nu;\xi,\eta) &= \eta^2 + 3\nu\xi \\ |\Delta(\nu;\xi,\eta)| &\approx 2^{2k_2-2n}\end{aligned}\right\} \Rightarrow \left|\nu + \frac{\eta^2}{3\xi}\right| \approx \frac{2^{2k_2-2n}}{|\xi|} \approx 2^{k_2-2n}$$

small support

$$\left(\int_1^2 |m_{\ell,n}^\nu(\xi,\eta)|^2\, d\nu\right)^{1/2} \lesssim 2^{-3\ell/2}\, 2^{-n/2}$$

$2^{-\ell\varepsilon}$ - L^2-bound if $n > \ell\varepsilon'$.

Recap : L^2- methods

$$m_\ell^\nu(\xi,\eta) := \int e^{i 2^{3\ell} \phi^\nu_{\xi,\eta}(t)} \frac{\beta(t)}{t}\, dt$$

where $\phi^\nu_{\xi,\eta}(t) := -t\xi - t^2\eta + t^3\nu$

L^2- methods give satisfactory bounds

- non-stationary case
- stationary super high freq
- $\mathcal{O}(1)$ freq

with bad decay but improved ν- support

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L^p- methods : • region near miss decay $2^{-3\ell/2}$

• $|\xi| \lesssim 1$ or/and $|\eta| \lesssim 1$

Variable Schrödinger-type propagators

On L^p, want to use the oscillation of $m_\ell^\nu(\xi,\eta)$, not just decay.

$$m_\ell^\nu(\xi,\eta) = \frac{2^{-3\ell/2}}{|(\phi_{\xi,\eta}^\nu)''(t_\pm(\nu;\xi,\eta)|^{1/2}} e^{i2^{3\ell}\phi_\pm(\nu;\xi,\eta)} a_\pm(\nu;\xi,\eta)$$

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This gives raise to the propagators

$$U^\ell_\pm f(x,y,\nu) = \int_{\substack{|\xi|\lesssim 1\\ |\eta|\lesssim 1}} e^{i(x\xi+y\eta+2^{3\ell}\phi_\pm(\nu;\xi,\eta)} a(x,y,\nu;\xi,\eta)\,\hat{f}(\xi,\eta)\,d\xi dy$$

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$$U_{\pm}^{\ell} f(x,y,r) = \int_{\substack{|\xi|\lesssim 1 \\ |\eta|\lesssim 1}} e^{i\left(x\xi + y\eta + 2^{3\ell}\,\phi_{\pm}(r;\xi,\eta)\right)} a(x,y,r;\xi,\eta)\,\hat{f}(\xi,\eta)\,d\xi d\eta$$

How is the **phase function** ?

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For each fixed r_0, $(\xi,\eta) \longmapsto (\xi, \eta, \partial_r \phi_{\pm}(r_0;\xi,\eta))$

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(ii) eigenvalues of $\partial_r \partial^2_{(\xi,\eta)} \phi_{\pm}(r;\xi,\eta)$ have **opposite signs**

Universal local smoothing bounds

Stein: $\| u_{\pm}^{\ell} f \|_{L^p(\mathbb{R}^2 \times [1,8])} \lesssim 2^{3\ell(\frac{1}{2}-\frac{1}{p})} \| f \|_{L^p(\mathbb{R}^2)}$

for $2 \le p \le 4$.

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Framework of Marina's talk!

Nikodym non-compression bounds

Theorem (Chen-Gan-Guo-Hickman-Iliopoulou-Wright, 2024)

If $\phi_\pm(r;\xi,\eta)$ satisfies **Nikodym non-compression** hypothesis, there exist $p_0 \in (2,4]$, $\varepsilon(p_0) > 0$ s.t.

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- decoupling can be used in that region, gain coming from the small support.

¡GRACIAS!