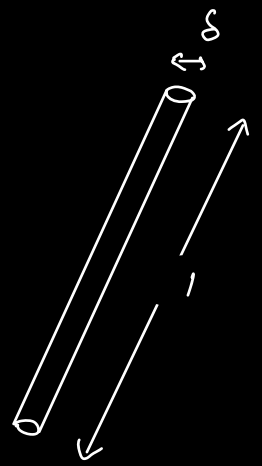
# Union of tubes

Joint with Zahl.

$0 < \delta << 1$ a small scale

$\delta$-tube $T$: length $= 1$, radius $= \delta$.




two $\delta$-tubes $T_1$ and $T_2$ are **distinct** if $|T_1 \cap T_2| \leq \frac{1}{2}|T_1|$.

$\mathbb{T}$: a set of distinct $\delta$-tubes contained in $B_1 \subseteq \mathbb{R}^n$

Question:

how large is the volume of union of tubes in $\mathbb{T}$?

$$\Big|\bigcup_{T \in \mathbb{T}} T\Big| \geq ?$$

Why do we care?

- Kakeya problem
- Geometry of wave packets
- Bourgain-Demeter decoupling theorem.

Kakeya Conj: If $\mathbb{T}$ contains a $\delta$-tube in each $\delta$-separated direction, then $\left|\bigcup_{T\in\mathbb{T}} T\right| \approx 1$.

(K) $\Rightarrow \#\mathbb{T} \approx \delta^{-(n-1)}$

$\approx$ up to a factor of $\delta^{\varepsilon}$, $\varepsilon \to 0$ as $\delta \to 0$.

Question ✡: For what $\mathbb{T}$,

$$\left|\bigcup_{T\in\mathbb{T}} T\right| \approx \min\{\#\mathbb{T}\cdot|T|,\ 1\}$$

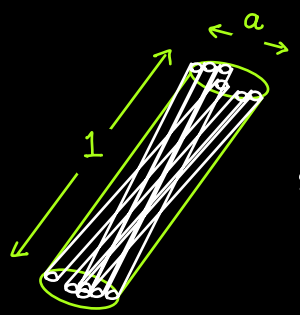
Enemy example for ✡:

① $U \subseteq B_1$ rectangular box that contains $\geq 1$ $\delta$-tube

$\mathbb{T}$: all distinct $\delta$-tubes in $U$.

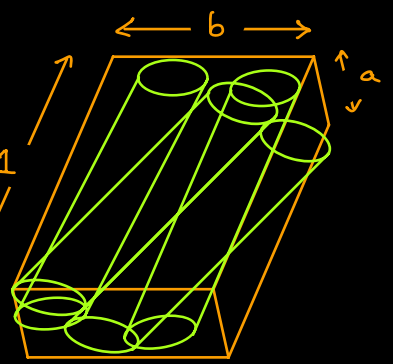
Then $\#\mathbb{T} = \left(\frac{|U|}{|T|}\right)^2$ and $\left|\bigcup_{T\in\mathbb{T}} T\right| \approx |U|$

In $\mathbb{R}^3$ :



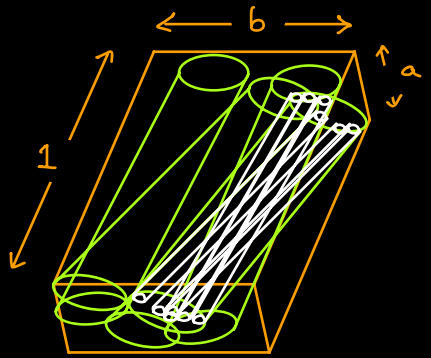

max # of distinct $\delta$-tubes in $T_a$ is

$$\approx \left(\frac{a}{\delta}\right)^4 = \left(\frac{|T_a|}{|T|}\right)^2$$




$b \geq a$.

max # of distinct $T_a$ in an $a \times b \times 1$-plank is $\left(\frac{b}{a}\right)^2 = \left(\frac{a \cdot b}{a^2}\right)^2$




max # of distinct $\delta$-tubes in an $a \times b \times 1$-plank is

$$\approx \left(\frac{a}{\delta}\right)^4 \cdot \left(\frac{b}{a}\right)^2 = \left(\frac{a \cdot b}{\delta^2}\right)^2 .$$

Q: What are other enemy examples ? ( in dim ≥ 4).

Focus on $n = 2, 3$. $\mathbb{T}[U] = \{ T \in \mathbb{T} : T \subseteq U \}$.

Def: $\mathbb{T}$ is a set of $\delta$-tubes $\subseteq B_1 \subseteq \mathbb{R}^n$.

① $\mathbb{T}$ satisfies Katz-Tao Wolff Axiom (KT) if

$\forall$ rectangular box $U \subseteq B_1$,

$$\#\mathbb{T}[U] \lesssim \frac{|U|}{|T|}. \Rightarrow \#\mathbb{T} \lesssim |T|^{-1}.$$

② $\mathbb{T}$ satisfies Frostman Wolff Axiom (F) if

$\forall$ box $U \subseteq B_1$,

$$\#\mathbb{T}[U] \lesssim |U| \cdot \#\mathbb{T} \Rightarrow \#\mathbb{T} \gtrsim |T|^{-1}.$$

Rmk: (a) Allows parallel $\delta$-tubes:

$\theta \in S^{n-1}$, $\mathbb{T}$ : all distinct $\delta$-tubes in direction $\theta$. $\mathbb{T}$ is both (KT) and (F).

