Geometric maximal functions

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MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

LECTURE 2: MULTI PARAMETER MAXIMAL FUNCTIONS

Lecture 2: Multi parameter Maximal Functions

Joint with:



Josh Flahl (UBC)

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Last Time: • Spherical maximal function :- $n \geq 3$,

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- GEOMETRIC / COMBINATORIAL PROOF OF WEAK FORM OF

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- Geometric / combinatorial proof of weak form of

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Today:

- Similar ideas to study multiparameter variant

Strong Spherical maximal function (H – Jahl '25).

Def^{n}: For $n \geq 2$, given $f \in C(\mathbb{R}^n)$ define

$$M_E f(x) := \sup_{r \in (0,\infty)^n} \left| \int_{S^{n-1}} f(x - \delta_r y)\, d\sigma(y) \right|$$

where $\delta_r y = (r_1 y_1, \ldots, r_n y_n)$ - Strong spherical maximal function.

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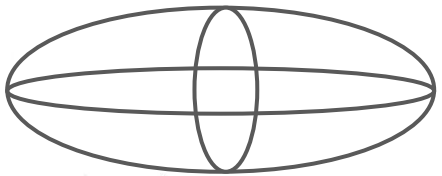
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Alternatively,

$$M_E f(x) := \sup_{r \in (0,\infty)^n} \left| \fint_{E(x,r)} f \right|$$ takes maximal averages

over axis-parallel ellipsoids :-

$$E(x,r) := \left\{ y \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{(y_j - x_j)^2}{r_j^2} = 1 \right\}$$



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Problem: Determine $1 \leq p \leq \infty$ such that

$$\| M_E f \|_{L^p(\mathbb{R}^n)} \lesssim_p \| f \|_{L^p(\mathbb{R}^n)}$$ holds.

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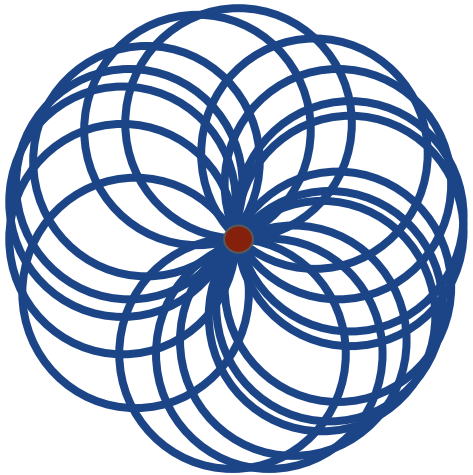
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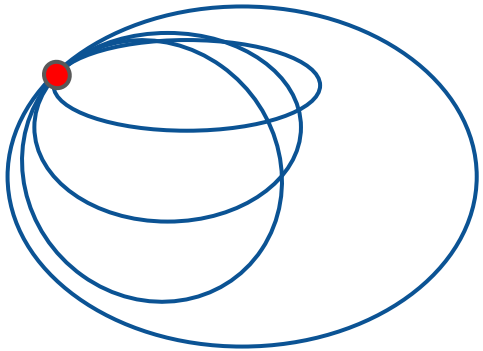
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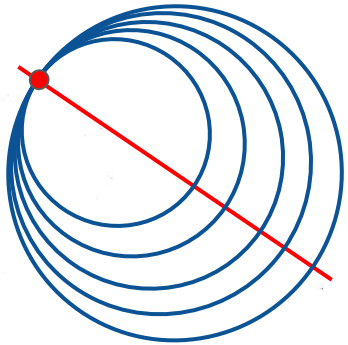
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Lemma: There exists $F \subseteq \mathbb{R}^n$ with $|F| > 0$ and an assignment $x \longmapsto r(x) \in [1,2]^n$ for each $x \in F$ such that

i) $0 \in E(x, r(x))$;

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Remark: V chosen to avoid coordinate vectors.
(preferred directions for M_E).

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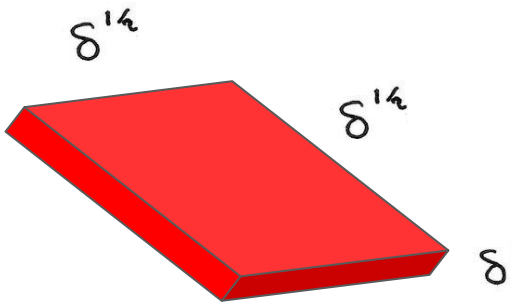
- **Dim of sol^n space:** # Variables $-$ # Constraints $= 2n - n = n$.

- Knapp condition (ctd). - $f := \chi_{R(\delta)}$

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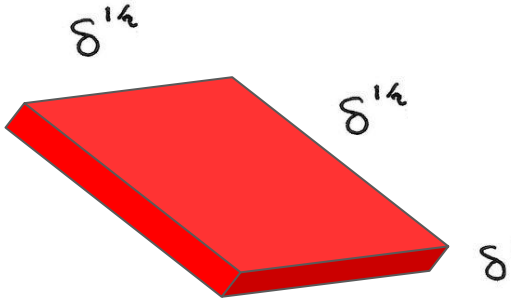
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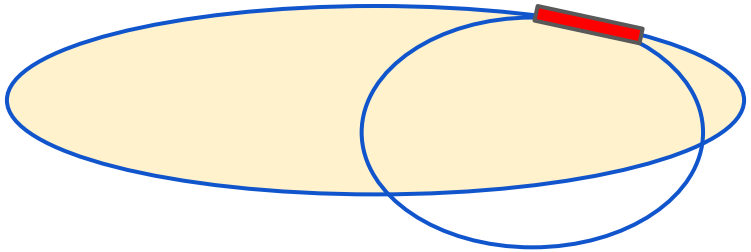
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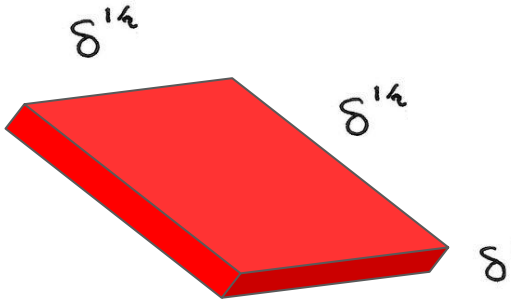


