Geometric maximal functions

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MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

Lecture 3 : Fourier Methods.

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Joint with :



David Beltran

(Valencia)



Jennifer Duncan

(UAM)



Shaoming Guo

(Chern Institute)



Andreas Seeger

(Madison).

Lecture 3 : Fourier Methods.

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Spherical maximal function re-revisited : $M_S f(x) := \sup_{1 \le r \le 2} |Af(x,r)|$

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$\hat{\sigma}(\xi) := \int_{S^{n-1}} e^{-2\pi i \xi \cdot y}\, d\sigma(y)$ satisfies $\boxed{|\hat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{n-1}{2}}}$.

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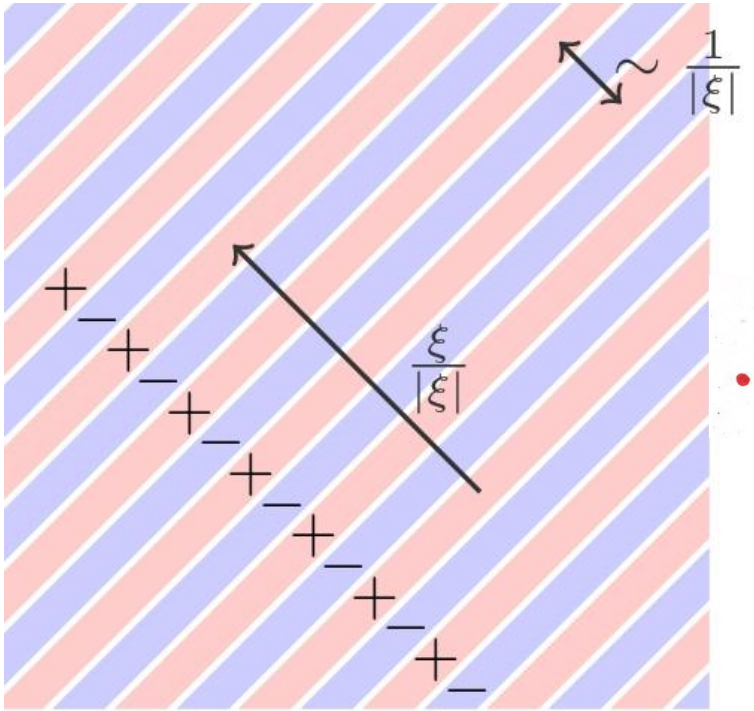
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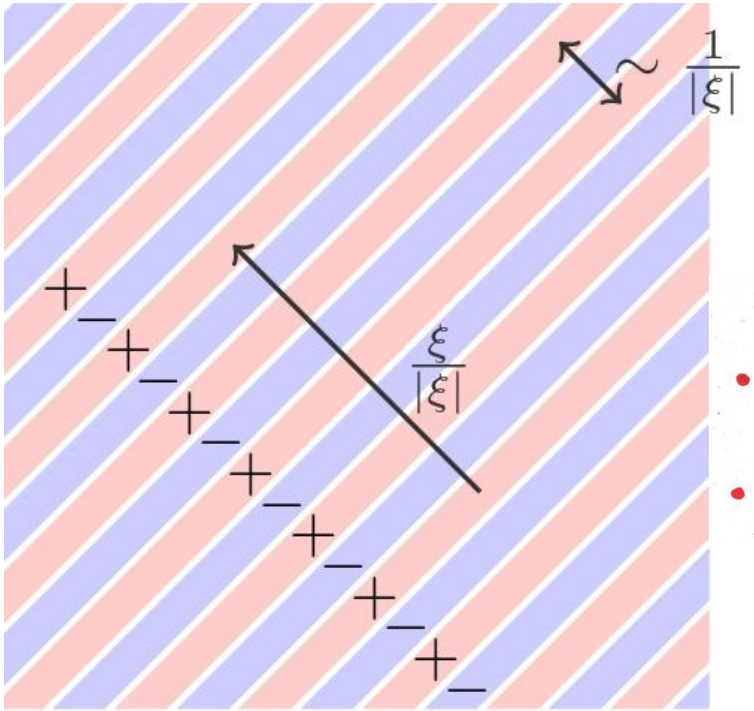




