Review of Geometry of Differential Spaces*

Jędrzej Śniatycki
Department of Mathematics and Statistics,
University of Calgaru, Calgary, Alberta Canada.

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I would like to thank the Organizing Committe: Professor Manuel de Leon, Professor Marcelo Epstein and Professor Victor M. Jimènez Morales, for the kind invitation to address this Workshop. I regret that I am unable to travel to Madrid to take full participation in the Workshop. Professor Jimenez and I have planned to use an internet connection for delivery of my lecture, but the time difference made the schedule rather unrealistic.

1 Introduction

I am going to talk about the theory of differential spaces proposed by Sikorski in 1967, [13], and further developed in his book published in 1972, [14]. Sikorski's theory is one of many approaches to study systems with singularities; for example, see [11], [5], [15], [1] and [12]. I find Sikorski's approach very easy comparing to those of other authors. He defines the differential structure $C^\infty(S)$ on a topological space S as a family of functions on S deemed to be smooth. The topological space S endowed with a differential structure $C^\infty(S)$ is called a differential space. Once a differential structure $C^\infty(S)$ is specified, we study geometric constructs on S in terms of their compatibility with $C^\infty(S)$.

For example, a map $\varphi : S \to R$ between differential spaces $(S, C^\infty(S))$ and $(R, C^\infty(R))$ is smooth if, for every $f \in C^\infty(R)$, the pull-back $\varphi^* f = f \circ \varphi \in C^\infty(S)$. The map $\varphi : S \to R$ is a diffeomorphism if it is smooth, invertible, and its inverse $\varphi^{-1} : R \to S$ is smooth.

We may also impose additional conditions on the differential space $(S, C^\infty(S))$ under investigation. For example, a differential space $(S, C^\infty(S))$ is a manifold of dimension n if it is locally diffeomorphic to open subsets of $\mathbb{R}^n$.

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2 Differential structure and smooth maps

Definition 1 *A differential structure on a topological space S is a family $C^\infty(S)$ of functions on S such that:*
1. $\{f^{-1}(a,b) \mid f \in C^\infty(S)$ and $a < b \in \mathbb{R}\}$ is a subbasis for the topology of S.
2. If $f_1, ..., f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$ then $F(f_1, ..., f_n) \in C^\infty(S)$.
3. For $f : S \to \mathbb{R}$ such that, for every $x \in S$, there exist an open neighbourhood V of x in S and $f_x \in C^\infty(S)$ such that $f_{x|V} = f_{|V}$.

The first condition of Definition 1 relates the topology of S to its differential structure. The remaining conditions ensure that if S is a topological manifold then S endowed with a differential structure $C^\infty(S)$ is a smooth manifold in the usual sense.

Definition 2 *A map $\varphi : S \to R$ between differential spaces $(S, C^\infty(S))$ and $(R, C^\infty(R))$ is smooth (of class C^∞) if, for every $f \in C^\infty(R)$, its pull-back $\varphi^* f = f \circ \varphi$ is in $C^\infty(S)$. A smooth map $\varphi : S \to R$ is a diffeomorphism if its invertible, and its inverse $\varphi^{-1} : R \to S$ is smooth.*

Given a set S, we can define a differential structure on S as follows. First, choose a family $\mathcal{F}_S$ of functions on S such that

$$\{f^{-1}(a,b) \mid f \in \mathcal{F}_S \text{ and } a < b \in \mathbb{R}\}$$

is a subbasis for the desired topology of S.

