RADIAL AND RESTRICTED PROJECTIONS

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ABSTRACT. These are lecture notes for a mini course given in El Escorial in June 2025. The purpose is to cover two results in projection theory which were recently applied by Wang and Zahl [12] to solve the *sticky Kakeya conjecture* in $\mathbb{R}^3$. The results in question concern *radial projections in* $\mathbb{R}^2$ and *restricted projections in* $\mathbb{R}^3$.

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1. RADIAL PROJECTIONS

In this section we will study the *radial projections* $\pi_x \colon \mathbb{R}^2 \setminus \{x\} \to S^1$ defined by

$$\pi_x(y) := \frac{x-y}{|x-y|}, \qquad y \in \mathbb{R}^2 \setminus \{x\}.$$

The purpose of the section is to sketch the proof of the following theorem:

Theorem 1.1. *Let $E, F \subset \mathbb{R}^2$ be disjoint compact sets with $\mathcal{H}^1(E) > 0$ and $\mathcal{H}^1(F) > 0$. Assume moreover that $\mathcal{H}^1(E \setminus \ell) > 0$ for every line $\ell \subset \mathbb{R}^2$. Then,*

$$\mathcal{H}^1(\{x \in E : \dim_{\mathrm{H}} \pi_x(F) \geqslant s\}) > 0, \qquad s \in [0,1). \tag{1.2}$$

There is a weaker version of Theorem 1.1, Lemma 1.3 below (proved initially in [6]), which is needed in the proof of Theorem 1.1. The proof of Lemma 1.3 is elementary but still so technically challenging that I omit the full details. I instead give a simple heuristic idea underlying the proof.

Lemma 1.3. *The conclusion of Theorem 1.1 holds for $s \in [0, \frac{1}{2})$.*

Heuristic idea. The following box-dimension variant is much easier to prove than (1.2):

$$\mathcal{H}^1(\{x \in E : \overline{\dim}_{\mathrm{B}} \pi_x(F) \geqslant \tfrac{1}{2}\}) > 0. \tag{1.4}$$

We can do this rigorously. First, it follows easily from the hypothesis $\mathcal{H}^1(E \setminus \ell) > 0$ that there exists $\tau \in (0,1]$ such that $\mathcal{H}^1(E \setminus [\ell]_\tau) > 0$ for all lines $\ell \subset \mathbb{R}^2$. Now, let $0 < r \leqslant \frac{1}{2}\min\{\tau, \operatorname{dist}(E,F)\}$, and let $x_0 \in E$ and $y_0 \in F$ be such that

$$\mathcal{H}^1(E \cap B(x_0, r)) > 0 \quad \text{and} \quad \mathcal{H}^1(F \cap B(y_0, r)) > 0.$$

Date: May 26, 2025.

Let ℓ be the line connecting x_0 to y_0. Since $E \cap B(x_0, r) \subset [\ell]_{\tau/2}$, and $\mathcal{H}^1(E \setminus [\ell]_\tau) > 0$, we may find another point $x_1 \in E \setminus [\ell]_\tau$ such that $\mathcal{H}^1(E \cap B(x_1, r)) > 0$. We claim the following: if $z_0 \in E \cap B(x_0, r)$ and $z_1 \in E \cap B(x_1, r)$, then

$$\max \left\{ \overline{\dim}_{\mathrm{B}} \pi_{z_0}(F \cap B(y_0, r)), \overline{\dim}_{\mathrm{B}} \pi_{z_1}(F \cap B(y_0, r)) \right\} \geqslant \tfrac{1}{2}.$$

The reason is the product set inequality $\overline{\dim}_{\mathrm{B}}(A \times B) \leqslant \overline{\dim}_{\mathrm{B}} A + \overline{\dim}_{\mathrm{B}} B$. The set $F \cap B(y_0, r)$ is contained in the "product" of the sets $\pi_{z_0}(F \cap B(y_0, r))$ and $\pi_{z_1}(F \cap B(y_0, r))$. Therefore one of the "components" of the product needs to have box dimension at least $\frac{1}{2}\overline{\dim}_{\mathrm{B}}(F \cap B(y_0, r)) \geqslant \frac{1}{2}$.

From the claim above it follows that either $\overline{\dim}_{\mathrm{B}} \pi_x(F) \geqslant \frac{1}{2}$ for all $x \in E \cap B(x_0, r)$, or the same holds for all $x \in E \cap B(x_1, r)$. Both cases give (1.4). □

To prove Theorem 1.1, it would be great if we could prove the following "Proposition":

"Proposition". *For every $s \in [0, 1)$ there exists $\eta = \eta(s) > 0$ such that the following holds. Let $E, F \subset B(1)$ be disjoint compact sets with $\mathcal{H}^1(E) > 0$ and $\mathcal{H}^1(F) > 0$. Assume $\sigma \geqslant \frac{1}{4}$, and*

$$\mathcal{H}^\sigma(\pi_x(F)) > 0 \text{ for } \mathcal{H}^1\text{-positively many } x \in E, \tag{1.5}$$

and

$$\mathcal{H}^{\sigma+\eta}(\pi_x(F)) = 0 \text{ for } \mathcal{H}^1 \text{ a.e. } x \in E. \tag{1.6}$$

Then $\sigma \geqslant s$.

I'm not sure if the "Proposition" is literally true: a more technical version is the "key lemma" in [9], see [9, Lemma 2.9]. The plan is nonetheless to use the "Proposition" to complete the proof of Theorem 1.1. A more complicated version of this argument allows one to prove Theorem 1.1 with the aid of [9, Lemma 2.9].

Once this is done, I will rigorously prove a δ-discretised version of the "Proposition" (this is still different from [9, Lemma 2.9], which would be too technical for these notes).