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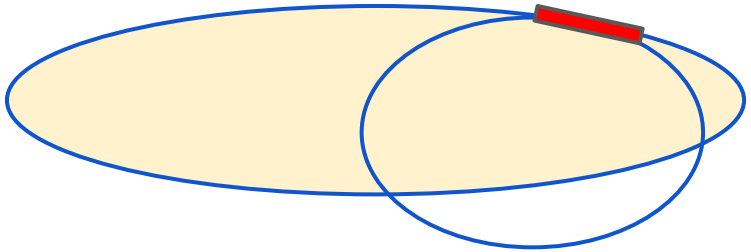
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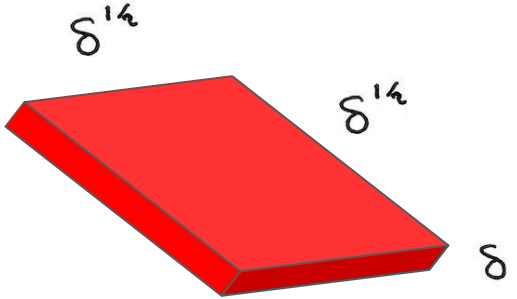
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Conjecture (Strong Spherical Maximal Conjecture): For $n \geq 2$ and $p > \frac{n+1}{n-1}$,

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- **Progress:**
  - $n = 2$ known: Pramanik-Yang-Zahl + { Chen-Guo-Yang, Lee-Lee-Oh }
    - Pramanik-Yang-Zahl: Geometric
    - Chen-Guo-Yang, Lee-Lee-Oh: Fourier analytic.

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- **Goal:** $\| M_E^\delta f \|_{L^p(\mathbb{R}^n)} \lesssim_p \| f \|_{L^p(\mathbb{R}^n)}$ uniformly in $\delta > 0$.

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Remark: Combining with Lee – Lee – Oh gives $n = 3$ case of SSM Conjecture.

- Multiplicity formulation: $\left\| \sum_{E \in \mathcal{E}} \chi_{E^\delta} \right\|_{L^{p'}(\mathbb{R}^3)} \lesssim_\varepsilon \delta^{1 - 3/p - \varepsilon} [\#\mathcal{E}]^{\frac{1}{p'}}$

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where $\mathcal{E}$ is any family of axis-parallel ellipsoids satisfying:

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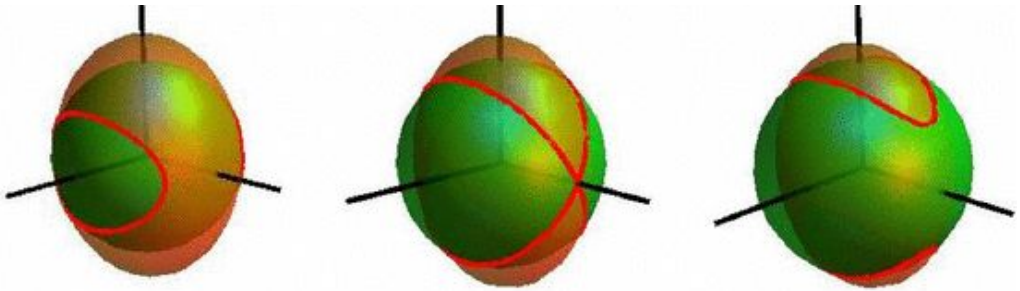
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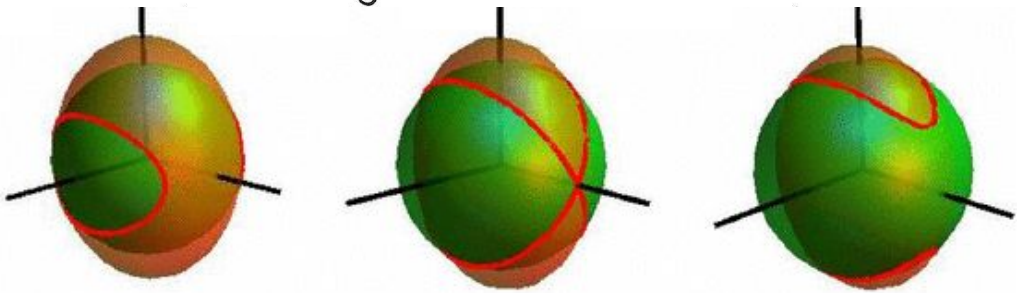
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  - N.B. These curves can be awkward to explicitly understand in general.



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Favourable!

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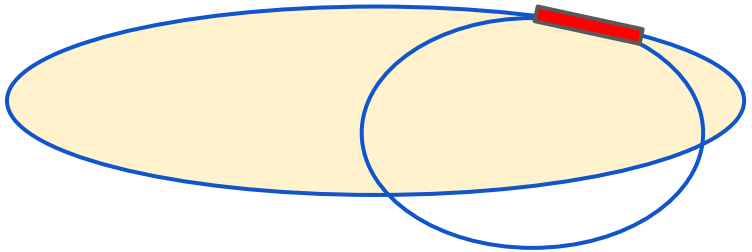
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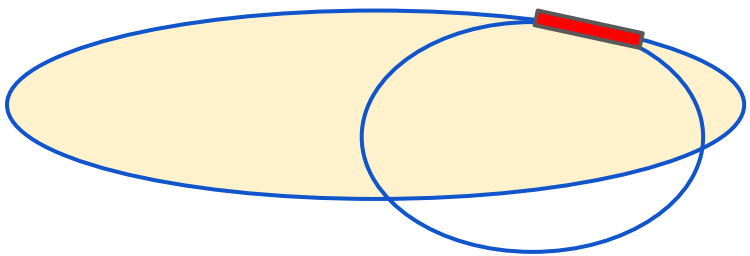


