Geometric maximal functions

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Goal: Analogues of Hardy-Littlewood maximal inequality in these settings:

$$\|M_{HL} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq \infty.$$

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- (Hopefully) some tasty titbits to keep experts happy.

Lecture 1 : Spherical Maximal Theorem Revisited.

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Joint with:



Ajša Jančar (Edinburgh)

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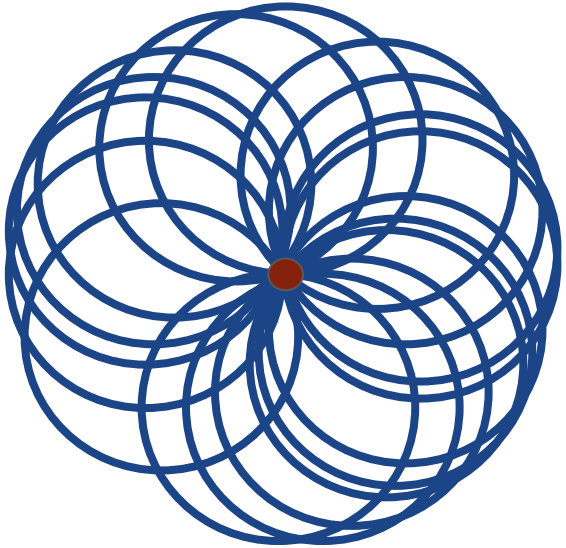
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Remarks:

- **Stein's proof**: Fourier analysis!
- **Today**: Geometric/combinatorial proof of a weak form of Theorem 1 (H – Jancar).

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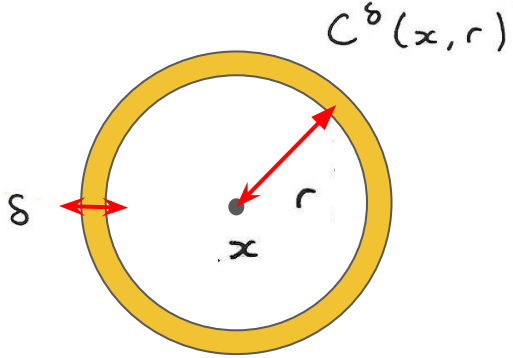
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Goal : $\|M_S^\delta f\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$ uniformly in $\delta > 0$.

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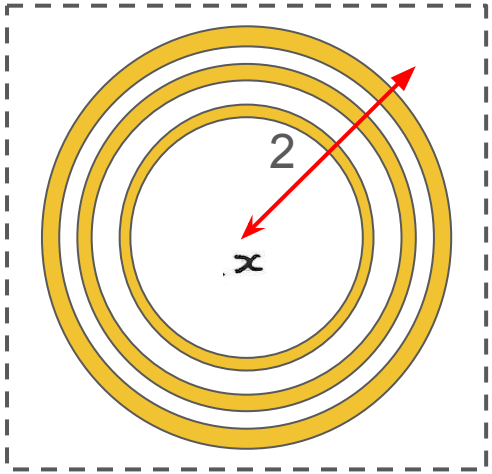
- Does not recover full strength of Stein's theorem.
- Proof (H-Jancar) showcases basic geometric/combinatorial techniques useful for solving other problems (see Lecture 2!).

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Goal: Bound linear operators $f \mapsto \fint_{C^\delta(x,r(x))} f$ uniformly over all $r : [-1/2, 1/2]^3 \to [1, 2]$.

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- $M^{\delta}_{s,\mathrm{loc}} f$ morally "locally constant" at scale δ :-

  i.e. $M^{\delta}_{s,\mathrm{loc}} f(x_1) \sim M^{\delta}_{s,\mathrm{loc}} f(x_2)$ for $|x_1 - x_2| \lesssim \delta$.

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- Thus, $\| M^{\delta}_{S,\mathrm{loc}} f \|_{L^p([-1/2,1/2]^3)} \lesssim \delta^{3/p-1} \Big(\sum_{C \in \mathcal{C}} \Big| \int_{C^\delta} f \Big|^p \Big)^{1/p}$

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Tension : input "continuous" output "discrete".