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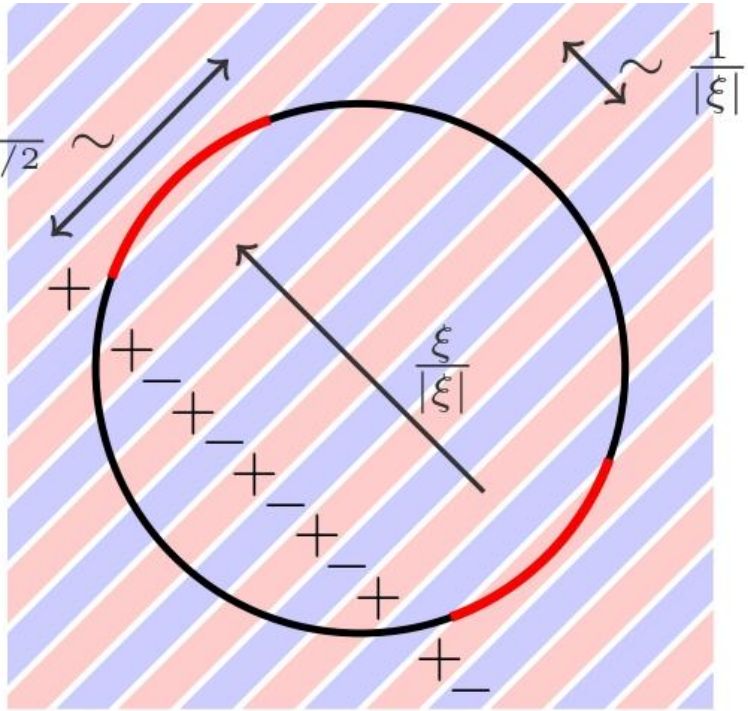


- Linear wave $x \longmapsto e^{2\pi i x \cdot \xi}$
- Frequency $= |\xi|$

$\Rightarrow$ Wave length $= 1/|\xi|$.

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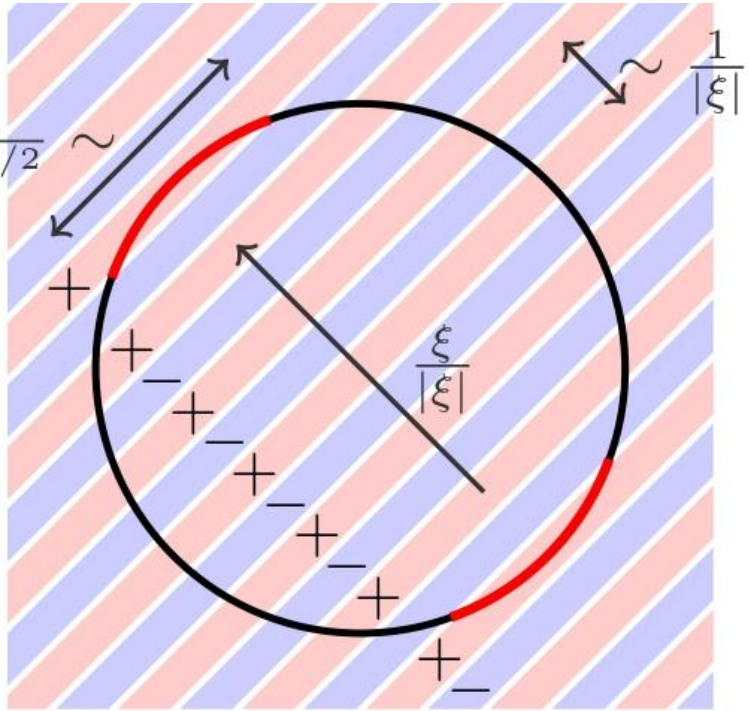
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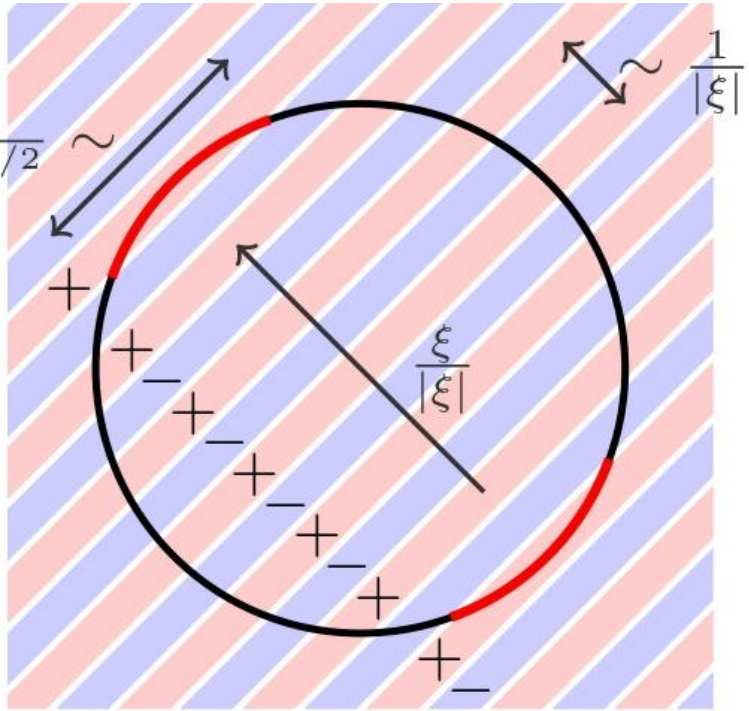