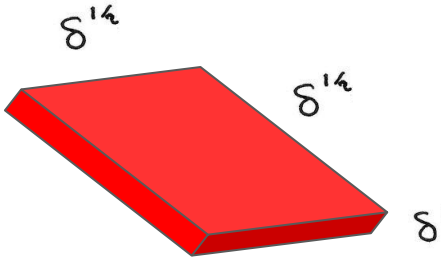
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– $E_1^\delta \cap E_2^\delta \approx$ "thickened cap" of dimension $\delta^{1/2} \times \delta^{1/2} \times \delta$



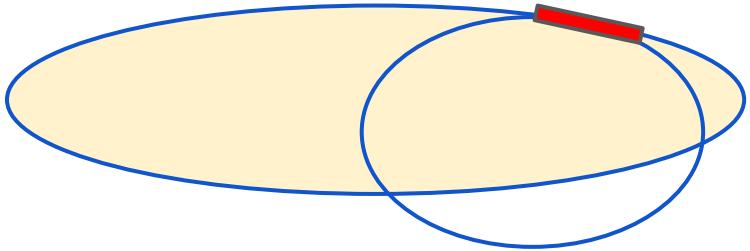


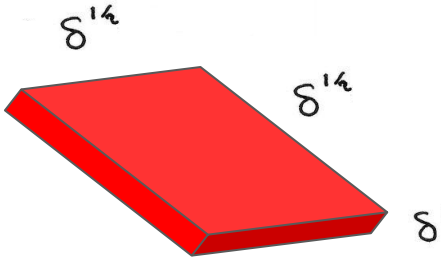
$$\Big\| \sum_{E \in \mathcal{E}} \chi_{E^\delta} \Big\|_{L^2(\mathbb{R}^3)} \lesssim_\varepsilon \delta^{-1/2-\varepsilon} [\#\mathcal{E}]^{1/2}.$$

- Expanding: $\Big\| \sum_{E \in \mathcal{E}} \chi_{E^\delta} \Big\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \sum_{E_1, E_2 \in \mathcal{E}} \chi_{E_1^\delta} \chi_{E_2^\delta} = \sum_{E_1, E_2 \in \mathcal{E}} |E_1^\delta \cap E_2^\delta|$
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Case 2: First order tangent pair (E_1, E_2).

- $E_1^\delta \cap E_2^\delta \approx$ "thickened cap" of dimension $\delta^{1/2} \times \delta^{1/2} \times \delta$
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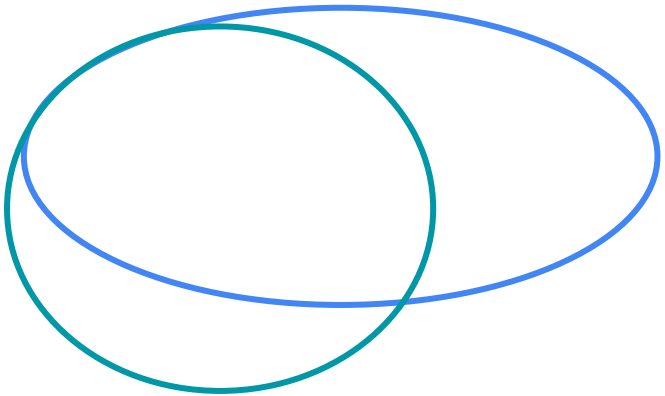
Favourable !

Case 3:  Osculating pair (E_1, E_2)

i.e. (E_1, E_2) tangent and share a principal curvature

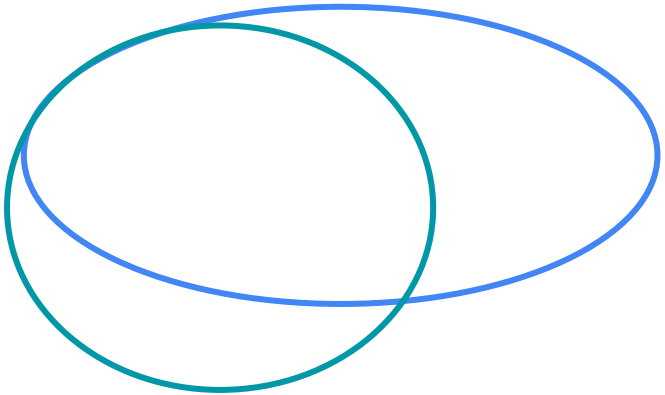
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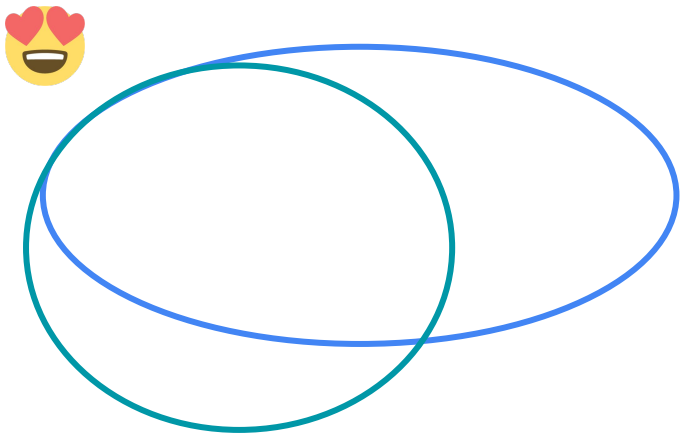
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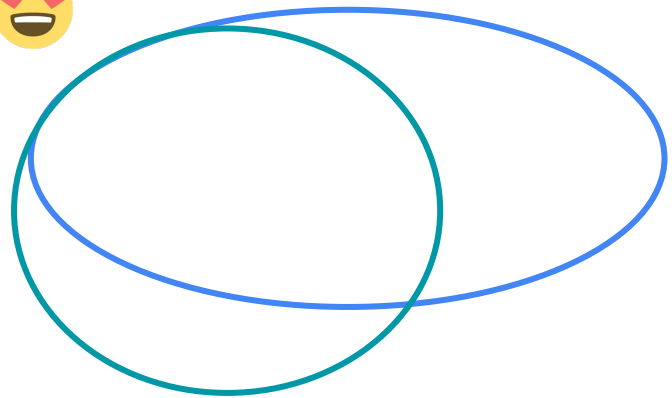