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5. Pigeonholing / weak-type endpoint :- Take $(a_c)_{c \in \mathcal{C}}$ binary

Multiplicity Formulation: $\left\| \sum_{C \in \mathcal{C}} \chi_{C^\delta} \right\|_{L^{p'}(\mathbb{R}^3)} \lesssim_\varepsilon \delta^{1-3/p-\varepsilon} [\#\mathcal{C}]^{1/p'}$

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where $\mathcal{C}$ is any family of spheres satisfying:

- Centre $x(C) \in [-1/2, 1/2]^3$;
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- $\left\| \sum_{C \in \mathcal{C}} \chi_{C^\delta} \right\|_{L^p(\mathbb{R}^3)}$ measures distribution of multiplicity function i.e. how frequently C^δ can overlap, counted w/ multiplicity.

Critical exponent: $p_{crit} = 3/2$, $p'_{crit} = 3$

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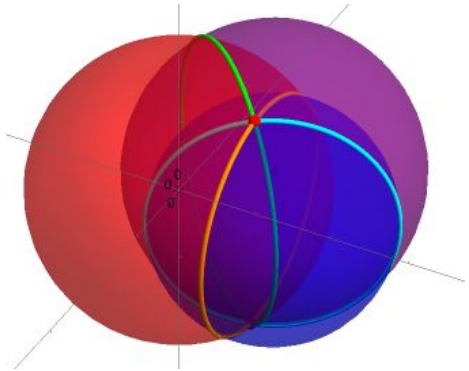
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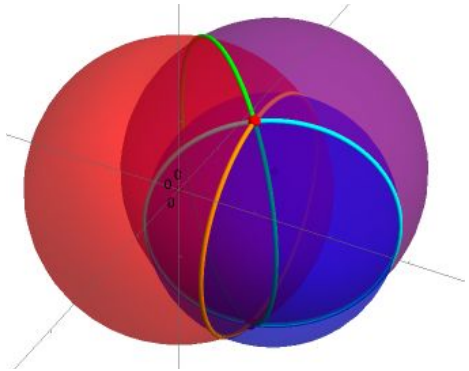
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$$\sum_{\substack{C_1, C_2, C_3 \in \mathcal{C} \\ \text{transversal}}} |C_1^\delta \cap C_2^\delta \cap C_3^\delta| \lesssim \delta^3 [\#\mathcal{C}]^3 \lesssim \delta^3 (\delta^{-3})^2 \#\mathcal{C} = \delta^{-3} \#\mathcal{C}$$

Favourable !!

Critical exponent: $p_{crit} = 3/2$, $p'_{crit} = 3$

$$\boxed{\left\| \sum_{C \in \mathcal{C}^\delta} \chi_{C^\delta} \right\|_{L^3(\mathbb{R}^3)} \lesssim_\varepsilon \delta^{-1-\varepsilon} [\#\mathcal{C}]^{1/3}}.$$

Expanding: $$\left\| \sum_{C \in \mathcal{C}^\delta} \chi_{C^\delta} \right\|_3^3 = \int_{\mathbb{R}^3} \sum_{C_1, C_2, C_3 \in \mathcal{C}} \chi_{C_1^\delta} \chi_{C_2^\delta} \chi_{C_3^\delta} = \sum_{C_1, C_2, C_3 \in \mathcal{C}} |C_1^\delta \cap C_2^\delta \cap C_3^\delta|$$

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Case 2: Colinear triple (C_1, C_2, C_3)

i.e. $x(C_1)$, $x(C_2)$, $x(C_3)$ lie along a line.

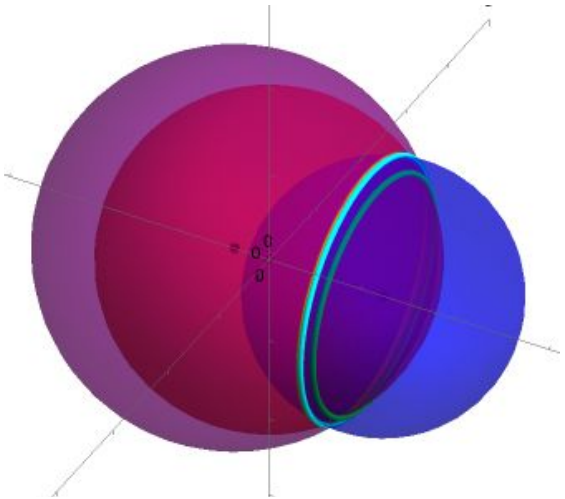
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Case 2: Colinear triple (c_1, c_2, c_3)

i.e. $x(c_1)$, $x(c_2)$, $x(c_3)$ lie along a line.