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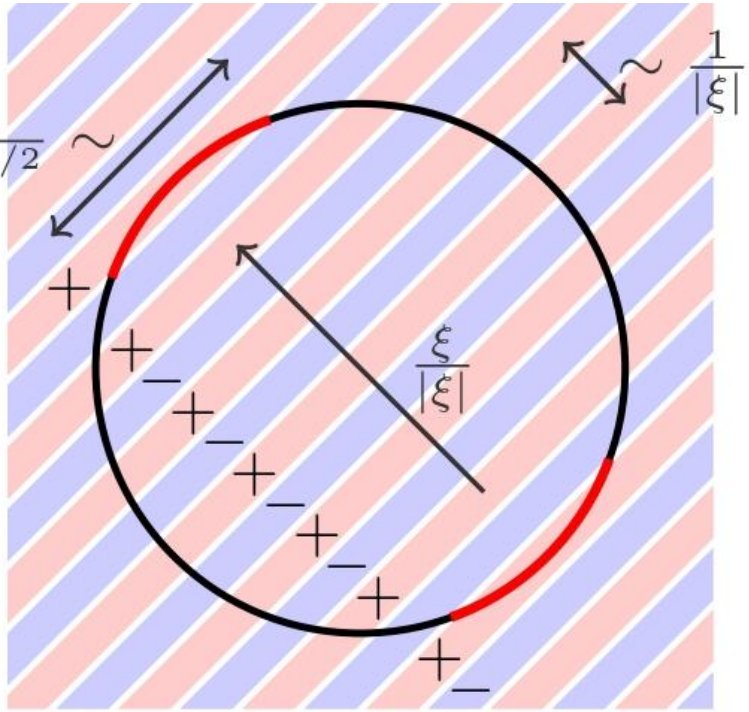


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Volume $\underbrace{(|\xi|^{-1/2})}_{\text{radius}}^{n-1} = |\xi|^{-\frac{n-1}{2}}$



Mollifying the singularity.