Proof of Theorem 1.1 assuming "Proposition". Let $E, F \subset B(1)$ be the compact and disjoint sets from Theorem 1.1. Write

$$s := \inf\{\sigma > 0 : \mathcal{H}^\sigma(\pi_x(F)) = 0 \text{ for } \mathcal{H}^1 \text{ a.e. } x \in E\}.$$

By Lemma 1.3 we know that $s \geqslant \frac{1}{2}$. We claim that $s = 1$. Assume to the contrary that $s < 1$, and let $\eta := \eta(s) > 0$ be the parameter given by the "Proposition". Then we may find $\sigma \in [\frac{1}{4}, s)$ such that $\sigma + \eta > s$. By the definition of "s",

$$\mathcal{H}^\sigma(\pi_x(F)) > 0 \text{ for } \mathcal{H}^1\text{-positively many } x \in E,$$

and

$$\mathcal{H}^{\sigma+\eta}(\pi_x(F)) = 0 \text{ for } \mathcal{H}^1\text{-a.e. } x \in E.$$

Now the "Proposition" implies that $\sigma \geqslant s$, which is immediately a contradiction. □

I proceed to state a δ-discretised version of the "Proposition" which can be proved in these notes without introducing too many technicalities. I emphasise that Proposition 1.10 does not imply the "Proposition" (as far as I know), but the proof of Proposition 1.10 will nonetheless display the key ideas present in the proof of [9, Lemma 2.9].

First, a definition:

Definition 1.7 ((δ, s, C)-set)**.** Let $\delta > 0$. A δ-separated set $P \subset \mathbb{R}^d$ is called a (δ, s, C)-set if

$$|P \cap B(x, r)| \leqslant Cr^s|P|, \qquad x \in \mathbb{R}^d,\ \delta \leqslant r \leqslant 1. \tag{1.8}$$

A set of δ-tubes $\mathcal{T}_x$ all containing a common point $x \in \mathbb{R}^2$ is called a (δ, s, C)-set if the directions of the tubes form a (δ, s, C)-subset of S^1.

If the value of the constant C is irrelevant, a (δ, s, C)-set will be called a (δ, s)-set.

Remark 1.9. A good way to think about a (δ, s, C)-set is to note that the normalised counting measure $\nu := |P|^{-1}\mathcal{H}^0|_P$ satisfies the Frostman condition $\nu(B(x, r)) \leqslant Cr^s$ on scales $\delta \leqslant r \leqslant 1$. It is also useful to note that if P is a non-empty (δ, s, C)-set, then $|P| \geqslant C^{-1}\delta^{-s}$: this follows by applying (2.16) to any $x \in P$ with $r := \delta$ (then $|P \cap B(x, r)| \geqslant 1$).

Proposition 1.10. *For every $s \in [0, 1)$ there exist $\delta_0 = \delta_0(s) > 0$ and $\eta = \eta(s) > 0$ such that the following holds for $\delta \in (0, \delta_0]$. Assume that $E, F \subset B(1)$ are non-empty $(\delta, 1)$-sets with*

$$\operatorname{dist}(E, F) \geqslant \tfrac{1}{2},$$

Let $\sigma \geqslant \frac{1}{4}$. Assume that for every $x \in E$ there is a family $\mathcal{T}_x$ of δ-tubes such that:

(a) *$\mathcal{T}_x$ is a (δ, σ)-set, the tubes in $\mathcal{T}_x$ contain x, and $F \subset \bigcup_{T \in \mathcal{T}_x} T$.*
(b) *$|\mathcal{T}_x| \leqslant \delta^{-\sigma-\eta}$.*
(c) *$T \mapsto |F \cap T|$ is roughly constant on $\mathcal{T}_x$.*

Then $\sigma \geqslant s$.

Remark 1.11. To see the correspondence between Proposition 1.10 and the "Proposition", note that (a) is roughly a δ-discretised counterpart of (1.5), and (b) is a counterpart of (1.6). It would be nicer to state Proposition 1.10 without the additional technical condition (c), but this is not straightforward: the problem is that a simple "pigeonholing" argument to make $T \mapsto |F \cap T|$ roughly constant may end up (seriously) breaking property (a). In the actual version of the proposition, [9, Lemma 2.9], property (a) is replaced by a more robust (but technical) condition which stays intact in such pigeonholing.

The proof of Proposition 1.10 uses two ingredients. The first one is a simple incidence bound, Lemma 1.12 below. The second one is a non-trivial estimate on the size of Furstenberg sets, stated as Theorem 1.14.

Lemma 1.12. *Let $P \subset B(1) \subset \mathbb{R}^2$ be a $(\delta, 1)$-set, and let $\mathcal{T}$ be any δ-separated family of δ-tubes.*[1] *Then,*

$$|\mathcal{I}(P, \mathcal{T})| \lesssim \log(1/\delta)|P||\mathcal{T}|^{1/2} + |\mathcal{T}|. \tag{1.13}$$

Proof. Write

$$|\mathcal{I}(P, \mathcal{T})| = \sum_{T \in \mathcal{T}} |P \cap T| \leqslant |\mathcal{T}|^{1/2} \left(\sum_{T \in \mathcal{T}} |\{(p, q) \in P^2 : p, q \in T\}| \right)^{1/2}.$$

Concerning the factor inside the brackets, we exchange the order of summation, and extract the "diagonal" terms $\{(p, p) : p \in T\}$:

$$\sum_{T \in \mathcal{T}} |\{(p, q) \in P^2 : p, q \in T\}| = \sum_{p \neq q} |\{T \in \mathcal{T} : p, q \in T\}| + |\mathcal{I}(P, \mathcal{T})|.$$

[1] This means that the central lines of any pair of distinct tubes have distance $\geqslant \delta$ in some natural metric of the affine Grassmannian $\mathcal{A}(2, 1)$.