Critical exponent: $p_{crit} = 3/2$, $p'_{crit} = 3$

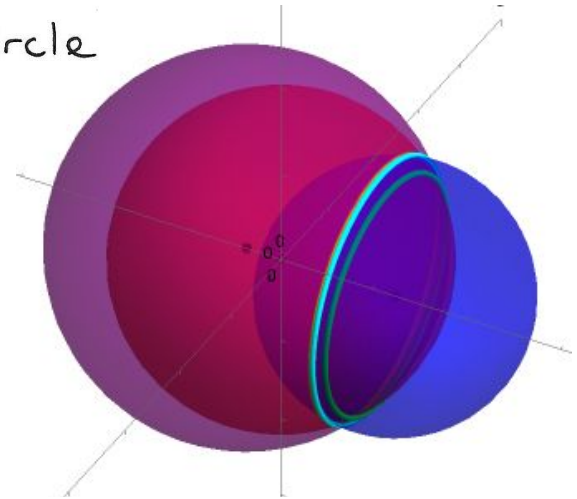
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Expanding: $$\Big\| \sum_{C \in \mathcal{C}^\delta} \chi_{C^\delta} \Big\|_3^3 = \int_{\mathbb{R}^3} \sum_{C_1, C_2, C_3 \in \mathcal{C}} \chi_{C_1^\delta} \chi_{C_2^\delta} \chi_{C_3^\delta} = \sum_{C_1, C_2, C_3 \in \mathcal{C}} |C_1^\delta \cap C_2^\delta \cap C_3^\delta|$$

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i.e. $x(C_1)$, $x(C_2)$, $x(C_3)$ lie along a line.

— $C_1^\delta \cap C_2^\delta \cap C_3^\delta \approx$ δ-neighbourhood of circle



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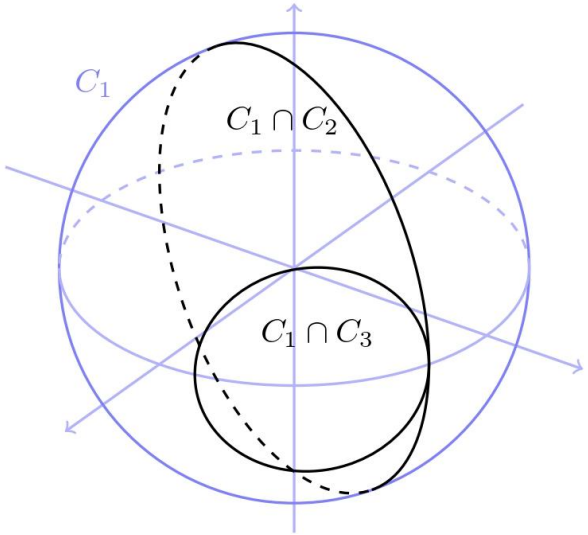
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Favourable !!

Case 3:  Tangent circle triple (C_1, C_2, C_3)

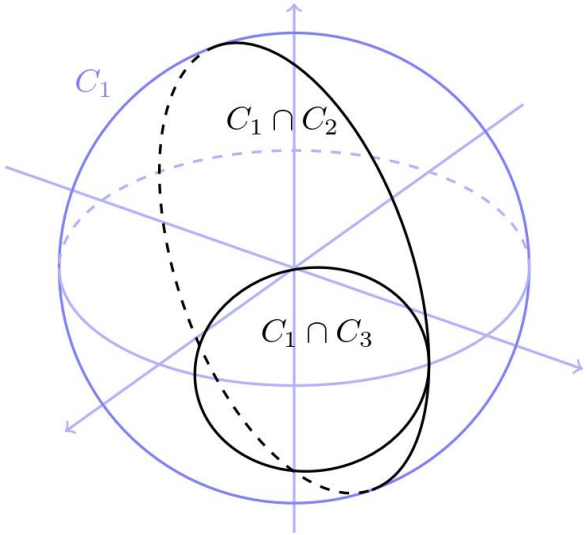
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Tangent circle triples are our main enemy!