Definition 3 *The* differential structure $C^\infty(S)$ generated by $\mathcal{F}_S$ *consists of functions $f : S \to \mathbb{R}$ such that, for every $x \in S$, there exists an open neighbourhood V of $x \in S$, an integer $k \in \mathbb{N}$, functions $f_1, ..., f_n \in \mathcal{F}_S$ and a function $F \in C^\infty(\mathbb{R}^k)$ such that*

$$h_{|V} = F(f_1, ..., f_n)_{|V}.$$

If S is a subset of a differential space R differential structure $C^\infty(R)$, and $\mathcal{F}_S = \{f_{|S} \mid f \in C^\infty(R)\}$ is the space of restrictions to S of smooth functions on R, then S endowed with the differential structure $C^\infty(S)$ generated by $\mathcal{F}_S$ is called a *differential subspace of* R. In particular, if $R = \mathbb{R}^n$ and $C^\infty(R) = C^\infty(\mathbb{R}^n)$ is the standard differential structure of $\mathbb{R}^n$, then S endowed with the differential structure $C^\infty(S)$ generated by $\mathcal{F}_S$ is a differential subspace of $\mathbb{R}^n$.

Definition 4 *A differential space $(S, C^\infty(S))$ is locally Euclidean if, for every $x \in S$, there exist an open neighbourhood V of $x \in S$ and $n \in \mathbb{N} \cup \{0\}$, such that the differential subspace $(V, C^\infty(V))$ is diffeomorphic to a differential subspace of $\mathbb{R}^n$.*

In the following, refering to a differential space $(S, C^\infty(S))$, we omit $C^\infty(S)$ and say a differential spaces S, provided there is no danger of confusing which differental structure $C^\infty(S)$ is implied.

The definition of a differential structure $C^\infty(S)$ on a topological space S, allows various operations like products, fibre products, quotients etc.

Theorem 5 *Let S be a differential space with differential structure $C^\infty(S)$ and let $\{U_\alpha\}$ be an open cover of S. If S is Hausdorff, locally compact and second countable, then there exists a countable partition of unity $\{f_i\} \subseteq C^\infty(S)$ subordinate to $\{U_\alpha\}$ and such that the support of each f_i is compact.*

Proof. See Theorem 2.2.4 in [18]. ■

2.1 Tangent bundle[1]

Following the principle that the differential structure $C^\infty(S)$ of a differential space S encodes all the geometric information about S, we define tangent vectors in terms of their actions on $C^\infty(S)$.

Definition 6 Vectors tangent *to a differential space S at a point $x \in S$ are derivations of $C^\infty(S)$ at x, that is, they are linear maps $\mathrm{v}_x : C^\infty(S) \to \mathbb{R} : f \mapsto \mathrm{v}_x f$ that satisfy Leibniz's rule at x:*

$$\mathrm{v}_x(f_1 f_2) = (\mathrm{v}_x f_1) f_2(x) + f_1(x)(\mathrm{v}_x f_2).$$

The tangent space *to S at a point x it the space $T_x S$ of all derivations* $\mathrm{v}_x : \mathcal{C}^\infty(S) \to \mathbb{R}$ *at x. The* tangent bundle *of S is*

$$TS = \coprod_{x \in S} T_x S. \tag{1}$$

The tangent bundle projection *is the map $\tau_S : TS \to S : \mathrm{v}_x \mapsto \tau(\mathrm{v}_x) = x$.*

Since the point x of attachment of a vector $\mathrm{v}_x \in TS$ is determined by the tangent bundle projection $\tau_S : TS \to S$, we may write $\mathrm{v}_x = \mathrm{v}$ and $x = \tau(\mathrm{v})$.

.Every $f \in C^\infty(S)$ gives rise to two functions on TS: the pull-back by the tangent bundle projection

$$\tau_S^* f = f \circ \tau_S : TS \to \mathbb{R},$$

and the differential

$$\mathrm{d}f : TS \to \mathbb{R} : \mathrm{v} \mapsto \mathrm{d}f(\mathrm{v}) = \mathrm{v}f.$$

[1] It should be noted that the tangent bundle of a differential space as defined here is not a locally trivial fibration. Therefore, some authors are using the term "pseudobundle".