(b) Kakeya assumption (K) $\Rightarrow$ both (KT) and (F).

**Thm** ( $n=2$ Córdoba 1970s , $n=3$ W.-Zahl 2025+ )

① If $\mathbb{T}$ is (KT) , then $\left|\bigcup_{T\in\mathbb{T}} T\right| \approx \#\mathbb{T}\cdot|T|$

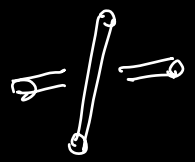
② If $\mathbb{T}$ is (F), then $\left|\bigcup_{T\in\mathbb{T}} T\right| \approx 1$

**Proof for $n=2$:**

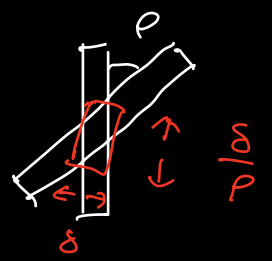
$$\left(\#\mathbb{T}\cdot|T|\right)^2 = \left(\int \sum_{T\in\mathbb{T}} \mathbb{1}_T\right)^2$$

$$\leq \left|\bigcup_{T\in\mathbb{T}} T\right| \cdot \left(\sum_{T_1,T_2\in\mathbb{T}} |T_1\cap T_2|\right)$$

fix $T_1$, $\sum_{T_2\in\mathbb{T}} |T_1\cap T_2|$



$$= \sum_{\substack{\delta\leq\rho\leq 1\\ \text{dyadic}}} \sum_{\angle(T_1,T_2)\approx\rho} |T_1\cap T_2| .$$




If $\angle(T_1,T_2)\approx\rho$, $T_1\cap T_2\neq\emptyset$ then

$$\begin{cases} |T_1\cap T_2| \approx \frac{\delta^2}{\rho} \\ T_2 \in N_\rho T_1 \end{cases}$$

$$\Rightarrow \#\mathbb{T}[N_\rho T_1] \leq \frac{\rho}{\delta} \text{ if } \mathbb{T} \text{ is (KT)}$$

$\lesssim \rho \cdot \#\mathbb{T}$ if $\mathbb{T}$ is (F).

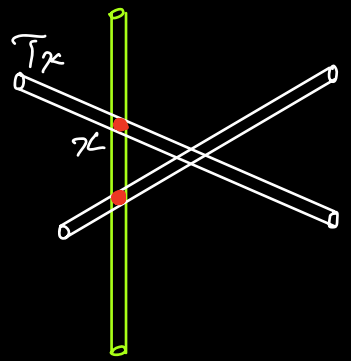
Excercises:

① $T$ : $\delta$-tube. $Y(T) \subseteq T$ a union of $\delta$-balls.

$\forall$ $\delta$-ball $x \subseteq Y(T)$, $\exists$ a $\delta$-tube $T_x$ through $x$

and Angle $(T_x, T) \approx 1$.

Let $\mathbb{T} = \{T_x\}_{x \subseteq Y(T)}$, then $\mathbb{T}$ is (kT).




② Generalize (kT) and (F) for $\delta$-slabs ($\delta \times \underbrace{1 \times \cdots \times 1}_{\geq 2}$)

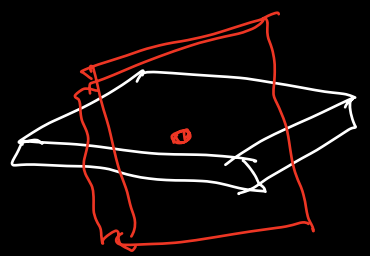
and prove an analogue of Córdoba's theorem.

③ Suppose $S$ is a $\delta$-slab, $\forall$ $\delta$-ball $x \subseteq S$,

$\exists$ $\delta$-slab $S_x$ through $x$, and

Angle $(S_x, S) \approx 1$, then

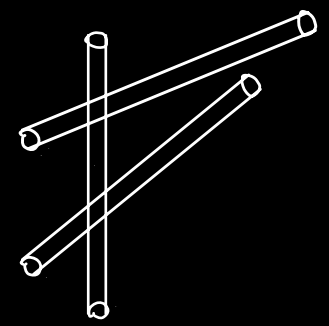
$$\left| \bigcup_{x \subseteq S} S_x \right| \approx 1.$$



dim 3 :

Wolff: (1995) If $\mathbb{T}$ is (KT) and (F), then $\Rightarrow \#\mathbb{T} \approx \delta^{-2}$

$$\left| \bigcup_{\mathbb{T}} T \right| \gtrsim \delta^{\frac{1}{2}}.$$



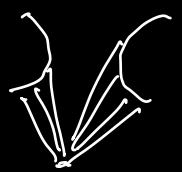
Heisenberg group in $\mathbb{C}^3$: (Katz-Laba-Tao 2000)

$$H = \{ |z_1|^2 + |z_2|^2 - |z_3|^2 = 1 \}$$

Through $p = (1,0,0)$, $|z_2|^2 - |z_3|^2 = 0$ is cone

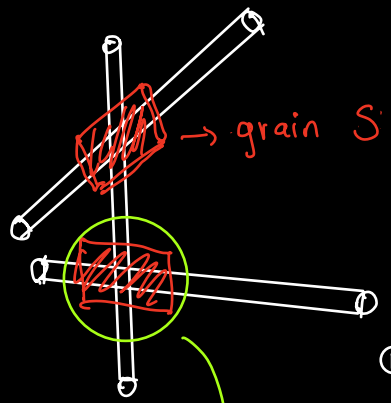
Contains 1 $\mathbb{R}$-dim set of $\mathbb{C}$-lines