Singular operator M_S
(conv. w/ spherical measure)

$\longmapsto$

Infinite family of
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- For $\lambda \in 2^{\mathbb{N}}$, $M_S^\lambda f(x) := \sup_{1 \le r \le 2} |A^\lambda f(x,r)|$ where

$$A^\lambda f(x,r) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \beta(|\xi|/\lambda)\, \widehat{\sigma_r}(\xi)\, \hat{f}(\xi)\, d\xi.$$

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- Triangle inequality $\Rightarrow$

$$\|M_S f\|_{L^p(\mathbb{R}^n)} \le \sum_{\lambda \in 2^{\mathbb{N}}} \|M_S^\lambda f\|_{L^p(\mathbb{R}^n)} + \|M_S^{\le 1} f\|_{L^p(\mathbb{R}^n)}$$

Low frequency piece.

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Goal: For $\varepsilon(p) > 0$, $\|M_S^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_p \lambda^{-\varepsilon(p)} \|f\|_{L^p(\mathbb{R}^n)}$.

Uncertainty Principle: Frequency localisation at scale λ <~> Spatial uncertainty at scale $\delta = \lambda^{-1}$

- $M_s f$: Spheres $C(x,r)$ $\longmapsto$ $M_s^{\lambda} f$: δ-annuli $C^{\delta}(x,r)$

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  - Also carries high frequency oscillation!

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where • $\mathcal{C}$ is a family of spheres satisfying:-

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Vital to understand cancellation

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- Understand: $\langle \tilde{\chi}_{c_1^\delta}, \tilde{\chi}_{c_2^\delta} \rangle = \langle \mathcal{F} \tilde{\chi}_{c_1^\delta}, \mathcal{F} \tilde{\chi}_{c_2^\delta} \rangle$

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- Could bound: $|\langle \tilde{\chi}_{c_1^\delta}, \tilde{\chi}_{c_2^\delta} \rangle| \leq \langle |\tilde{\chi}_{c_1^\delta}|, |\tilde{\chi}_{c_2^\delta}| \rangle$
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Heuristic! – Fraccaroli diagram.



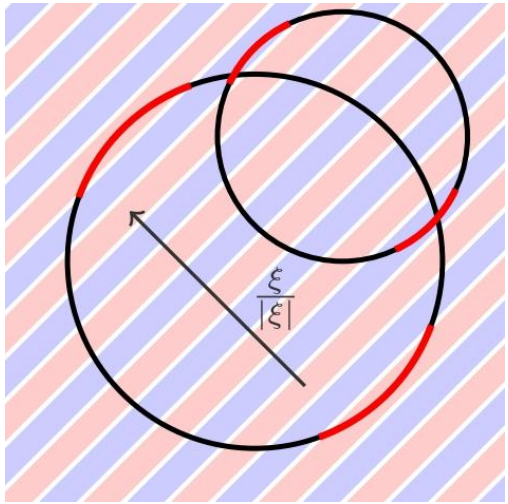
Fraccaroli Diagram.

$\langle \mathcal{F}\tilde{X}_{C_1^s}, \mathcal{F}\tilde{X}_{C_2^s} \rangle$

Fraccaroli Diagram. $\langle \mathcal{F}\tilde{X}_{C_1^s}, \mathcal{F}\tilde{X}_{C_2^s} \rangle = \int_{\hat{\mathbb{R}}^n} \hat{\sigma}_{C_1}(\xi)\, \overline{\hat{\sigma}_{C_2}(\xi)}\, |\beta(|\xi|/\lambda)|^2\, d\xi$

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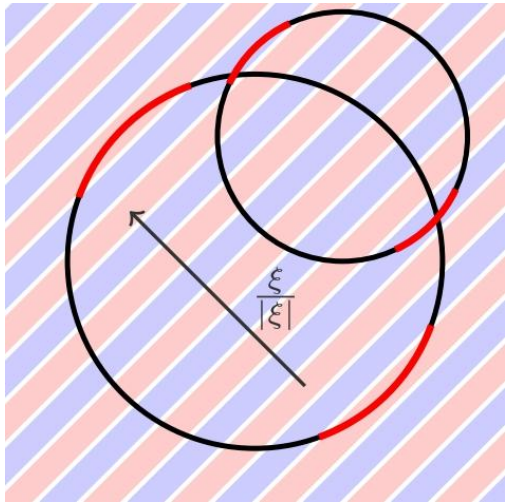
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Transversal Case