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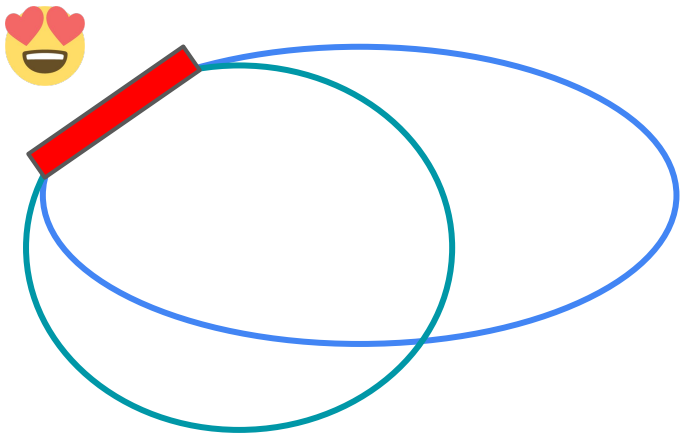




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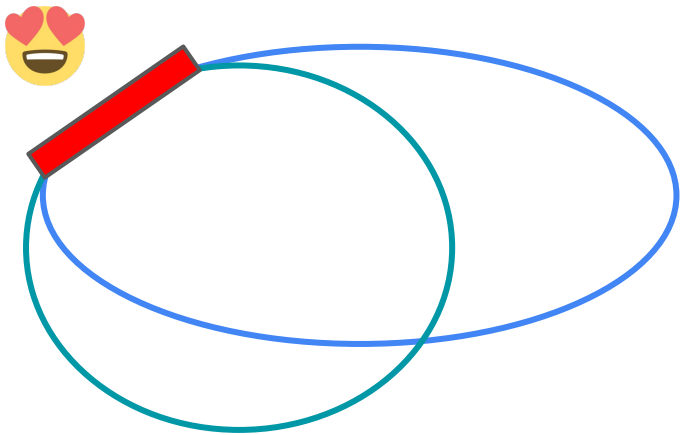
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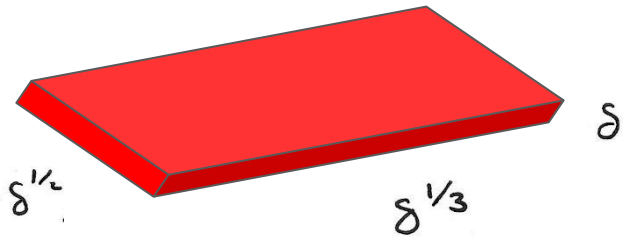


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Osculating pairs are our main enemy !!

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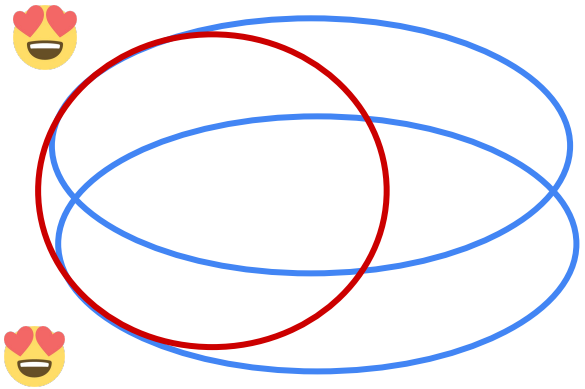
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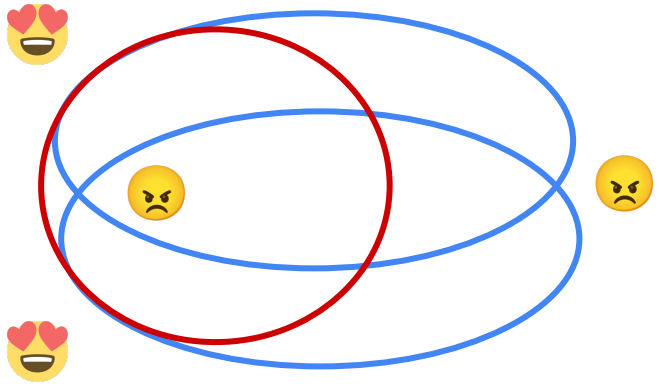
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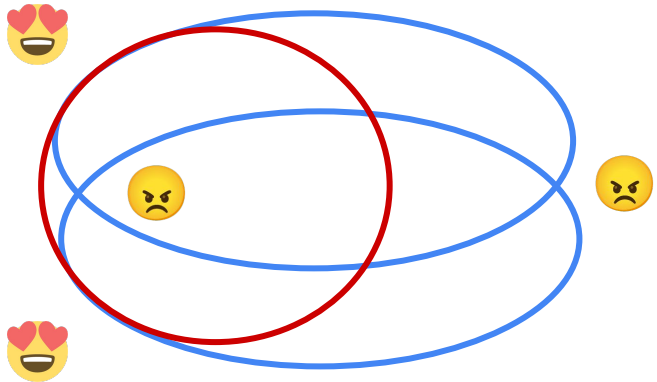
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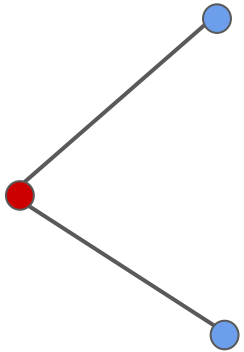
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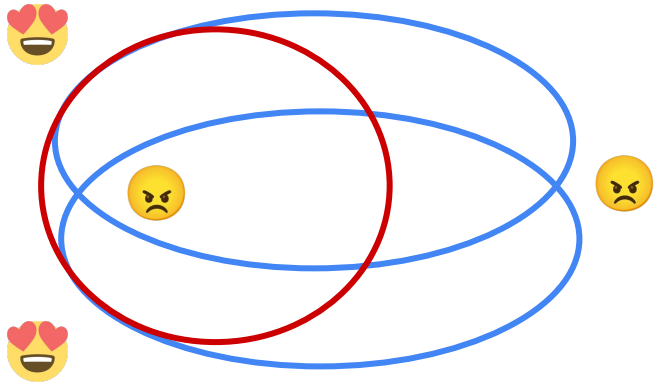
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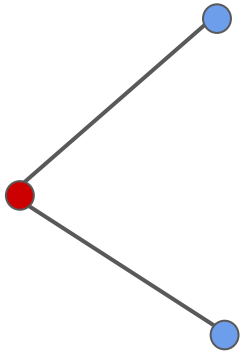