H Contains a 4-$\mathbb{R}$-dim set of $\mathbb{C}$-lines



H satisfies an analogue of (KT) and (F) in $\mathbb{C}^3$.

$$\dim_{\mathbb{R}} H = 5 = \frac{5}{2} \cdot \dim_{\mathbb{R}} \mathbb{C}.$$



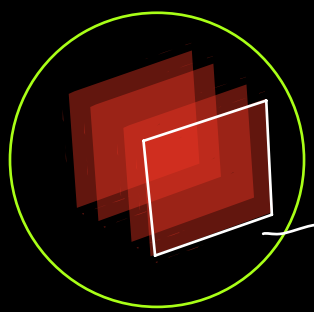

## Graininess

$$\begin{array}{ccc} U_T & \approx & U_S \\ \pi & & \text{slabs} \end{array}$$

① Graininess for Heisenberg group.

$H = \{ |z_1|^2 + |z_2|^2 - |z_3|^2 = 1 \}$

$p = (1, 0, 0)$



e.g.

$$N_\delta H \cap B(p, \delta^{\frac{1}{2}}) \approx \{ S_k \}_{k = 1, \cdots, \delta^{-\frac{1}{2}}}.$$

$S_k = \{ z_1 = 1 + (k\delta) i \, , |z_2|, |z_3| \leq \delta^{\frac{1}{2}} \}$

② Later we'll see extremal examples of sticky Kakeya sets have the same picture.

# Projection theory

| "Sticky" sets | vs | General sets |
| --- | --- | --- |
| Sets with similar properties at all scales ⇒ rigid structure | | Arbitrary sets |
| ( + Ideas of Orponen ) | | ( + Shmerkin's multi-scale decomposition ) |
| + Bourgain's discretized Sum-product (R vs ℂ) | | |

Theorems proved using this scheme ( "sticky" ⇒ "general" )

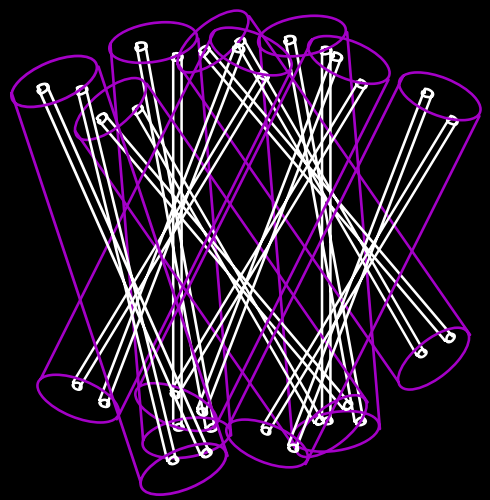
① Baby Furstenberg set estimate ( ε-improvement . Orponen-Shmerkin )

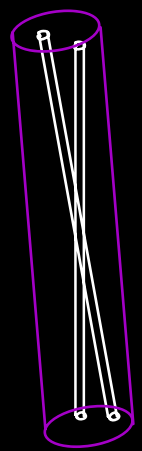
② Furstenberg set conjecture in $\mathbb{R}^2$ ( Orponen-Shmerkin . Ren-W. )
sticky case general case

③ Kakeya set Conjecture in $\mathbb{R}^3$

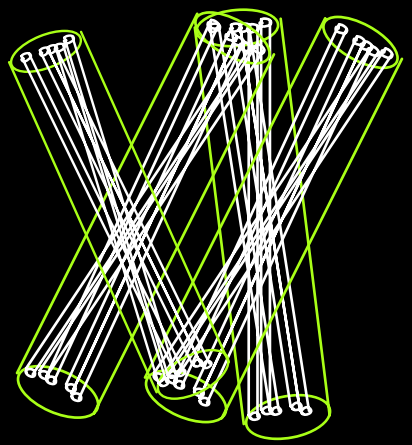
# "Sticky" : $\mathbb{T}$ satisfies (F) at all scales.

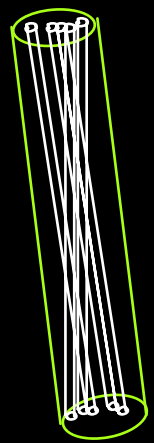
$\mathbb{T}$ satisfies (F) but not (F) at all scales.





$\mathbb{T}$ satisfies (F) at all scales.





Def: $\mathbb{T}$ satisfies (F) at all scales:

**Thm** (W. - Zahl | 2022+)

If $\mathbb{T} \subseteq B_1 \subseteq \mathbb{R}^3$ is (F) at all scales, then $|\bigcup_{\mathbb{T}} T| \approx 1$.

Proof ideas:

Let $\sigma > 0$ be the smallest number s.t.

$|\bigcup_{\mathbb{T}} T| \gtrsim \delta^{\sigma}$ for all $\mathbb{T}$ satisfying (F) at all scales.

Goal: $\sigma = 0$.