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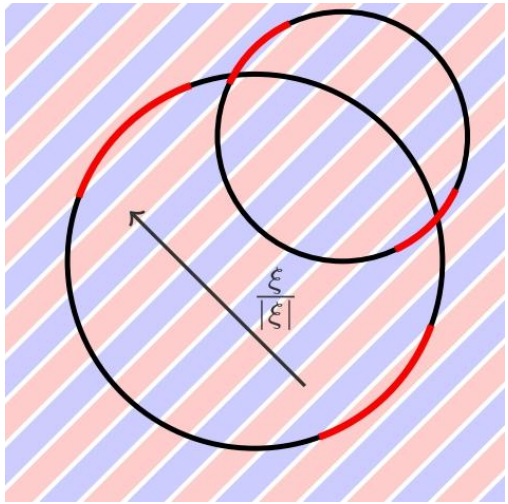


Transversal Case

- Contributions from arcs tangent to wave

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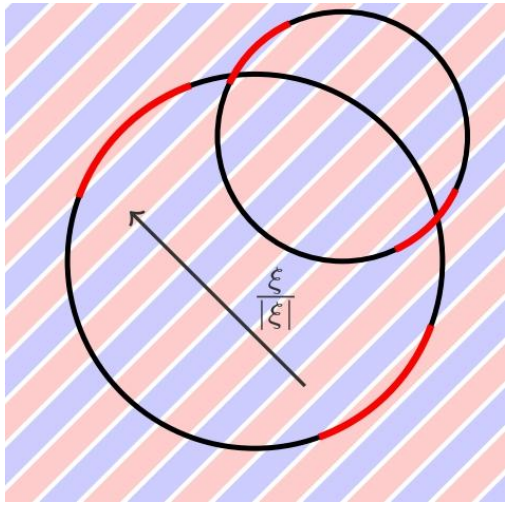


Transversal Case

- Contributions from arcs tangent to wave
- Tangent arcs disjoint

Fraccaroli Diagram.

$$\langle \mathcal{F}\tilde{\chi}_{C_1^\delta}, \mathcal{F}\tilde{\chi}_{C_2^{\delta'}} \rangle = \int_{\hat{\mathbb{R}}^n} \hat{\sigma}_{C_1}(\xi)\, \overline{\hat{\sigma}_{C_2}(\xi)}\, |\beta(|\xi|/\lambda)|^2\, d\xi$$

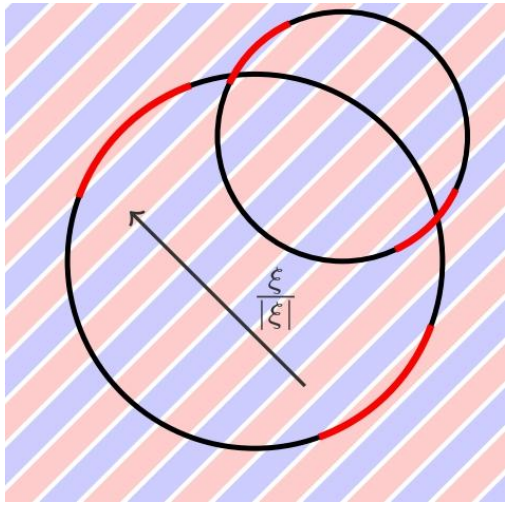


Transversal Case

- Contributions from arcs tangent to wave
- Tangent arcs disjoint
- $\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^{\delta'}} \rangle = 0$