For $p, q \in P$ with $p \neq q$, we note that $|\{T \in \mathcal{T} : p, q \in T\}| \lesssim |p-q|^{-1}$, using the fact that $\mathcal{T}$ consists of δ-separated elements. Now, using the $(\delta, 1)$-set property of P,

$$\sum_{p \neq q} \frac{1}{|p-q|} \lesssim \sum_{p \in P} \sum_{\delta \leqslant 2^{-j} \leqslant 1} 2^j \cdot |P \cap B(p, 2^{-j})| \lesssim \log(1/\delta) \cdot |P|^2.$$

This concludes the proof. □

The following theorem is the main result in [8]. It is an "ϵ-improvement" towards the *Furstenberg set conjecture*, which was fully resolved in 2023 by Ren and Wang [11]. In particular, the full strength of [11] is not required to prove Theorem 1.1, and in fact Theorem 1.1 is used (implicitly) in [11].

Theorem 1.14 (Baby Furstenberg set theorem)**.** *For every $\sigma \in (0,1)$, there exist $\delta_0 > 0$ and $\epsilon := \epsilon(\sigma) > 0$ such that the following holds for all $\delta \in (0, \delta_0]$. Assume that $P \subset B(1)$ is a $(\delta, 1)$-set, and for every $x \in P$, there is a non-empty (δ, σ)-set $\mathcal{T}_x$ of δ-tubes, each of which contains x. Then, $\bigcup_{x \in P} \mathcal{T}_x$ contains a δ-separated subset of cardinality $\geqslant \delta^{-2\sigma-\epsilon}$.*

Moreover, if $I \subset (0,1)$ is compact, then $\inf\{\epsilon(\sigma) : \sigma \in I\} > 0$.

Remark 1.15. To understand where the numerology $\delta^{-2\sigma-\epsilon}$ comes from, let's see what Lemma 1.12 would imply in the context of the Furstenberg set theorem. Note that $|\mathcal{T}_x| \geqslant \delta^{-\sigma}$ by the (δ, σ)-set hypothesis, so $|\mathcal{I}(P, \mathcal{T})| \geqslant |P|\delta^{-\sigma}$. On the other hand $|\mathcal{I}(P, \mathcal{T})| \lessapprox |P||\mathcal{T}|^{1/2} + |\mathcal{T}|$. Combining these estimates yields $|\mathcal{T}| \gtrsim \delta^{-2\sigma}$.

This argument works whenever $|\mathcal{T}_x| \geqslant \delta^{-\sigma}$. Theorem 1.14 says that the ensuing $\delta^{-2\sigma}$-bound can be improved by exploiting the (δ, σ)-set hypothesis in a less trivial way.

Proof of Proposition 1.10. Fix $s < 1$, and let

$$\epsilon_0 := \inf\left\{\epsilon(\sigma) : \sigma \in \left[\tfrac{1}{4}, s\right]\right\} > 0,$$

where $\epsilon(\sigma) > 0$ is the parameter given by Theorem 1.14. Then, let

$$\eta := \eta(s) := \min\{\epsilon_0/4, (1-s)/2\}. \tag{1.16}$$

We claim that Proposition 1.10 holds with this constant η, and for $\delta > 0$ sufficiently small.

Fix $\sigma \geqslant \frac{1}{4}$ as in the statement of Proposition 1.10. Counter assumption: $\sigma < s$.

Let N_x be the constant from property (c). We would prefer N_x to be independent of x, and this can be achieved by a simple pigeonholing argument: for every $\epsilon > 0$, assuming that $\delta > 0$ is small enough, there exists a $(\delta, 1, \delta^{-\epsilon})$-subset $E' \subset E$ such that $N_x \sim N$ for all $x \in E'$. There is no significant difference between E and E' during the rest of the proof, so we assume that $N_x \equiv N$ for all $x \in E$ to begin with.

Let $\mathcal{T}$ be a maximal family of distinct elements in $\bigcup_{x \in E} \mathcal{T}_x$. Then, by putting together property (c) and Lemma 1.12,

$$N|\mathcal{T}| \lesssim |\mathcal{I}(F, \mathcal{T})| \lesssim \log(1/\delta)|F||\mathcal{T}|^{1/2} + |\mathcal{T}|. \tag{1.17}$$

Note that since $F \subset \cup \mathcal{T}_x$ (for any $x \in E$), we have

$$N \sim |F|/|\mathcal{T}_x| \gtrsim \delta^{-1+\sigma+\eta} \geqslant \delta^{-1+s+\eta} \overset{(2.17)}{\gg} 1. \tag{1.18}$$

Consequently the second term in (2.13) cannot dominate the left hand side. We may therefore deduce from (2.13), and using $|\mathcal{T}_x| \leqslant \delta^{-\sigma-\eta}$, that

$$N|\mathcal{T}| \lessapprox \sqrt{\delta^{-1}|F||\mathcal{T}|} \quad \Longrightarrow \quad |\mathcal{T}| \lessapprox \frac{|F|^2}{N^2} \overset{(1.18)}{\lesssim} \delta^{-2\sigma-2\eta}.$$

(Here $\lessapprox$ hides factors of type $(\log(1/\delta))^C$). On the other hand, the families $\mathcal{T}_x$ satisfy the hypotheses of the Baby Furstenberg set theorem, therefore $|\mathcal{T}| \geqslant \delta^{-2\sigma-\epsilon(\sigma)} \geqslant \delta^{-2\sigma-\epsilon_0}$. Since $2\eta < \epsilon_0/2$ by (2.17), a contradiction ensues for $\delta > 0$ small enough. Therefore $\sigma \geqslant s$, and the proof of Proposition 1.10 is complete. □

2. Restricted projections

The purpose of this section is to study linear projections (as opposed to the radial projections in Section 1). We will prove something called a *restricted projection theorem in* $\mathbb{R}^3$. The version here (Theorem 2.2) was originally established in [3], but the proof here is simpler (modulo using heavier machinery) and based on [2].