Definition 7 *We endow TS with the differential structure $C^\infty(TS)$ generated by $\mathcal{F}_{TS} = \{\tau_S^* f,\ \mathrm{d}f \mid f \in C^\infty(S)\}$. In this differential structure the tangent bundle projection is smooth.*

Definition 8 *The tangent map (derived map) of a map $\varphi : S \to R$ between differential spaces with differential structures $C^\infty(S)$ and $C^\infty(R)$, respectively, is a map $T\varphi : TS \to TR : \mathrm{v} \mapsto T\varphi(\mathrm{v})$ such that, $\tau_R(T\varphi(\mathrm{v})) = \tau_S(\mathrm{v})$ and, for every $f \in C^\infty(R)$*

$$T\varphi(\mathrm{v})f = \mathrm{v}(\varphi^* f) = \mathrm{v}(f \circ \varphi).$$

Proposition 9 *The tangent map $T\varphi : TS \to TR$ is smooth in the differential structures $C^\infty(TS)$ and $C^\infty(TR)$ generated by $\mathcal{F}_{TS} = \{\tau_S^* f, \mathrm{d}f \mid f \in C^\infty(S)\}$ and $\mathcal{F}_{TR} = \{\tau_R^* f, \mathrm{d}f \mid f \in C^\infty(R)\}$, respectively.*

3 Derivations

Definition 10 *Let S be a differential space. A (global)*[2] *derivation of $C^\infty(S)$ is a linear map $X : C^\infty(S) \to C^\infty(S) : f \mapsto Xf$ satisfying Leibniz's rule*

$$X(f_1 f_2) = (Xf_1)f_2 + f_1(Xf_2) \tag{2}$$

for every $f_1,\ f_2 \in C^\infty(S)$.

Let $\mathrm{Der}\, C^\infty(S)$ denote the space of deriviations of $C^\infty(S)$. It is a Lie algebra with Lie bracket

$$[X_1, X_2]f = X_1(X_2 f) - X_2(X_1 f) \tag{3}$$

for every $X_1,\ X_2 \in \mathrm{Der}\, C^\infty(S)$ and every $f \in C^\infty(S)$. In addition, $\mathrm{Der}\, C^\infty(S)$ is a module over the ring $C^\infty(S)$ with $[fX_1, X_2] = f[X_1, X_2]$ and

$$[X_1, fX_2] = (X_1 f)X_2 + f[X_1, X_2] \tag{4}$$

for every $X_1,\ X_2 \in \mathrm{Der}\, C^\infty(S)$ and every $f \in C^\infty(S)$.

Definition 11 *A section of the tangent bundle projection $\tau_S : TS \to S$ is a map $\sigma : S \to TS$ such that $\tau_S \circ \sigma = \mathrm{id}_S$.*

[2]If there is a possibility of confusion of tangent vectors,which are derivations at points of S, with derivations defined here, we add a descriptor "global".

Every derivation $X \in \operatorname{Der} C^\infty(S)$ corresponds to a section of the tangent bundle projection $\tau_S : TS \to S$

$$X : S \to TS : x \mapsto X(x),$$

where $X(x)f = (Xf)(x)$ for every $f \in C^\infty(S)$.

In Example 40, we show a differential space S such that TS is not spanned by derivations $X \in \operatorname{Der} C^\infty(S)$.

Suppose that the map $\varphi : R \to S$ in Definition 8 is a diffeomorphism, that is $\varphi^{-1} : R \to S$ exists and is smooth. For every derivation $X \in \operatorname{Der} C^\infty(R)$ there exists a unique derivation $\varphi_* X \in \operatorname{Der} C^\infty(S)$,

$$\varphi_* X : C^\infty(S) \to C^\infty(S) : f \mapsto (\varphi_* X)f = (\varphi^{-1})^*(X(\varphi^* f)), \tag{5}$$

which is φ-related to X. It is called the *push-forward* of X by φ. Moreover,

$$\varphi_* : \operatorname{Der} C^\infty(R) \to \operatorname{Der} C^\infty(S) : X \mapsto \varphi_* X$$

is a Lie algebra diffeomorphism.

4 Integral curves of derivations and vector fields

From this section on, we make an additional assumption that the differential spaces under consideration are locally Euclidean. This means that they are locally diffeomorphic to differential subspaces of Euclidean spaces.