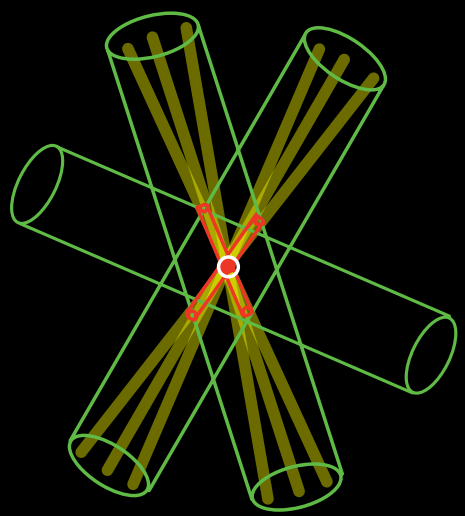
Let $\mathbb{T}$ be $\begin{cases} \text{(F) at all scales.} \\ |\bigcup_{\mathbb{T}} T| \approx \delta^{\sigma}, \quad \#\mathbb{T} \approx \delta^{-2} = \underbrace{\#\mathbb{T}_\rho}_{\approx \rho^{-2}} \cdot \underbrace{\#\mathbb{T}[T_\rho]}_{\approx (\frac{\delta}{\rho})^{-2}} \end{cases}$

① extremal multiplicity at all scales

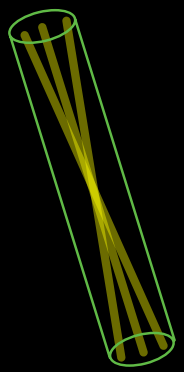
② graininess + reduction to sum-product type problem

③ projection theory (radial projection + restricted projection).

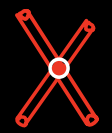
① Multiplicity:



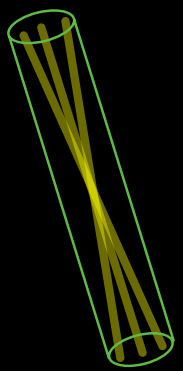
(A)
≃
(holds in general)



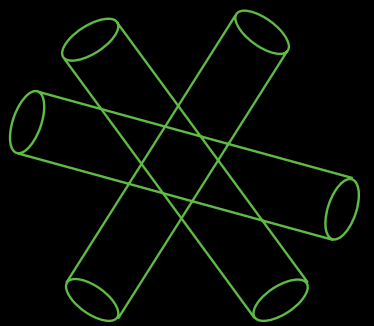
×

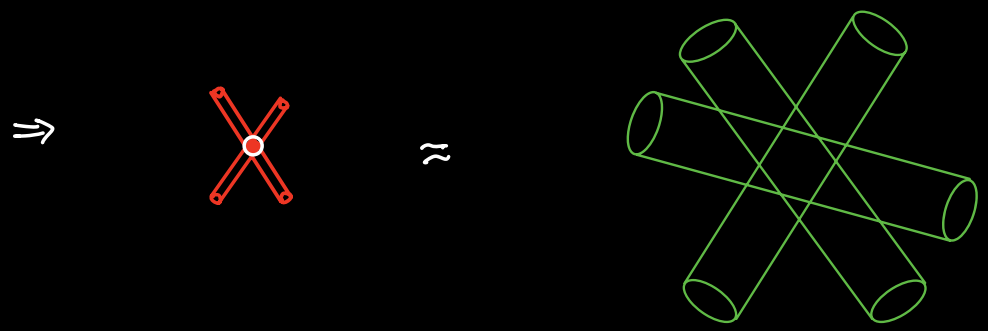


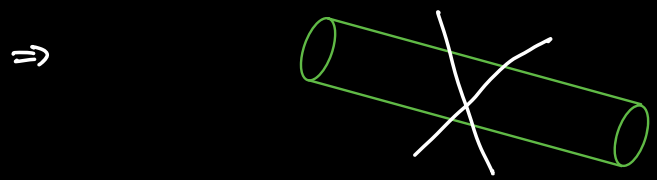
(B)
≃
(extremal example for "sticky" case)