We set up some notation. For $\theta \in [0,\pi)$, let $\pi_\theta \colon \mathbb{R}^2 \to \mathbb{R}$ be the map $\pi_\theta(x,y) := (x,y) \cdot (\cos\theta, \sin\theta)$. Up to isometry, π_θ is the orthogonal projection to the line $\operatorname{span}(\cos\theta, \sin\theta)$. We will also consider the maps $\rho_\theta \colon \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\rho_\theta(x,y,z) := (x,y,z) \cdot (\cos\theta, \sin\theta, 1) = \pi_\theta(x,y) + z, \qquad \theta \in [0,\pi).$$

Up to isometry, these are the orthogonal projections to the 1-dimensional subspaces

$$\ell_\theta := \operatorname{span}(\cos\theta, \sin\theta, 1) \subset \mathbb{R}^3$$

contained in the cone $\mathcal{C} = \{(z,t) \in \mathbb{R}^2 \times \mathbb{R} : |z| = |t|\}$. This observation will be important when we use the *decoupling theorem for the cone* to study the projections ρ_θ.

Theorem 2.1 (Marstrand's projection theorem)**.** *Let $K \subset \mathbb{R}^2$ be a compact set with $\dim_{\mathrm{H}} K = 1$. Then there exists $\theta \in [0,\pi)$ such that $\dim_{\mathrm{H}} \pi_\theta(K) = 1$.*

Theorem 2.2 (Restricted projection theorem in $\mathbb{R}^3$)**.** *Let $K \subset \mathbb{R}^3$ be a compact set with $\dim_{\mathrm{H}} K = 1$. Then, there exists $\theta \in [0,\pi)$ such that $\dim_{\mathrm{H}} \rho_\theta(K) = 1$.*

Remark 2.3. In fact, Theorem 2.2 holds for much more general (1-parameter) families of projections to lines in $\mathbb{R}^3$, see [10]. Here we just treat the family $\{\rho_\theta\}_{\theta\in[0,\pi)}$ for simplicity.

Remark 2.4. Theorem 2.2 implies Theorem 2.1: for $K \subset \mathbb{R}^2$ compact, apply Theorem 2.2 to $\bar{K} := K \times \{0\}$, and note that

$$\pi_\theta(K) = \rho_\theta(\bar{K}), \qquad \theta \in [0,\pi).$$

Of course using Theorem 2.2 to prove Theorem 2.1 is rather overkill: Theorem 2.1 has several simpler proofs, see for example [5, Corollary 9.4].

We proceed with the proof of Theorem 2.2. The plan is to reduce matters to a δ-discretised incidence result, Theorem 2.5. First, a word on terminology: a *δ-slab* is a rectangle of dimensions $1 \times 1 \times \delta$ in $\mathbb{R}^3$. If $\mathcal{W}$ is a (finite) family of δ-slabs, a point $p \in \mathbb{R}^3$ is called *m-rich relative to $\mathcal{W}$* if $|\{W \in \mathcal{W} : p \in W\}| \geqslant m$.

Theorem 2.5. *There exists an absolute constant $\mathbf{c} > 0$ such that the following holds. Let $\delta \in (0, \frac{1}{2}]$, and let $\varnothing \neq \Theta \subset [0,\pi)$ be a δ-separated set. For every $\theta \in \Theta$, let $\mathcal{W}_\theta$ be a collection of disjoint δ-slabs perpendicular to ℓ_θ. Assume that there exists a radius $R \geqslant 1$ such that*

$$|\{W \in \mathcal{W}_\theta : W \cap B(x,r) \neq \varnothing\}| \leqslant \mathbf{c}(\delta/r), \qquad x \in \mathbb{R}^3,\ R\delta \leqslant r \leqslant 1. \tag{2.6}$$

Write $\mathcal{W} := \bigcup_{\theta\in\Theta} \mathcal{W}_\theta$, and let $H \subset \mathbb{R}^3$ be a set of $|\Theta|$-rich points relative to $\mathcal{W}$. Then,

$$\operatorname{Leb}(H) \lesssim_{\epsilon,R} \delta^{-2-\epsilon} \cdot |\mathcal{W}|/|\Theta|^6, \qquad \epsilon > 0. \tag{2.7}$$

Before proving Theorem 2.5, I discuss why it implies Theorem 2.2. The proof employs the following useful "covering theorem" due to Katz and Tao [4, Lemma 7.5] (the statement in [4] is a tiny bit different, but see [7, Lemma 2.4] for this version):

Lemma 2.8. *Let $A \subset \mathbb{R}^d$ be a set with $\mathcal{H}^s(A) = 0$. Then, for every dyadic scale $\delta_j = 2^{-j}$ there exists a set P_j (which can be taken to be a union of δ_j-cubes) satisfying the following non-concentration condition*

$$|P_j \cap B(x,r)|_{\delta_j} \leqslant (r/\delta_j)^s, \qquad x \in \mathbb{R},\ \delta_j \leqslant r \leqslant 1. \tag{2.9}$$

Moreover, $A \subset \bigcup_j P_j$.

Remark 2.10. The non-concentration condition (2.9) is often called the *Katz-Tao (δ_j, s)-set property*, in contrast to the *(δ, s)-set property* introduced in Definition 1.7.

Proof of Theorem 2.2 assuming Theorem 2.5. Assume, to reach a contradiction, that there exists a compact set $K \subset B(1) \subset \mathbb{R}^3$ with $\dim_{\mathrm{H}} K = 1$ such that nonetheless

$$\dim_{\mathrm{H}} \rho_\theta(K) < s < 1, \qquad \theta \in [0,\pi).$$

In particular $\mathcal{H}^s(\rho_\theta(K)) = 0$ for all $\theta \in K$ for all $\theta \in [0,\pi)$, so we may apply Lemma 2.8 to each set $A = \rho_\theta(K)$. This produces, for each $\theta \in [0,\pi)$, a countable sequence of sets $P_j(\theta)$, each of which satisfies the Katz-Tao (δ_j, s)-set property (2.9). After some pigeonholing, one may reduce matters to the following situation: there is a single set $P(\theta) \subset \mathbb{R}$ covering $\rho_\theta(K)$ for each $\theta \in [0,\pi)$, and indeed each $P(\theta)$ is a Katz-Tao (δ, s)-set for a common (arbitrarily small) scale $\delta > 0$. We take this situation as our starting point.