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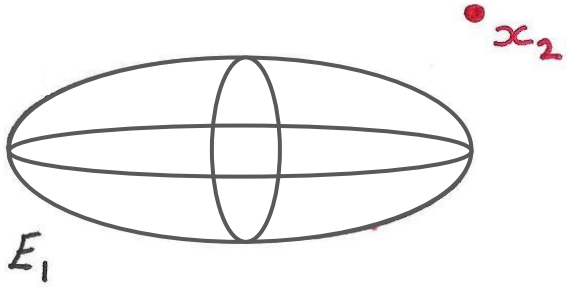
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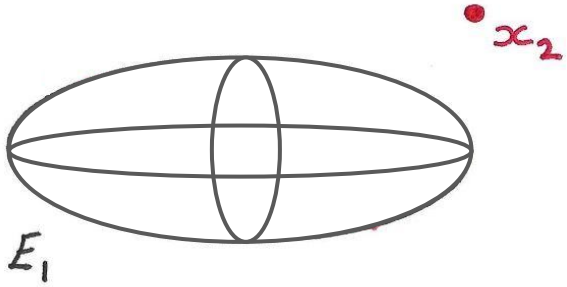
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- Not easy to make this precise / quantitative!

Key observation: Fix ellipsoid E_1; centre $x_2 \in \mathbb{R}^3$ with $|x(E_1) - x_2| \sim 1$.



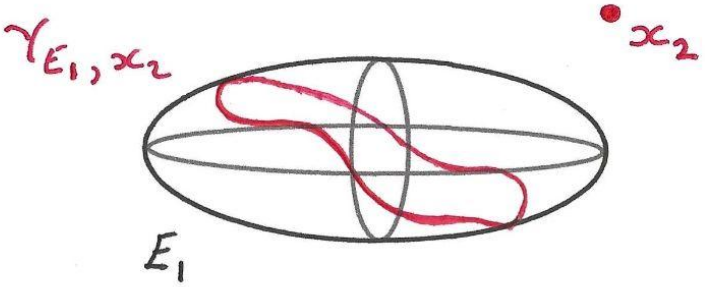
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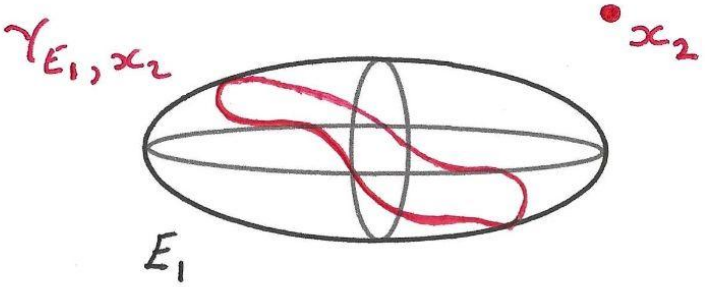
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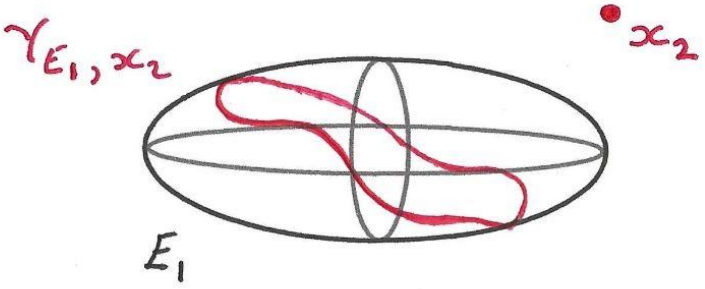
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Heuristic (Parameter counting).

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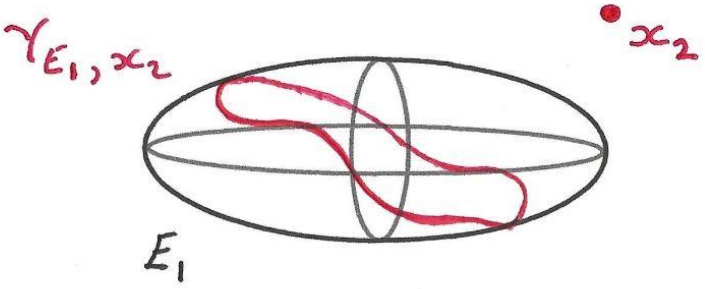


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- **Variables**: — Choice of radii $r \in [1,2]^3$ for E_2 & point $p \in E_1$ of osculation.
  — # Variables $= 3 + 2 = 5$.

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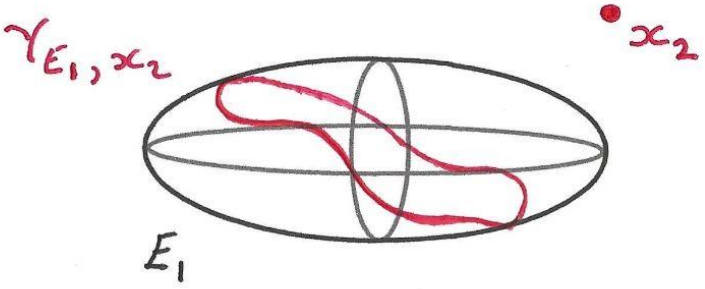
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  — # Constraints $= 1 + 2 + 1 = 4$