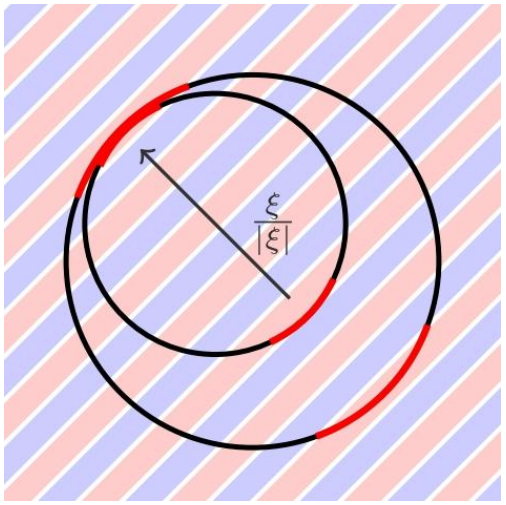
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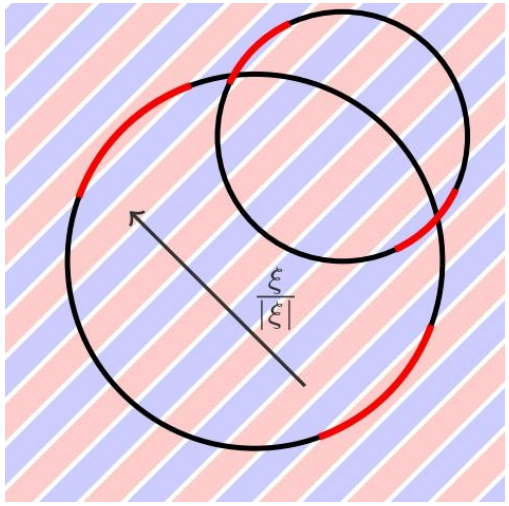
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Tangential Case

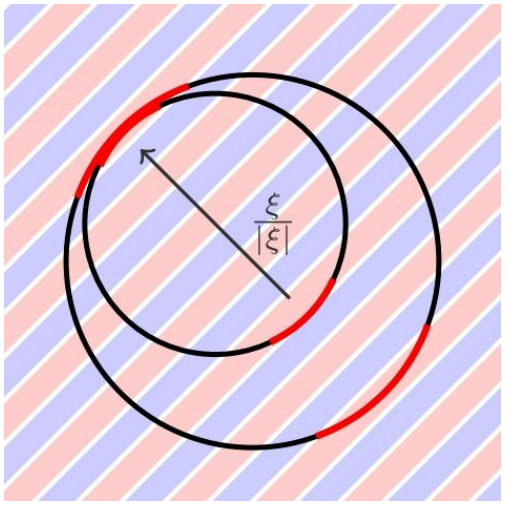
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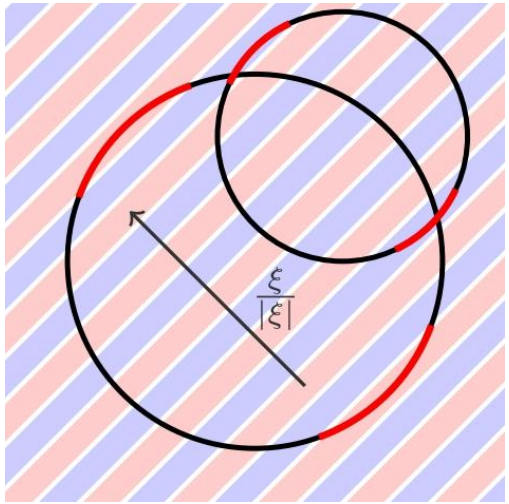