Now we proceed to apply Theorem 2.5. For $\theta \in [0,\pi)$, recall from Lemma 2.8 that $P(\theta)$ is a union of δ-intervals. Denote this family of intervals $\mathcal{P}(\theta)$. Let

$$\mathcal{W}_\theta := \{B(1) \cap \rho_\theta^{-1}(2I)\}_{I \in \mathcal{P}(\theta)}$$

be the family of 2δ-slabs obtained as ρ_θ-pre-images of the (doubled) intervals in $\mathcal{P}(\theta)$. Then $\mathcal{W}_\theta$ satisfies the non-concentration condition (2.6) for all $R \geqslant 1$ large enough, depending only on $s < 1$, since

$$|\{W \in \mathcal{W}_\theta : W \cap B(x,r)\}| \lesssim (r/\delta)^s, \qquad x \in \mathbb{R}^3,\ \delta \leqslant r \leqslant 1,$$

due to the Katz-Tao (δ, s)-set property of $P(\theta)$.

Let $\Theta \subset [0,\pi)$ be a maximal δ-separated subset, and let $\mathcal{W} := \bigcup_{\theta \in \Theta} \mathcal{W}_\theta$. Note that each point $x \in H := [K]_\delta$ (the δ-neighbourhood of K) is $|\Theta|$-rich relative to $\mathcal{W}$. Therefore, (2.7) yields (for any $\epsilon > 0$)

$$\mathrm{Leb}(H) \lesssim_\epsilon \delta^{-2-\epsilon} |\mathcal{W}|/|\Theta|^6 \lesssim \delta^{-2-\epsilon} \cdot \delta^{-1-s}/\delta^{-6} = \delta^{3-s-\epsilon}. \tag{2.11}$$

However, $\mathrm{Leb}(H) = \mathrm{Leb}([K]_\delta) \gtrapprox \delta^2$ since $\dim_{\mathrm{H}} K = 1$, so (2.11) cannot hold for $\delta, \epsilon > 0$ small enough. □

We then move towards the proof of Theorem 2.5. We need a few pieces of information about the geometry of the Fourier transform. If $R \subset \mathbb{R}^d$ is a rectangle, then

$$|\widehat{\mathbf{1}_R}| \approx \mathrm{Leb}(R) \cdot \mathbf{1}_{R^*},$$

where R^* is the *dual rectangle of R centred at* 0. The rectangle R^* has the same orientation as R, but the side-lengths of R^* are the reciprocals of the side-lengths of R, see Figure 2 for an illustration in $\mathbb{R}^2$. Thus,

$$\mathrm{Leb}(R) = \mathrm{Leb}(R^*)^{-1}.$$

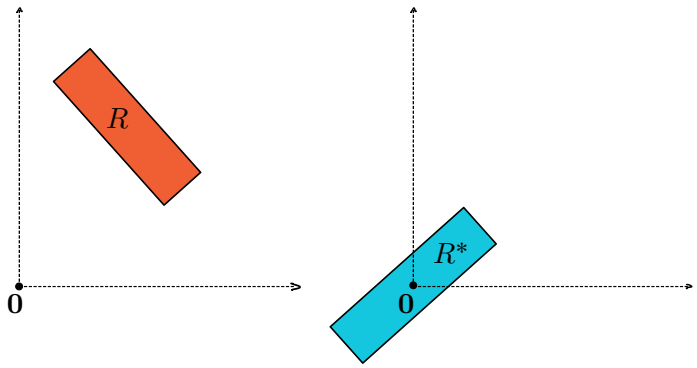


FIGURE 1. A rectangle $R \subset \mathbb{R}^2$ and its dual rectangle R^*.

Moreover, R^* is always centred at 0, no matter where R lies: if R_1, R_2 are translates of each other, then $R_1^* = R_2^*$.

Exercise 1. *Let $P, R \subset \mathbb{R}^d$ are rectangles such that $P \subset \mathbb{R}$. Show $R^* \subset \lambda_d P^*$, where $\lambda_d \geqslant 1$ only depends on the ambient dimension d.*

Example 2.12. *The dual rectangle of a δ-slab W is a rectangle W^* of dimensions $1 \times 1 \times \delta^{-1}$ centred at 0. Recall that the slabs $\mathcal{W}_\theta$, in Theorem 2.5, were all perpendicular to $\ell_\theta = \operatorname{span}(\cos\theta, \sin\theta, 1)$, so in particular they are translates of each other. This implies that (a) all the slabs in $\mathcal{W}_\theta$ have the same dual rectangle W_θ^*, and (b) the long δ^{-1}-side of the W_θ^* is parallel to ℓ_θ. Thus, the dual rectangles W_θ^*, $\theta \in [0, \pi)$, are contained in $B(\delta^{-1}) \cap [\mathcal{C}]_1$.*

Another useful fact from Fourier analysis is that $\widehat{fg} = \hat{f} * \hat{g}$. In particular, if R is a rectangle, then

$$|\widehat{f\mathbf{1}_R}| \leqslant |\hat{f}| * (\operatorname{Leb}(R) \cdot \mathbf{1}_{R^*}) = |\hat{f}| * (\operatorname{Leb}(R^*)^{-1} \cdot \mathbf{1}_{R^*}) = |\hat{f}| * \varphi_{R^*}. \tag{2.13}$$

Here $\varphi_{R^*} = \operatorname{Leb}(R^*)^{-1} \cdot \mathbf{1}_{R^*}$ is an "approximate identity adapted to R^*" (note that $\|\varphi_{R^*}\|_{L^1} \equiv 1$, independently of R). A slightly imprecise way to interpret (2.13) is the following principle:[2] *multiplying f by R corresponds to averaging $|\hat{f}|$ over translates of R^**. In practice, we will apply this observation in the following form: if $R \subset \mathbb{R}^d$ is a rectangle, and f_R is the function with Fourier transform $\hat{f}_R = \hat{f}\mathbf{1}_R$ (i.e. f_R is the *Fourier restriction of f to R*), then

$$|f_R| = |\check{\hat{f}}_R| = |(\hat{f}\mathbf{1}_R)^\vee| \leqslant |f| * \varphi_{R^*}.$$

In particular, Fourier-restricting f never increases its L^∞-norm.