×

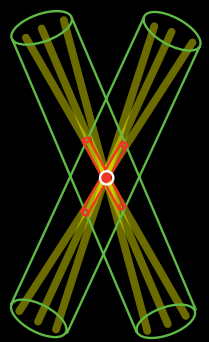


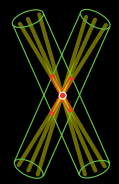


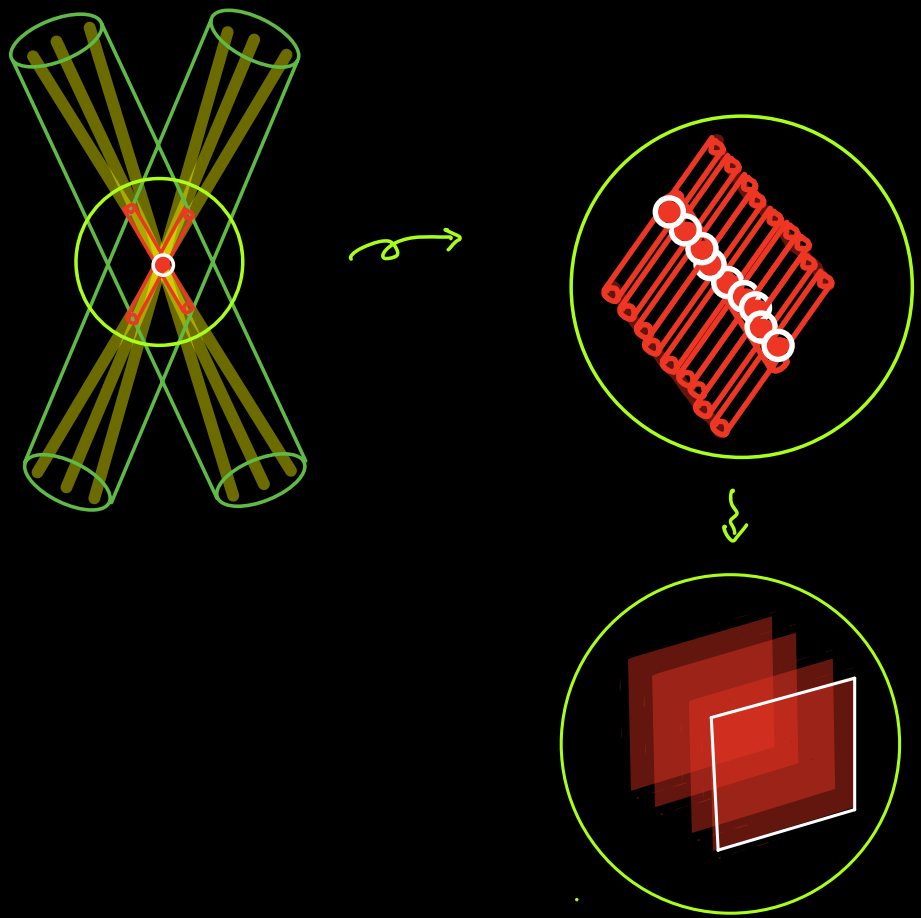


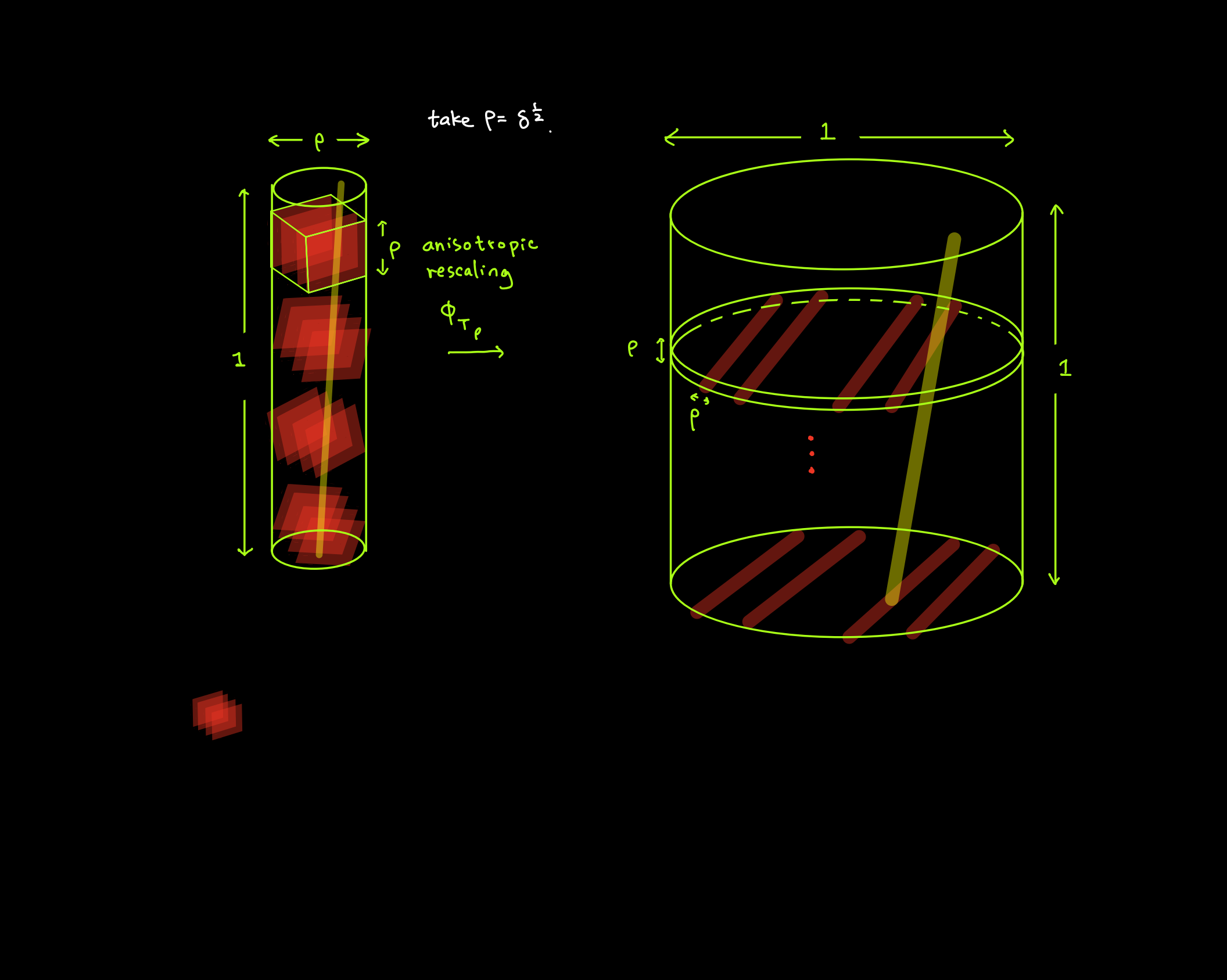
right picture for extremal.