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- Cannot "avoid" tangent circles by a "basic" count :

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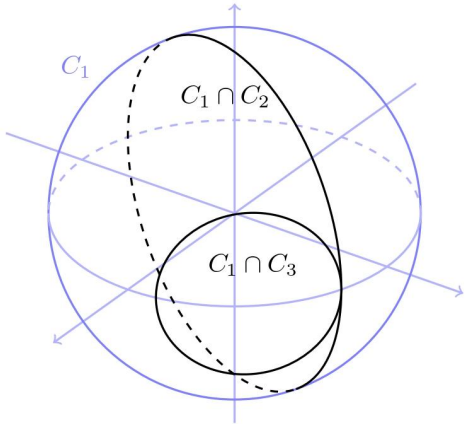
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- However, as $C_1, C_2, C_3 \in \mathcal{C}$ vary, few tangent circle tripks overall
- Not easy to make this precise / quantitative, however!

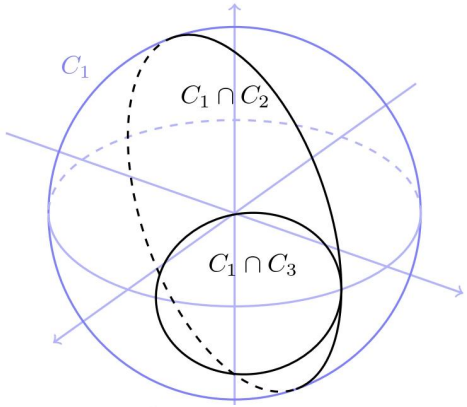
Key observation:

Sphere centres partially determine circle tangency.



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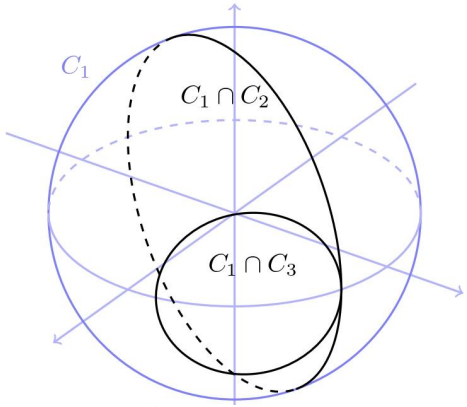
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- Given: (C_1, C_2, C_3) tangent circle triple
- Let: Π = affine plane through $x(C_1), x(C_2), x(C_3)$
  p = point of tangency b/w $C_1 \cap C_2$ and $C_1 \cap C_3$.

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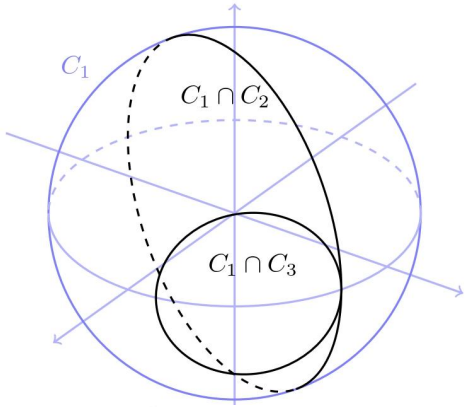
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- Then: $p \in \Pi$

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Sphere centres partially determine circle tangency.

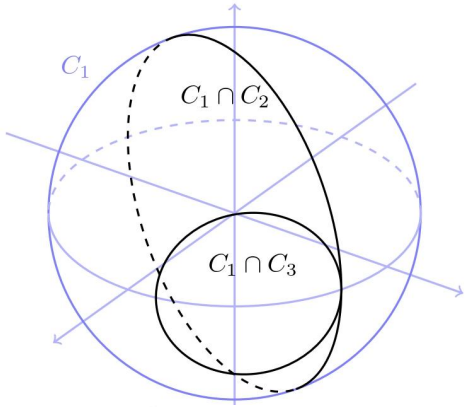


- Given: (C_1, C_2, C_3) tangent circle triple
- Let: Π = affine plane through $x(C_1), x(C_2), x(C_3)$
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- Use this information to circumvent enemy case — Guerrilla tactics!

Strategy: 1. Symmetry breaking / preferred orientation.