Proof of Theorem 2.5. We start by defining the function(s)

$$f := \sum_{\theta \in \Theta} f_\theta := \sum_{\theta \in \Theta} \sum_{W \in \mathcal{W}_\theta} \chi_W.$$

[2] This informal statement is pretty accurate if R is a rectangle centred at 0, but fails to take into account the effect of translating R.

Here $\chi_W \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ satisfies $\mathbf{1}_W \leqslant \chi_W \leqslant \mathbf{1}_{2W}$, so $\widehat{\chi_W}$ has rapid decay outside W^*. In particular, recalling Example 2.12, and ignoring Schwartz tails,

$$\operatorname{spt} \hat{f} \subset \bigcup_{\theta \in W} \bigcup_{W \in \mathcal{W}_\theta} W^* \subset B(\delta^{-1}) \cap [\mathcal{C}]_1.$$

We then start by observing that

$$|\Theta|^6 \cdot \operatorname{Leb}(H) \leqslant \int_H f^6 \, dx,$$

since each point in H was assumed to be $|\Theta|$-rich. The power "6" will enable us to use the sharp decoupling theorem for the cone.

We now define the following *high-low decomposition* of f:

$$\hat{f}_{\text{low}} = \hat{f} \cdot \chi_{B((R\delta)^{-1})} \quad \text{and} \quad \hat{f}_{\text{high}} = \hat{f} - \hat{f}_{\text{low}}. \tag{2.14}$$

Here $R \geqslant 1$ is the constant for which the non-concentration condition (2.6) holds, and $\chi_{B((R\delta)^{-1})}$ is a non-negative smooth cut-off to the ball of radius $(R\delta)^{-1}$ centred at the origin. Let us be a bit more precise. We require

$$\mathbf{1}_{B((R\delta)^{-1})} \leqslant \chi_{B((R\delta)^{-1})} \leqslant \mathbf{1}_{B(2(R\delta)^{-1})}. \tag{2.15}$$

We can further define $\chi_{B((R\delta)^{-1})}$ as the Fourier transform of $\varphi_{R\delta}(x) := (R\delta)^{-3}\varphi(x/(R\delta))$, where $\varphi \in \mathcal{S}(\mathbb{R}^3)$. The proof now splits into the low case and high case, depending on whether

$$|\Theta|^6 \cdot \operatorname{Leb}(H) \lesssim \int_H |f_{\text{low}}|^6 \quad \text{or} \quad |\Theta|^6 \cdot \operatorname{Leb}(H) \lesssim \int |f_{\text{high}}|^6.$$

Low case. We claim that if the absolute constant $\mathbf{c} > 0$ in Theorem 2.5 is small enough, then it is **impossible** that

$$|\Theta|^6 \cdot \operatorname{Leb}(H) \lesssim \int_H |f_{\text{low}}|^6. \tag{2.16}$$

In fact, we will prove that

$$\|f_{\text{low}}\|_{L^\infty} \lesssim \mathbf{c} \cdot |\Theta|, \tag{2.17}$$

where the implicit constants arise from the non-concentration condition (2.6). Once this has been established, we have

$$\int_H |f_{\text{low}}|^6 \lesssim \mathbf{c}^6 \cdot |\Theta|^6 \cdot \operatorname{Leb}(H),$$

and it is evident that (2.16) cannot hold for $\mathbf{c} > 0$ small enough.

To prove (2.17), start by noting that

$$f_{\text{low}} = \sum_{\theta \in \Theta} f_{\theta,\text{low}},$$

where $\hat{f}_{\theta,\text{low}} = \hat{f}_\theta \cdot \chi_{B(\delta^{-1}/R)}$. It suffices to show that $\|f_{\theta,\text{low}}\|_{L^\infty} \lesssim \mathbf{c}$. Recalling the definition of $\chi_{B(\delta^{-1}/R)}$ from below (2.14), we may write

$$f_{\theta,\text{low}} = f_\theta * \varphi_{R\delta} = \Big(\sum_{W \in \mathcal{W}_\theta} \chi_W \Big) * \varphi_{R\delta},$$

Thus, $f_{\theta,\text{low}}$ is obtained by "averaging" f_θ over balls of radius $R\delta$. By the non-concentration condition (1.13), the number of δ-slabs $W \in \mathcal{W}_\theta$ meeting any fixed ball of radius $R\delta$ is bounded as follows:

$$|\{W \in \mathcal{W}_\theta : 2W \cap B(x, R\delta) \neq \varnothing\}| \lesssim \mathbf{c}R.$$

Since $\operatorname{spt} \chi_W \subset 2W$, and $\operatorname{Leb}(W \cap B(x, R\delta)) \lesssim R^2\delta^3$, we obtain

$$\begin{aligned} f_{\theta,\text{low}}(x) &\lesssim (R\delta)^{-3} \int_{B(x,R\delta)} \Big(\sum_{W \in \mathcal{W}_\theta} \chi_W \Big)\, dy \\ &\lesssim \frac{R^2\delta^3 \cdot |\{W \in \mathcal{W}_\theta : 2W \cap B(x, R\delta) \neq \varnothing\}|}{R^3\delta^3} \overset{(2.6)}{\lesssim} \mathbf{c}. \end{aligned}$$

(The estimate above ignores the Schwartz tail of the function $\varphi_{R\delta}$. In a rigorous version of the proof, the non-concentration condition (2.6) needs to be applied at many scales $r \geqslant R\delta$.) This proves (2.17) and completes the "low case".