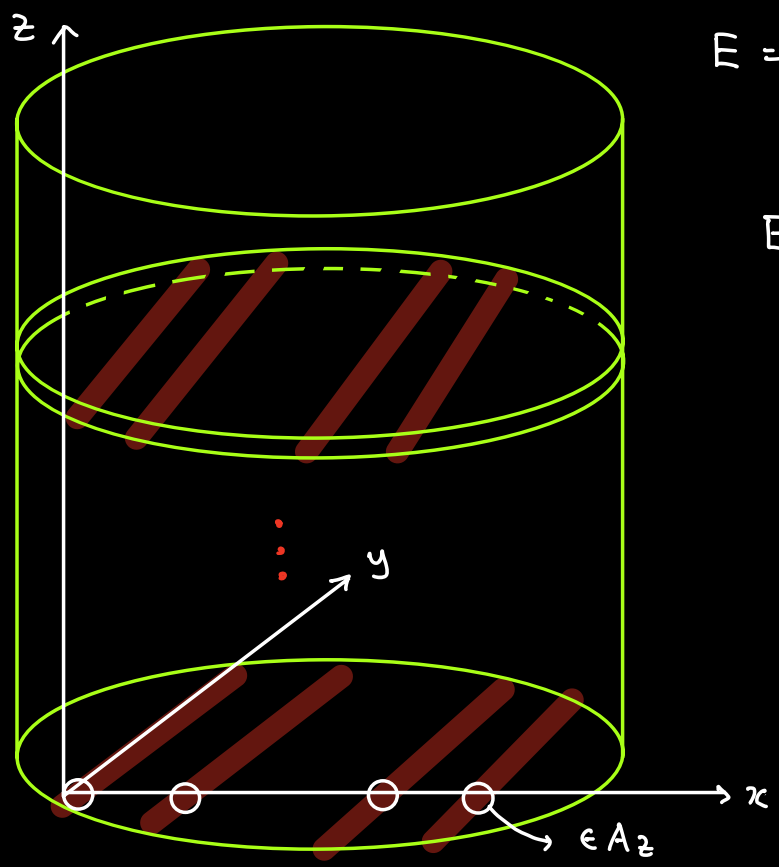
"sticky" kakeya sets



②  ⇒ graininess + reduction to Sum-product type problem.

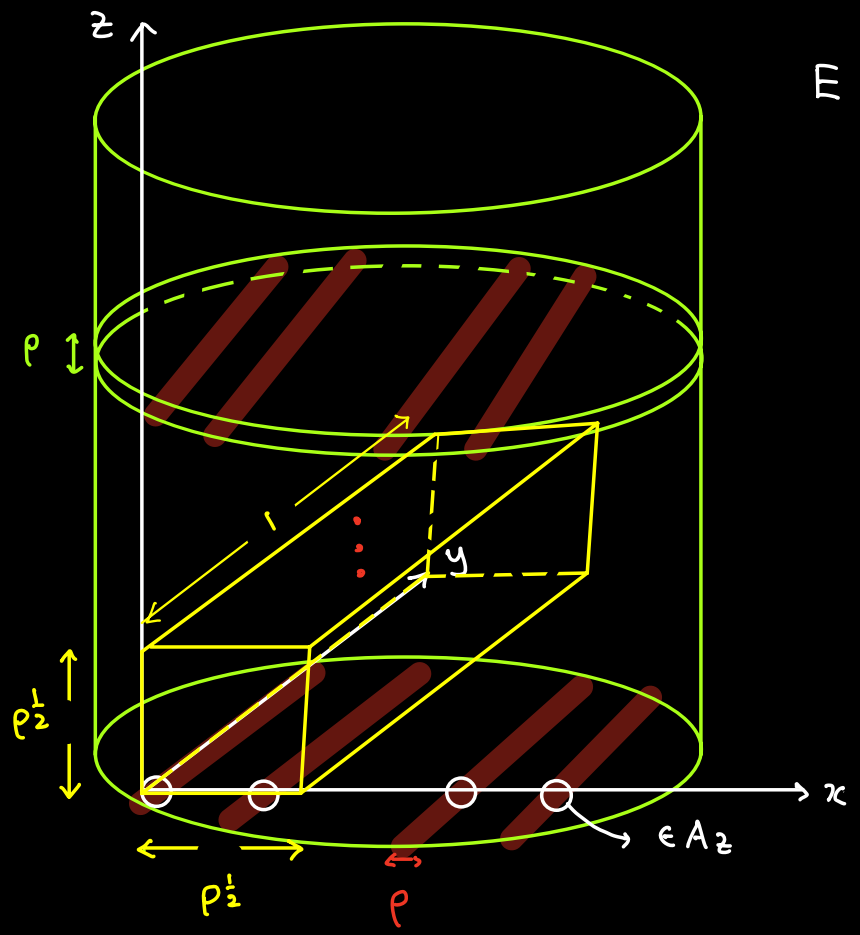



take $\rho = \delta^{\frac{1}{2}}$.
$\rho$
$\rho$
anisotropic rescaling
$\Phi_{T_\rho}$
1
1
$\rho$
$\rho$
1



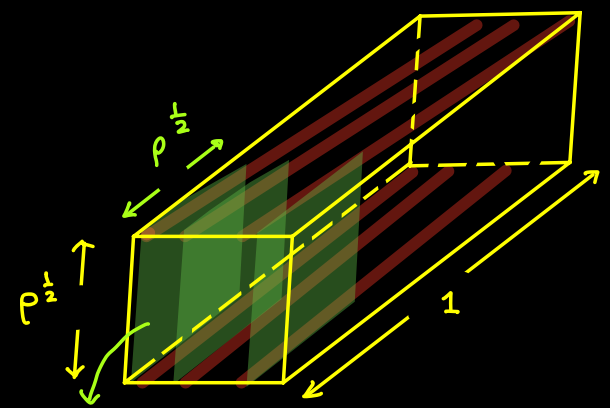

$E = \Phi_{T_\rho}(UT)$ can be parametrized as $\mathbb{T}[T_\rho]$

$$E = \{ (x + f(z)y, y, z) : y, z \in [0,1], \; x \in A_z \subseteq [0,1] \}$$

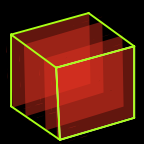



$$E = \bigcup_{T \in \mathbb{T}[T_\rho]} \Phi_{T_\rho}(T) \quad \text{can be parametrized as}$$