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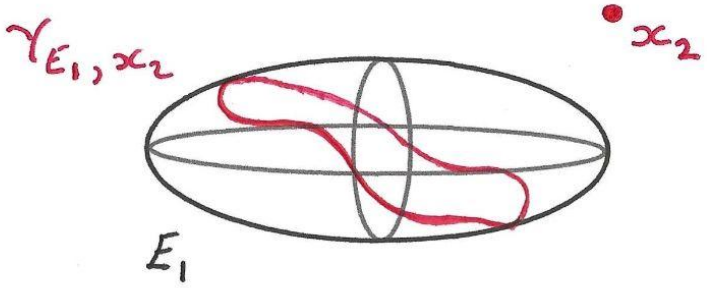
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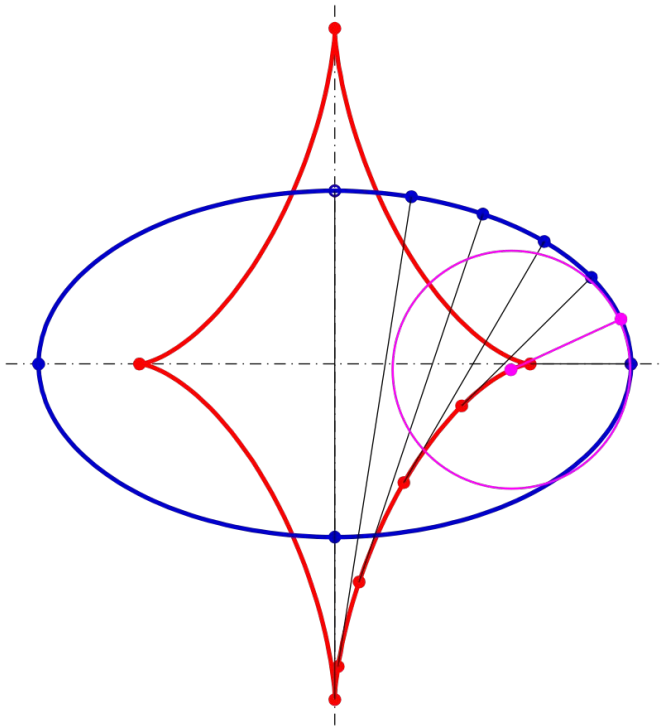
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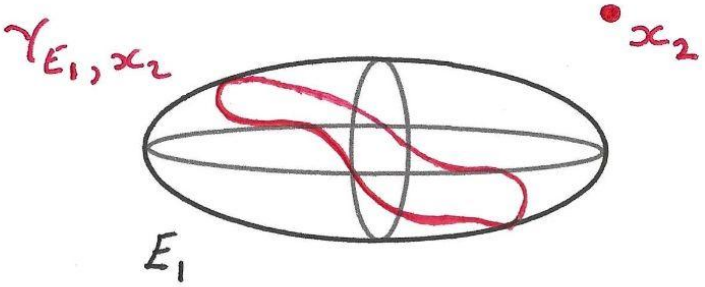
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- **Dim of sol^n space** : # Variables − # Constraints = $5 - 4 = 1$.

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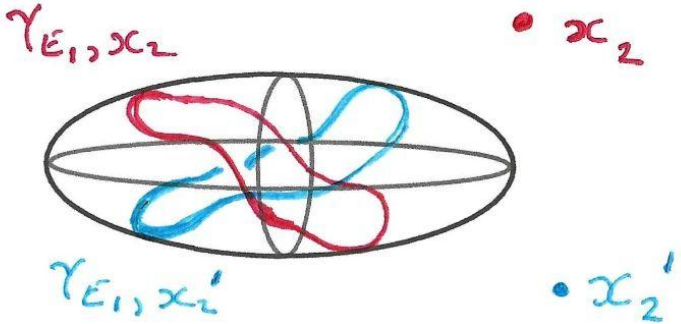




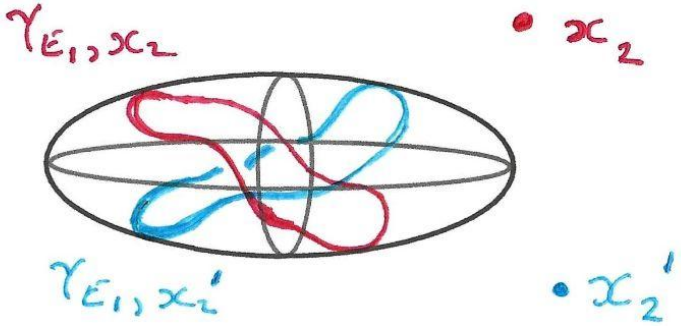
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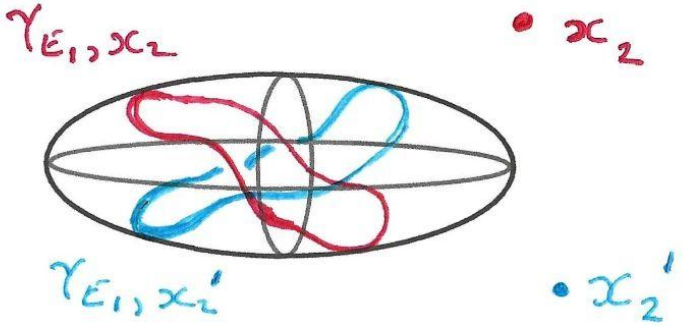
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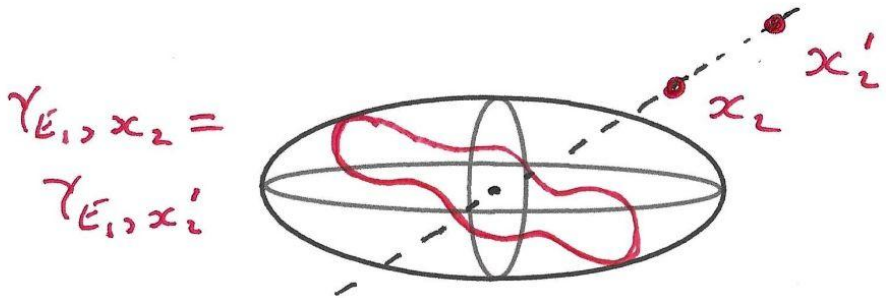


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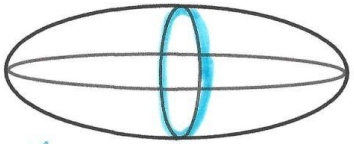