High case. To handle the high case, we employ the *ℓ^2-decoupling theorem for the cone $\mathcal{C}$*, proved by Bourgain and Demeter [1]. This theorem says the following. Let g be a function with

$$\operatorname{spt} \hat{g} \subset [\mathcal{C}]_1 \cap \{|\xi| \sim \delta^{-1}\}.$$

In particular, $g = f_{\text{high}}$ is admissible (to be precise $\operatorname{spt} \hat{f}_{\text{high}} \subset \{|\xi| \geqslant (R\delta)^{-1}\}$), but f itself is not: $\hat{f}$ is supported on $[\mathcal{C}]_1$ (up to Schwartz tails), but not in the annulus $\{|\xi| \sim \delta^{-1}\}$.

Let $\mathcal{P}$ be a boundedly overlapping cover of $[\mathcal{C}]_1 \cap \{|\xi| \sim \delta^{-1}\}$ by *δ-plates*. These are, by definition, rectangles of dimensions $1 \times \delta^{-1/2} \times \delta^{-1}$ whose longest side is parallel to some ℓ_θ, see Figure 2. As another piece of notation, for $\theta \in \Theta$, let $\theta^* := W^*$ be the (common) dual rectangle all the slabs in $\mathcal{W}_\theta$. Recall that θ is a rectangle (or a "tube") of dimensions $1 \times 1 \times \delta^{-1}$ parallel to ℓ_θ. For $P \in \mathcal{P}$, we write $\theta^* \in P$ if $\theta^* \cap \{|\xi| \sim \delta^{-1}\} \subset P$.

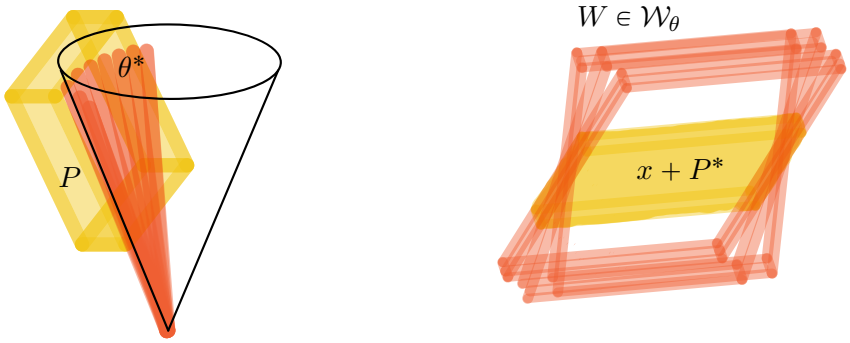


FIGURE 2. On the left: the plate $P \in \mathcal{P}$ and the dual rectangles θ^* with $\theta \in \Theta(P)$. On the right: a translate of the $(1 \times \delta^{1/2} \times \delta)$-rectangle P^*, and some δ-slabs $W \in \mathcal{W}_\theta$ with $\theta \in \Theta(P)$ which intersect $x + P^*$.

A few words on numerology. Note that $|\mathcal{P}| \sim \delta^{-1/2}$, and a single δ-plate $P \in \mathcal{P}$ may contain up to $\sim \delta^{-1/2}$ rectangles θ^*. For $P \in \mathcal{P}$ fixed, define the function g_P by $\hat{g}_P := g \cdot \mathbf{1}_P$.

With this notation, the ℓ^2-decoupling theorem for $\mathcal{C}$ says the following:

$$\|g\|_{L^6}^6 \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{P\in\mathcal{P}} \|g_P\|_{L^6}^2\Big)^3, \qquad \epsilon > 0. \tag{2.18}$$

To save space, I will abbreviate the "$\delta^{-\epsilon}$" in the sequel by the notation "$\lessapprox$". To estimate (2.18), we first apply Hölder's inequality, and recall that $|\mathcal{P}| \sim \delta^{-1/2}$:

$$\begin{aligned}\|f_{\text{high}}\|_{L^6}^6 \lessapprox \Big(\sum_{P\in\mathcal{P}} \|f_{\text{high},P}\|_{L^6}^2\Big)^3 &\leqslant |\mathcal{P}|^2 \sum_{P\in\mathcal{P}} \|f_{\text{high},P}\|_{L^6}^6 \\ &\lesssim \delta^{-1}\Big(\max_P \|f_{\text{high},P}\|_{L^\infty}^4\Big) \sum_{P\in\mathcal{P}} \|f_{\text{high},P}\|_{L^2}^2. \end{aligned} \tag{2.19}$$

The rightmost "L^2-part" will eventually be estimated by L^2-orthogonality, but let us first find an upper bound for $\|f_{\text{high},P}\|_{L^\infty}$ for $P \in \mathcal{P}$ fixed. The function $f_{\text{high},P}$ is the Fourier restriction of f_{high} to the $(1\times\delta^{-1/2}\times\delta^{-1})$-plate P, so $f_{\text{high},P}$ is (roughly speaking) obtained by averaging f_{high} over translates of the dual $(1\times\delta^{1/2}\times\delta)$-plate P^*. Moreover, since f_{high} is the Fourier restriction of f, we have $|f_{\text{high}}| \leqslant f$. All in all,

$$|f_{\text{high},P}(x)| \lesssim \fint_{x+P^*} f. \tag{2.20}$$

We need to do a little better than (2.20). Let $\Theta(P) := \{\theta \in \Theta : \theta^* \cap P \neq \varnothing\}$ and

$$f_{\Theta(P)} := \sum_{\theta\in\Theta(P)} f_\theta = \sum_{\theta\in\Theta(P)} \sum_{W\in\mathcal{W}_\theta} \chi_W.$$

Then, since $\operatorname{spt} \hat{f}_\theta \subset \theta^*$ (we ignore Schwartz tails), we have

$$\hat{f}_{\text{high},P} = \sum_{\theta\in\Theta} \hat{f}_\theta \cdot \chi_{\mathbb{R}^3\setminus B((R\delta)^{-1})} \cdot \mathbf{1}_P = \sum_{\theta\in\Theta(P)} \hat{f}_\theta \cdot \chi_{\mathbb{R}^3\setminus B((R\delta)^{-1})} \cdot \mathbf{1}_P = \hat{f}_{\Theta(P),\text{high},P}.$$