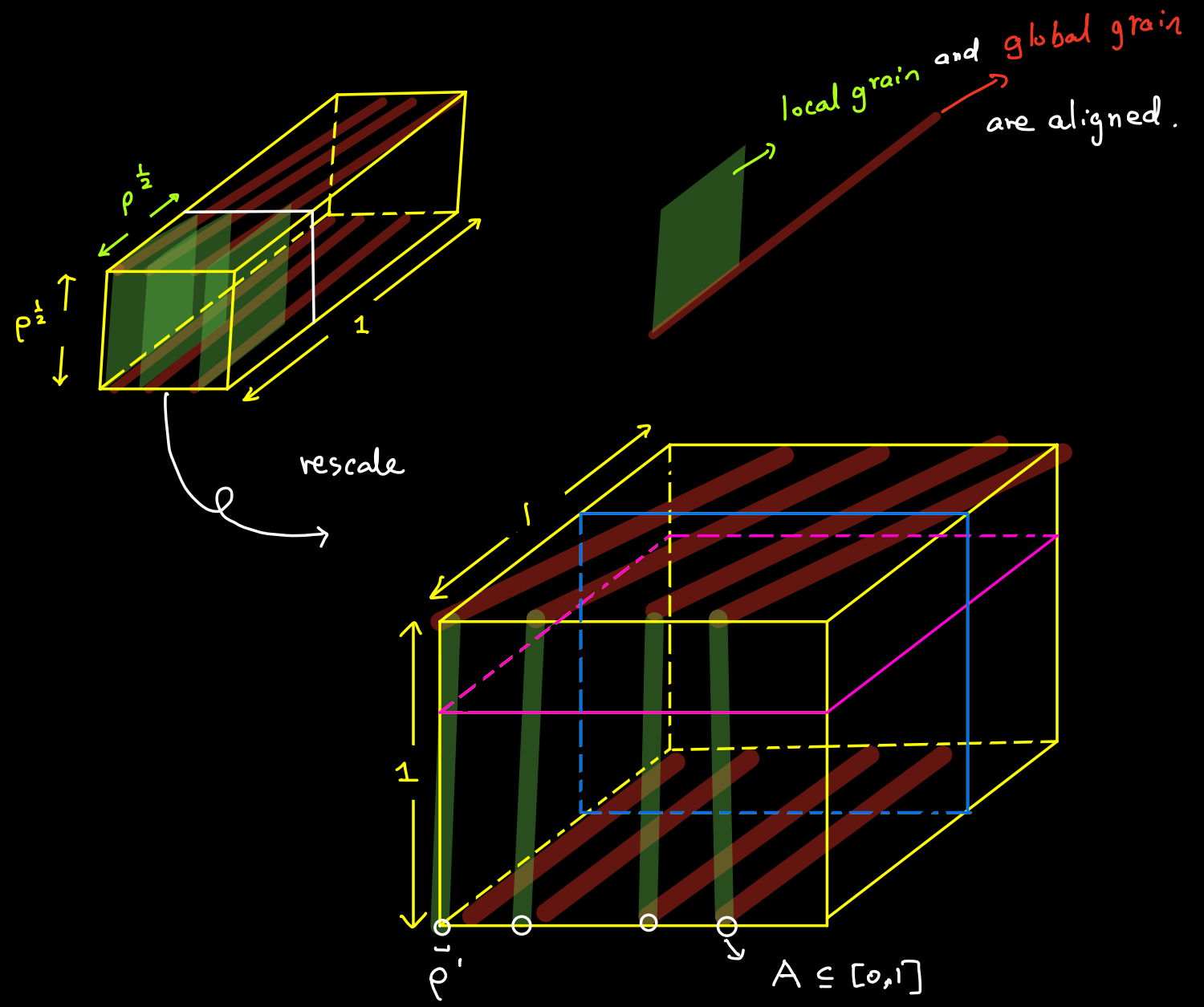
$$E = \{ (x + f(z)y,\ y,\ z) : y, z \in [0,1],\ x \in A_z \subseteq [0,1] \}$$