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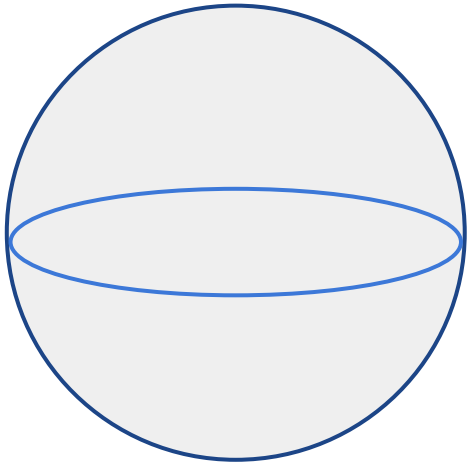
- Decompose $S^2 = \bigcup_{\alpha \in \mathcal{A}} S^2_\alpha$, each S^2_α a spherical $\frac{1}{100}$-cap.

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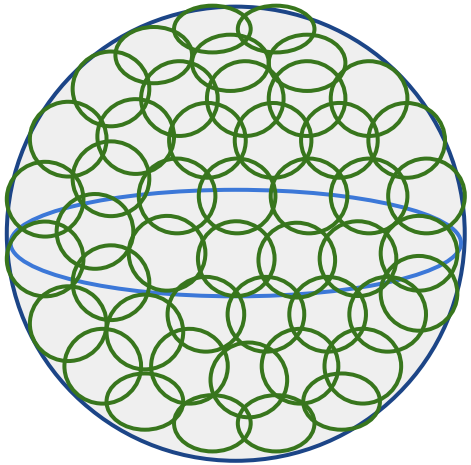
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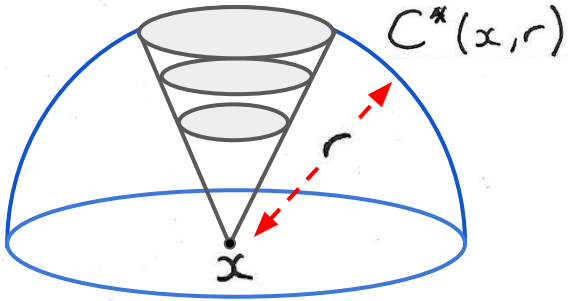
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- $M_S^{*,\delta} := M_S^{\alpha_*,\delta}$ takes maximal averages over δ-thickened "concentric" polar caps $C^*(x,r)$.



2. • Slicing (i.e. Fubini) $\Rightarrow$ suffices to show

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where $\mathcal{C}$ is any family of spheres satisfying:

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- N.B.:

Estimate	δ-power
Unsliced	δ^{-1}
Sliced	$\delta^{-1/3}$.

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  δ^3: intersection $\approx$ δ-ball

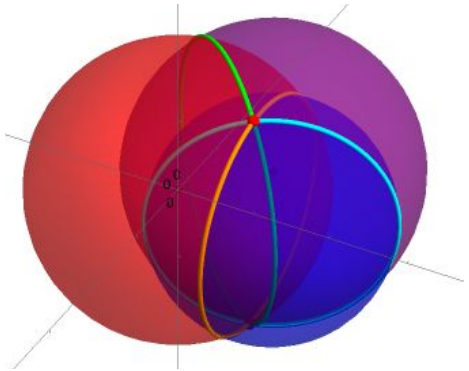
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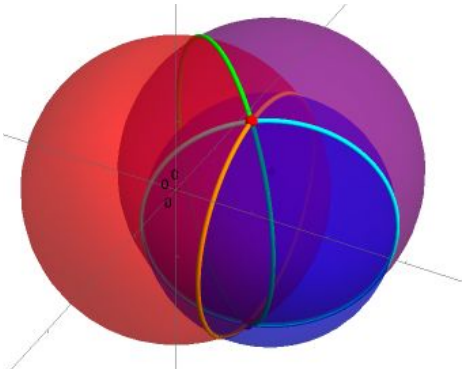
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  $(\delta^{-2})^2$: Fewer spheres!