It follows (from the argument leading to (2.20)) that (2.20) remains valid if we replace "f" on the right by $f_{\Theta(P)}$. To estimate the ensuing average integral, recall that if $\theta \in \Theta(P)$ is fixed, then $\theta^* \cap P \neq \varnothing$. This implies by elementary geometry that $\theta^* \subset \Lambda P$ for an absolute constant $\Lambda \geqslant 1$, see the left hand side of Figure 2. Consequently (see Exercise 1)

$$P^* \subset \Lambda'\theta^{**}$$

for another absolute constant $\Lambda' \geqslant 1$. Here θ^{**} is a translate of (any) slab $W \in \mathcal{W}_\theta$, so we may deduce that P^*, and $x + P^*$, can be covered by $\lesssim 1$ translates of W. Thanks to the disjointness of $\mathcal{W}_\theta$, it follows that $x + P^*$ only intersects $O(1)$ elements in $\mathcal{W}_\theta$. The geometry of $x + P^*$ vs. $W \in \mathcal{W}_\theta$ is depicted on the right hand side of Figure 2.

Since we just argued that $|\{W \in \mathcal{W}_\theta : W \cap (x + P^*) \neq \varnothing\}| \lesssim 1$ for $\theta \in \Theta$ fixed,

$$|f_{\text{high},P}(x)| \lesssim \fint_{x+P^*} f_{\Theta(P)} \lesssim \sum_{\theta\in\Theta(P)} 1 \lesssim \delta^{-1/2}.$$

Plugging this information into (2.19), we find that

$$\|f_{\text{high}}\|_{L^6}^6 \lessapprox \delta^{-3} \sum_{P\in\mathcal{P}} \|f_{\text{high},P}\|_{L^2}^2. \tag{2.21}$$

Finally, we estimate the L^2-norms. The main observation is that the functions $f_{\theta,\text{high}}$ have Fourier supports of bounded overlaop, where the overlap constant only depends on R. Indeed, recalling (2.15), and ignoring Schwartz tails,

$$\operatorname{spt} \hat{f}_{\theta,\text{high}} = \operatorname{spt} \hat{f}_\theta \cdot (1 - \chi_{B((R\delta)^{-1})}) \subset \theta^* \cap (\mathbb{R}^3 \setminus B((R\delta)^{-1})).$$

Now, since Θ is δ-separated, we have

$$|\{\theta \in \Theta : x \in \theta^* \cap (\mathbb{R}^3 \setminus B((R\delta)^{-1}))\}| \lesssim R, \qquad x \in \mathbb{R}^3.$$

Using this, and Plancherel, and noting that $\|f_\theta\|_{L^2}^2 \sim |\mathcal{W}_\theta| \cdot \delta$,

$$\begin{aligned} \sum_{P \in \mathcal{P}} \|f_{\text{high},P}\|_{L^2}^2 &= \sum_{P \in \mathcal{P}} \int \Big| \sum_{\theta \in \Theta(P)} \hat{f}_\theta \cdot \chi_{\mathbb{R}^3 \setminus B((R\delta)^{-1})} \cdot \mathbf{1}_P \Big|^2 \\ &\lesssim R \sum_{P \in \mathcal{P}} \sum_{\theta \in \Theta(P)} \|f_\theta\|_{L^2}^2 \lesssim R \sum_{\theta \in \Theta} \|f_\theta\|_{L^2}^2 \sim R|\mathcal{W}| \cdot \delta. \end{aligned}$$

Combining this bound with (2.21), and recalling that we are in the *high case*, we obtain

$$|\Theta|^6 \cdot \operatorname{Leb}(H) \lessapprox C\delta^{-2}|\mathcal{W}|.$$

Dividing by $|\Theta|^6$ completes the proof of Theorem 2.5. □

REFERENCES

[1] Jean Bourgain and Ciprian Demeter. The proof of the l^2 decoupling conjecture. *Ann. of Math. (2)*, 182(1):351–389, 2015.

[2] Shengwen Gan, Shaoming Guo, Larry Guth, Terence L. J. Harris, Dominique Maldague, and Hong Wang. On restricted projections to planes in $\mathbb{R}^3$. *arXiv e-prints*, page arXiv:2207.13844, July 2022.

[3] Antti Käenmäki, Tuomas Orponen, and Laura Venieri. A Marstrand-type restricted projection theorem in $\mathbb{R}^3$. *Amer. J. Math.*, 147(1):81–123, 2025.

[4] Nets Hawk Katz and Terence Tao. Some connections between Falconer's distance set conjecture and sets of Furstenburg type. *New York J. Math.*, 7:149–187, 2001.

[5] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability. 1st paperback ed.* Cambridge: Cambridge University Press, 1st paperback ed. edition, 1999.

[6] Tuomas Orponen. On the dimension and smoothness of radial projections. *Anal. PDE*, 12(5):1273–1294, 2019.

[7] Tuomas Orponen. Combinatorial proofs of two theorems of Lutz and Stull. *Math. Proc. Cambridge Philos. Soc.*, 171(3):503–514, 2021.

[8] Tuomas Orponen and Pablo Shmerkin. On the Hausdorff dimension of Furstenberg sets and orthogonal projections in the plane. *Duke Math. J.*, 172(18):3559–3632, 2023.

[9] Tuomas Orponen, Pablo Shmerkin, and Hong Wang. Kaufman and Falconer estimates for radial projections and a continuum version of Beck's theorem. *Geom. Funct. Anal.*, 34(1):164–201, 2024.

[10] Malabika Pramanik, Tongou Yang, and Joshua Zahl. A Furstenberg-type problem for circles, and a Kaufman-type restricted projection theorem in $\mathbb{R}^3$. Preprint, arXiv:2207.02259, July 2022.

[11] Kevin Ren and Hong Wang. Furstenberg sets estimate in the plane. *arXiv e-prints*, page arXiv:2308.08819, August 2023.

[12] Hong Wang and Joshua Zahl. Sticky Kakeya sets and the sticky Kakeya conjecture. *arXiv e-prints*, page arXiv:2210.09581, October 2022.

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