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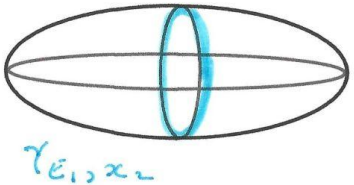
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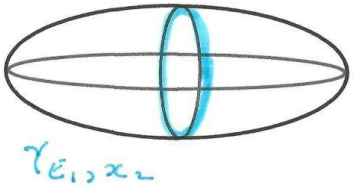
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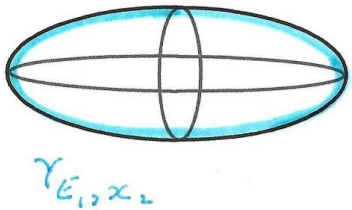


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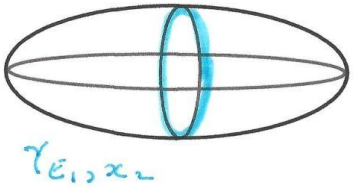


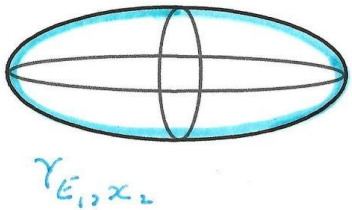


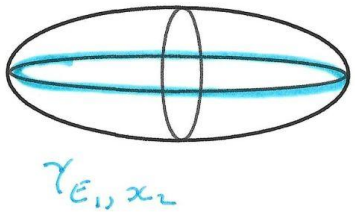
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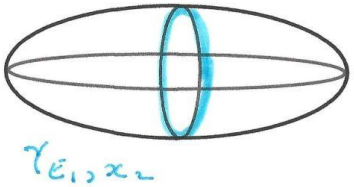


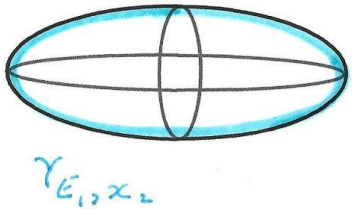


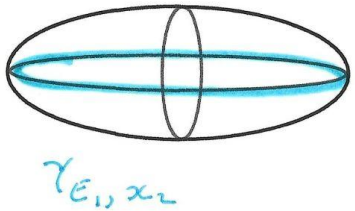
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Suggests slicing domain into lines!

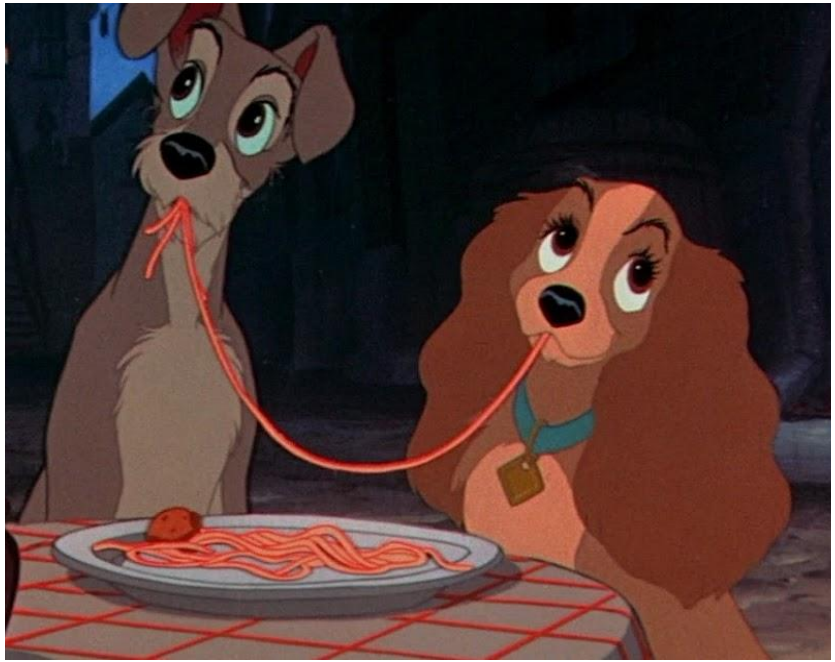
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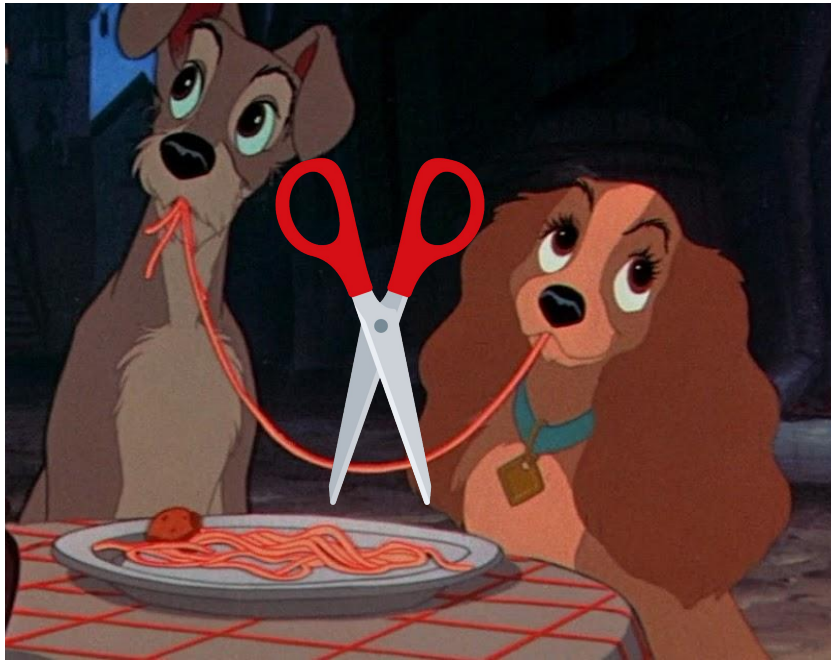
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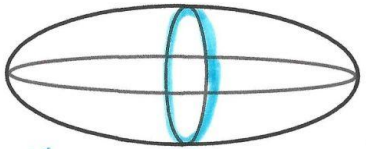
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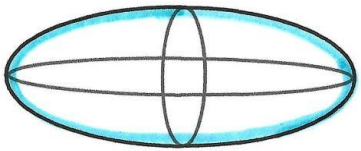