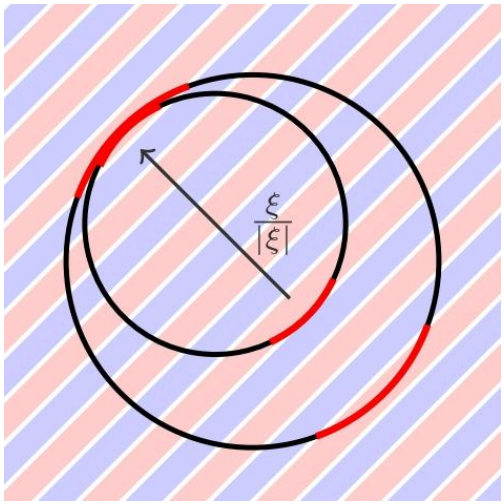
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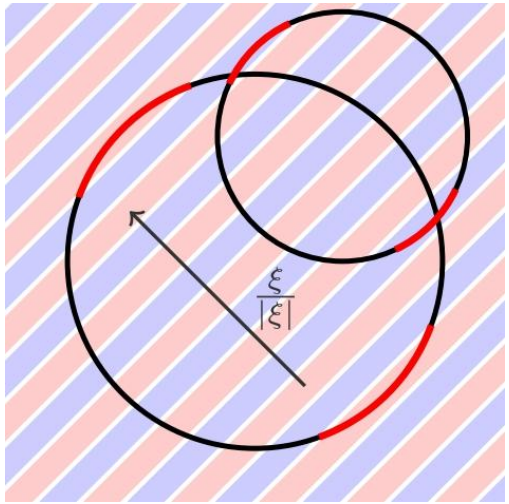
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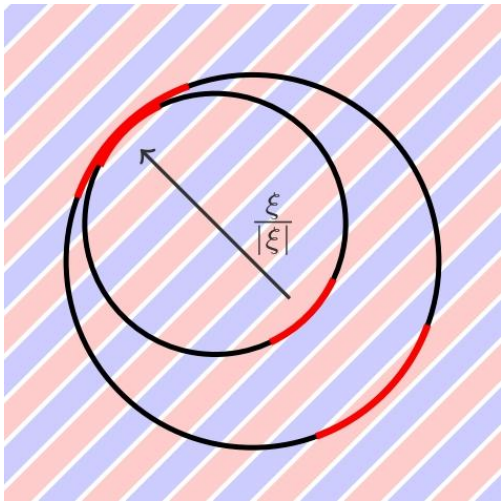
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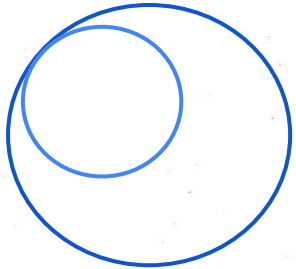
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Tangential Case

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- Nontrivial contribution.

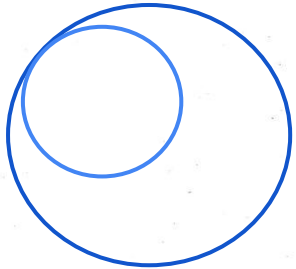
Key observation :

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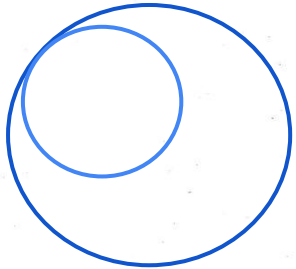
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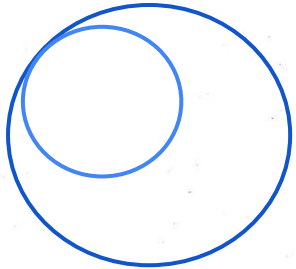
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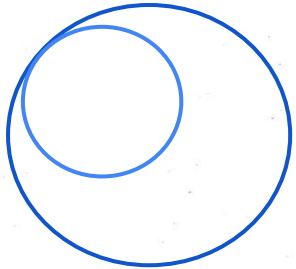
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Thus, $\Big\| \sum_{C \in \mathcal{C}} \tilde{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^n)} \lesssim |\#\mathcal{C}|^{1/2}$, as required. □

Helical Maximal Function.

Define: $M_H f(x) := \sup_{r>0} \left| \fint_{H(x,r)} f \right|, \quad f \in C(\mathbb{R}^3),$

where $H(x,r) := \{ x - r(\cos\theta, \sin\theta, \theta) : |\theta| \leq 1 \}$

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Theorem (Beltran – Guo – H – Seeger, Ko – Lee – Oh) : For $p > 3$

$$\| M_H f \|_{L^p(\mathbb{R}^3)} \lesssim_p \| f \|_{L^p(\mathbb{R}^3)}.$$

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Open Problem!