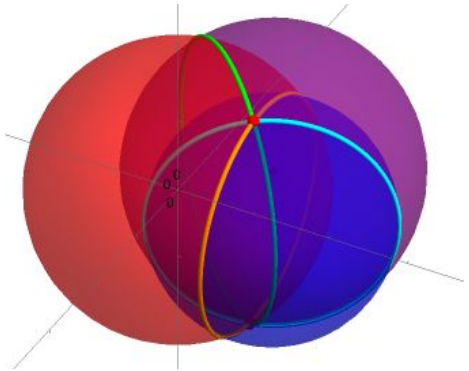


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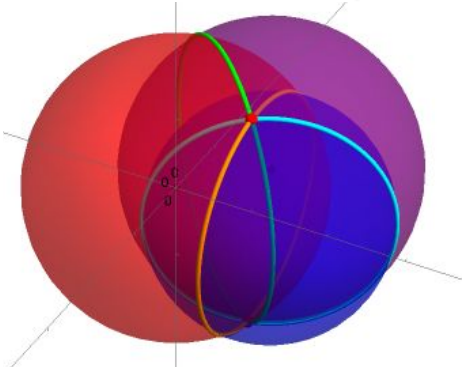
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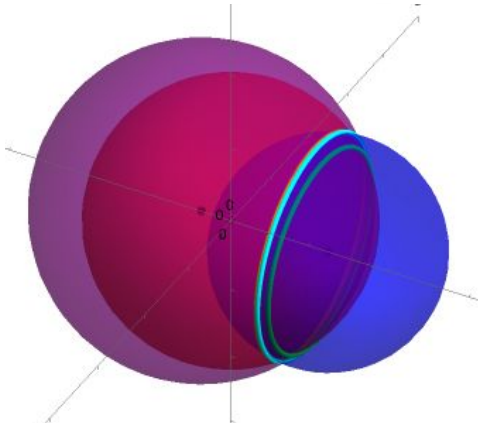
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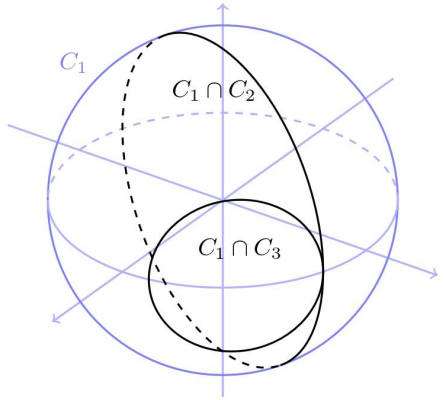
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- $\sum_{\substack{C_1, C_2, C_3 \in \mathcal{C} \\ \text{transversal}}} |C_1^{*,\delta} \cap C_2^{*,\delta} \cap C_3^{*,\delta}| \lesssim \underbrace{\delta^3}_{\substack{\text{intersection} \\ \approx \delta\text{-ball}}} [\#\mathcal{C}]^3 \lesssim \delta^3 \underbrace{(\delta^{-2})^2}_{\substack{\text{Fewer} \\ \text{spheres!}}} \#\mathcal{C} = \delta^{-1} \#\mathcal{C}$ Favourable!!

- $\sum_{\substack{C_1, C_2, C_3 \in \mathcal{C} \\ \text{colinear}}} |C_1^{*,\delta} \cap C_2^{*,\delta} \cap C_3^{*,\delta}| \lesssim \underbrace{\delta^2}_{\substack{\text{intersection} \\ \approx \delta\text{-neighbourhood} \\ \text{of circle}}} \underbrace{\delta^{-1}}_{\substack{\text{choice} \\ \text{of } C_3}} [\#\mathcal{C}]^2 \lesssim \delta^2 \delta^{-1} \underbrace{\delta^{-2}}_{\substack{\text{Fewer} \\ \text{spheres!}}} \#\mathcal{C} = \delta^{-1} \#\mathcal{C}$ Favourable!!

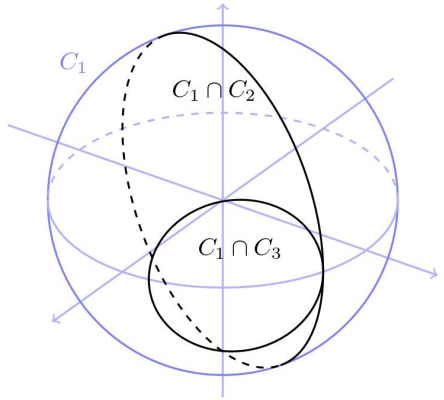
Case 3 :  Tangent circle triples

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Case 3: ⚠ Tangent circle triples