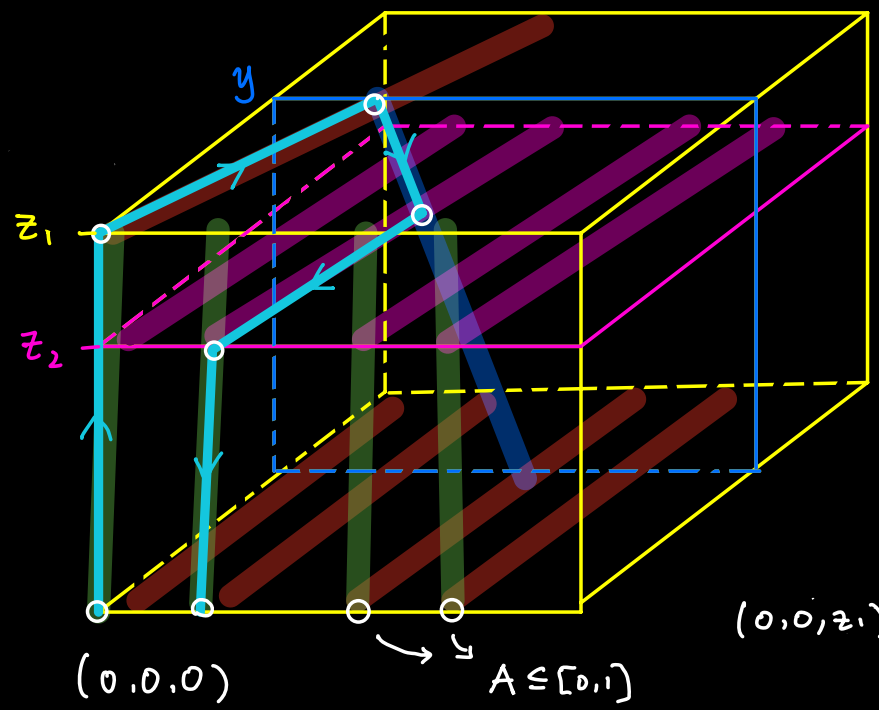


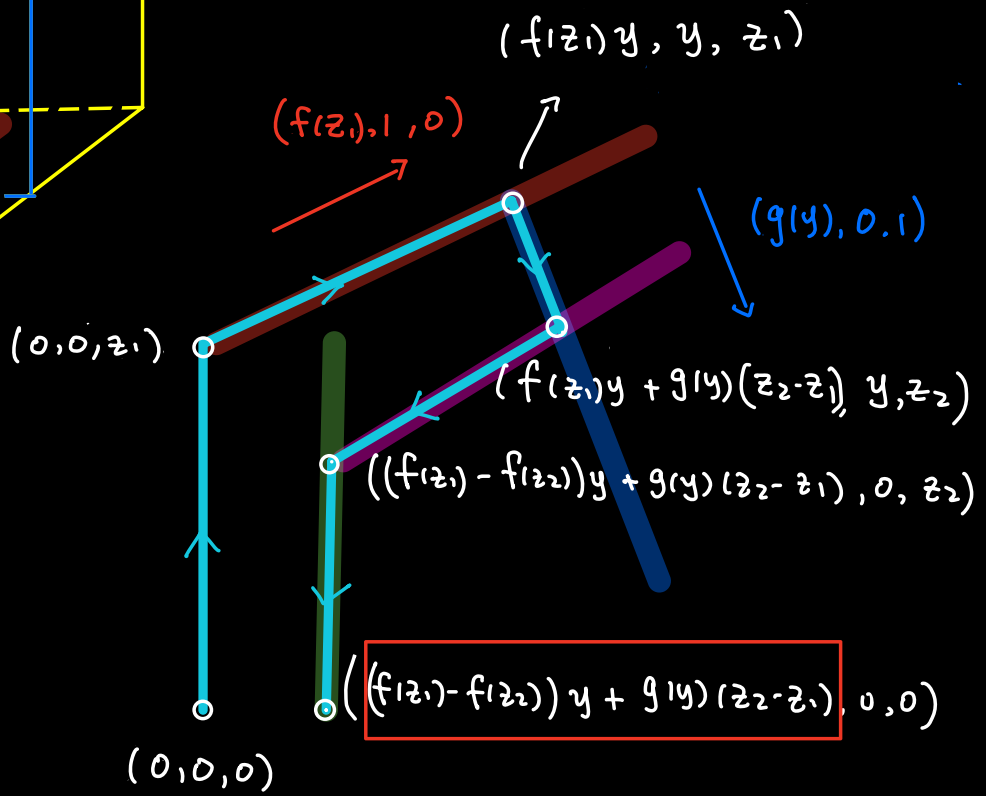

grains: $\rho \times \rho^{\frac{1}{2}} \times \rho^{\frac{1}{2}}$ slabs



Since $E = \bigcup_{T \in \mathbb{T}[T_\rho]} \Phi_{T_\rho}(T)$ is an extremal set of $(\frac{\delta}{\rho} = \rho)$-tubes satisfy (F) at all scales.


local grain and global grain
are aligned.
$\rho^{\frac{1}{2}}$
$\rho^{\frac{1}{2}}$
1
rescale
1
1
$\rho^{\frac{1}{2}}$
$A \subseteq [0,1]$


y
$z_1$
$z_2$
(0,0,0)
$A \subseteq [0,1]$


$(f(z_1)y, y, z_1)$
$(f(z_1),1,0)$
$(g(y),0,1)$
$(0,0,z_1)$
$(f(z_1)y + g(y)(z_2-z_1), y, z_2)$
$((f(z_1)-f(z_2))y + g(y)(z_2-z_1), 0, z_2)$
$((f(z_1)-f(z_2))y + g(y)(z_2-z_1), 0, 0)$
(0,0,0)

$$(f(z_1)-f(z_2))\,y + g(y)(z_2-z_1) \in A$$

$F = \{(f(z), z) : z \in [0,1]\}$ $\qquad$ $G = \{(y, g(y)) : y \in [0,1]\}$

$A + (F-F)\cdot G \subseteq A$, where $(F-F)\cdot G = \left\{ (f(z_1)-f(z_2))\,y + g(y)(z_2-z_1) \;:\; z_1, z_2, y \in [0,1] \right\}$

Apply **Radial projection theorem**.

to show either "dim" $A = 1 \Rightarrow |U_T| \approx 1$ (for $T \in \Pi$)

or $F$ and $G$ lies on orthogonal lines.

Apply **Restricted projection theorem**.

See Tuomas Orponen's talk.