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- Proof heavily Fourier analytic!

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Too weak to give a sharp result!

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- For "typical" directions, much stronger decay holds.

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Slow decay only occurs along Σ_2.

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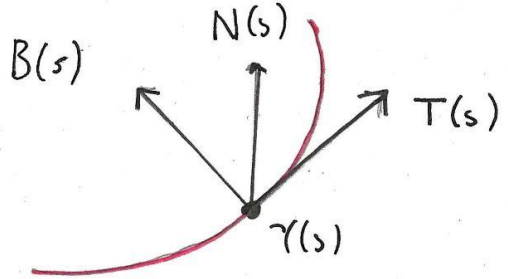
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Geometry of the slow decay cone.

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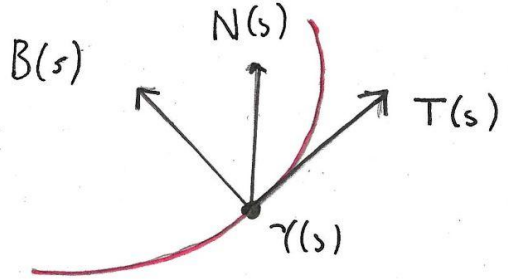


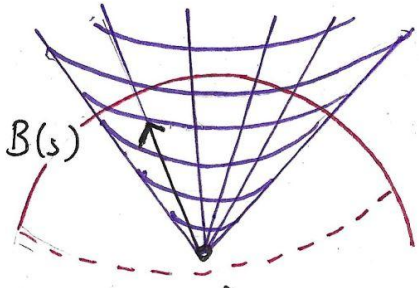
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B = binormal vector.





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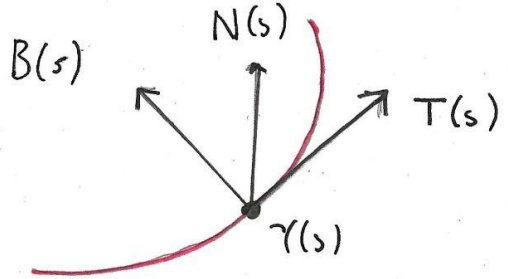
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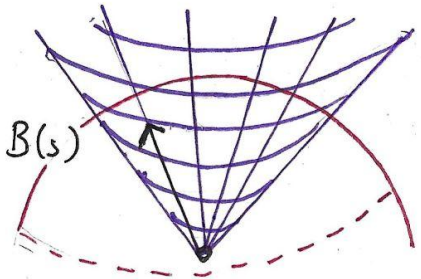
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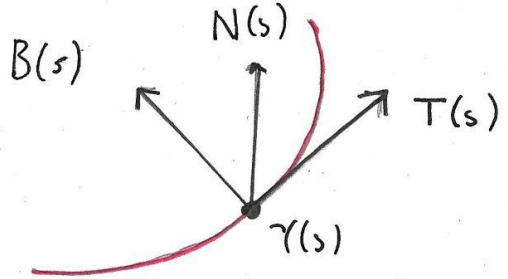
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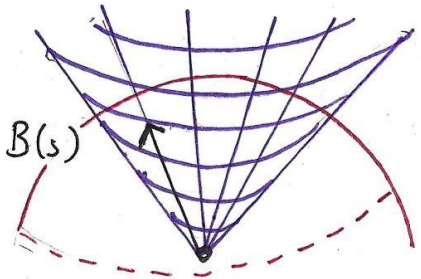
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• Decompose A^λ in frequency space $\hat{\mathbb{R}}^3$ according to distance to Σ_2. c.f. Pramanik–Seeger.

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    - Restriction theory tools :-

      Decoupling, square functions, Nikodym max.

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- For $n \ge 5$ new features (Counterexample to method).

Open problem: • Geometric proof of Helical maximal theorem (in $\mathbb{R}^3$) ?

Thanks!.