Claim: $$\sum_{C_1, C_2, C_3 \in \mathcal{C} \text{ tangent circle}} |C_1^{*,\delta} \cap C_2^{*,\delta} \cap C_3^{*,\delta}| = 0.$$



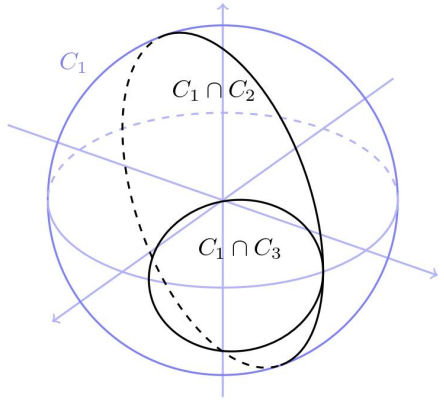
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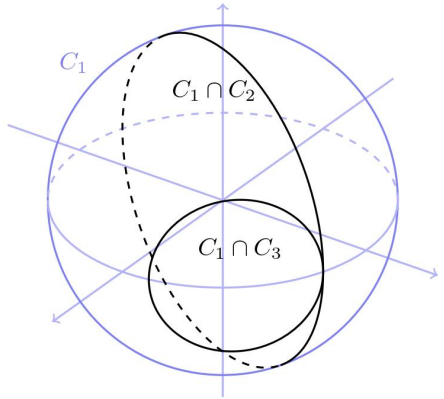
Circumvented the enemy case!

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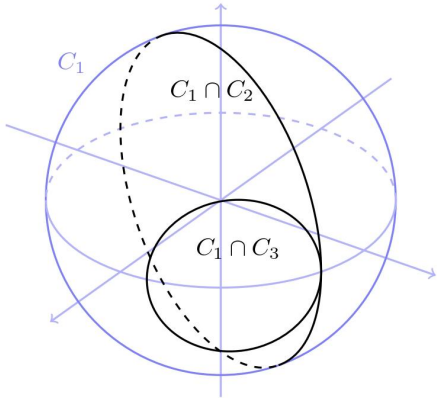


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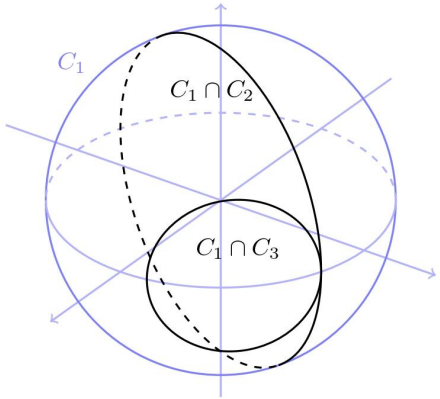
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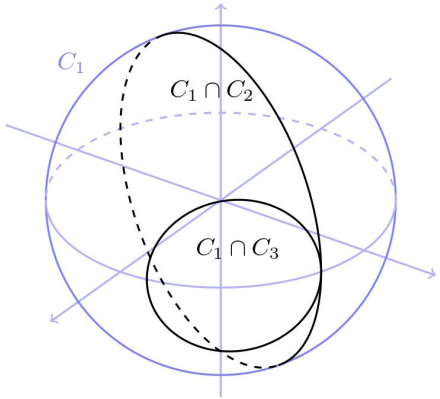
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However, we are only considering the polar cap $C_1^{*,\delta}$!

□

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1. (Maximally) transverse; 2. (Maximally) colinear; 3. Tangent circles

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- Summing contributions over θ, t_2, t_3 leads to $\log \delta^{-1}$-losses.
  (c.f. Córdoba's Kakeya argument).

In conclusion: Our geometric argument gives:-

Theorem 2 (Weak Stein): For $p \geq 3/2$, $\varepsilon > 0$,

$$\| M^{\delta}_{S,\mathrm{loc}} f \|_{L^p(\mathbb{R}^3)} \lesssim_{\varepsilon} \delta^{-\varepsilon} \| f \|_{L^p(\mathbb{R}^3)}.$$

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Open problem: Geometric proof of full Stein's spherical maximal theorem (or local variant)?

Schlag 1997: Geometric proof of circular max. theorem.

Next time : Adapt these methods to the multiparameter setting - new results!

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Thanks!