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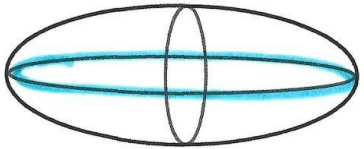
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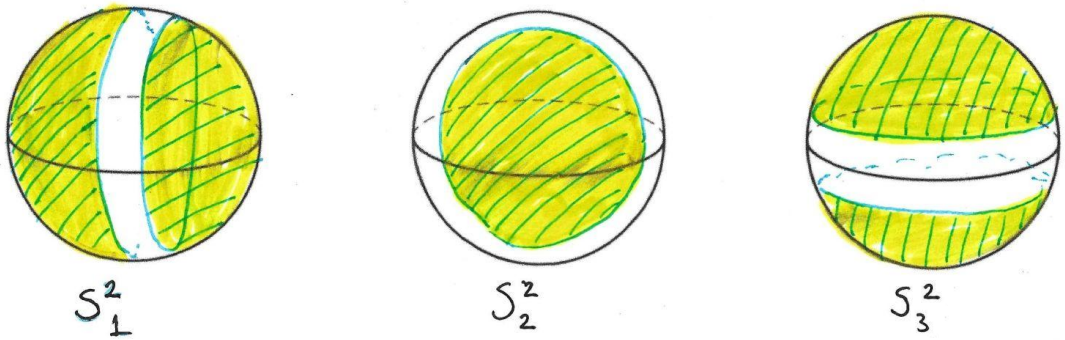


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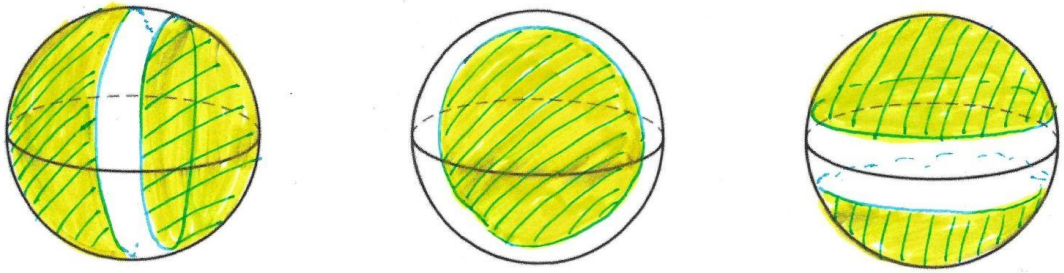
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1. • Decompose $S^2 = S^2_1 \cup S^2_2 \cup S^2_3$; $S^2_k := \{\omega \in S^2 : |\omega_k| \gtrsim 1\}$



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(Analogous to cap decomposition from Lecture 1).

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(Analogous to cap decomposition from Lecture 1).

• Triangle inequality => suffices to show

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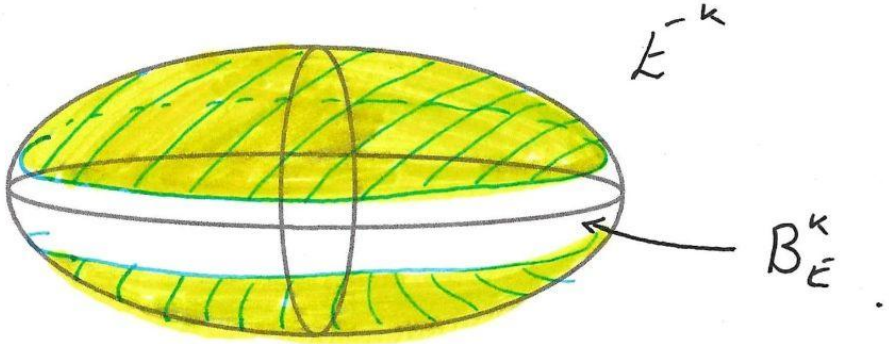
$$\| M^{k,s}_{E} f \|_{L^p([-1/2,1/2]^3)} \lesssim_\varepsilon \delta^{-\varepsilon} \| f \|_{L^p(\mathbb{R}^3)} .$$

$$\| M_E^{k,\delta} f \|_{L^p([-1/2,1/2]^3)} \lesssim_\varepsilon \delta^{-\varepsilon} \| f \|_{L^p(\mathbb{R}^3)} .$$

- What is $M_E^{k,\delta}$?

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- What is $M_E^{k,\delta}$? Takes maximal averages over $E^{-k,\delta}$ where
  - $E^{-k,\delta} :=$ δ-neighbourhood of E^k
  - $E^k := E \setminus B_E^k$ for $B_E^k =$ neighbourhood of axial plane



2. • Slicing (i.e. Fubini) => suffices to show

$$\| M_E^{k,\delta} f \|_{L^p((\ell_k + h) \cap [-1/2, 1/2]^3)} \lesssim_\varepsilon \delta^{-\varepsilon} \| f \|_{L^p(\mathbb{R}^3)}$$

for all $h \in \ell_k^\perp$ (here $\ell_1 = \left\langle \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\rangle$ etc ...).

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where $\mathcal{E}$ is any family of axis-parallel ellipsoids satisfying:

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- We have removed $B_{E_1}^k$ to form $\tilde{E}_1^k := E_1 \setminus B_{E_1}^k$

$\Rightarrow$ Osculating pairs do not contribute!

Remarks:

- Technical implementation involved (e.g. making precise/ quantitative parameter counting heuristics).
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  - Hard to describe $E_1 \cap E_2$ explicitly
  - Use implicit descriptions + real algebraic geometry.
- Proof works for all $n \geq 3$, giving $p \geq 2$ range.
  - Cannot go below $p = 2$ using line slicing.
  - For $n = 4$, expect $p > 5/2$ sharp (Open problem!)
  - Without line slicing, may need to treat osculating pairs directly.
  - Situation becomes more involved for larger n.

Next time : Fourier Analytic methods

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Thanks!