Definition 12 *Let S be a locally Euclidean differential space and X a derivation of $C^\infty(S)$. An integral curve of X originating at $x_0 \in S$ is a map $c : I \to S$, where I is a connected subset of $\mathbb{R}$ containing 0, such that $c(0) = x_0$ and*

$$\frac{d}{dt} f(c(t)) = (Xf)(c(t)) \textit{ for every } f \in \mathcal{C}^\infty(S) \textit{ and every } t \in I,$$

whenever the interior of I is not empty.

Integral curves of a given derivation X of $\mathcal{C}^\infty(S)$ starting at x_0 can be ordered by inclusion of their domains. In other words, if $c_1 : I_1 \to S$ and $c_2 : I_2 \to S$ are two integral curves of X, such that $c_1(0) = c_2(0) = x_0$, and $I_1 \subseteq I_2$, then $c_1 \preceq c_2$. An integral curve $c : I \to S$ of X is *maximal* if $c \preceq c_1$ implies that $c = c_1$.

Theorem 13 *Let S be a locally Euclidean differential space space and let X be a derivation of $\mathcal{C}^\infty(S)$. For every $x \in S$, there exists a unique maximal integral curve c of X such that $c(0) = x$.*

Proof. See [16] or the proof of Theorem 3.2.1 in [18]. ■

Remark 14 *Note that Definition 12 is somewhat different from the standard definition of an integral curve of a vector field on manifold. In particular, integral curves with domain consisting of a single point are permitted. These modifications allow for the generality of the statement of Theorem 13.*

We denote by $\mathrm{e}^{tX}(x)$ the point on the maximal integral curve of X, originating at x, corresponding to the value t of the parameter. Given $x \in S$, $\mathrm{e}^{tX}(x)$ is defined for t in an interval I_x containing zero, and $\mathrm{e}^{0X}(x)(x) = x$. If t, s, and $t+s$ are in I_x, $s \in I_{\mathrm{e}^{tX}(x)}$, and $t \in I_{\mathrm{e}^{sX}(x)}$, then

$$\mathrm{e}^{(s+t)X}(x) = \mathrm{e}^{sX}(\mathrm{e}^{tX}(x)) = \mathrm{e}^{tX}(x)(\mathrm{e}^{sX}(x)).$$

Proposition 15 *For every derivation X of the differential structure $\mathcal{C}^\infty(S)$ of a locally Euclidean differential space S, and a diffeomorphism $\varphi : S \to R$,*

$$\mathrm{e}^{t\varphi_* X} = \varphi \circ \mathrm{e}^{tX} \circ \varphi^{-1}.$$

Proof. See [18] ■

In the case when X is a derivation of $C^\infty(M)$, where M is a manifold, it is a vector field on M, and $\mathrm{e}^{tX} : x \mapsto \mathrm{e}^{tX}(x)$ is a local one-parameter group of local diffeomorphisms of M. For a subcartesian space S, $\mathrm{e}^{tX} : x \mapsto \mathrm{e}^{tX}(x)$ might fail to be a local diffeomorphism.

Definition 16 *A* vector field *on a locally Eucliean differential space S is a derivation X of $C^\infty(S)$ such that for every $x \in S$, there exist an open neighbourhood V of x in S and $\varepsilon > 0$ such that for every $t \in (-\varepsilon, \varepsilon)$, the map $e^{tX}(x)$ is defined on V, and its restriction to V is a diffeomorphism from V onto an open subset of S. In other words, X is a vector field on S if e^{tX} is a local 1-parameter group of local diffeomorphisms of S.*

We denote by $\mathfrak{X}(S)$ the familly of all vector fields on a locally Euclidean differential space S.

Proposition 17 *$\mathfrak{X}(S)$ is a Lie subalgebra of the Lie algebra* $\mathrm{Der}C^\infty(S)$ *of derivations of $C^\infty(S)$.*

Proof. See [21]. ■

Theorem 18 *Let S be a locally Ecliean differential space. A derivation X of $C^\infty(S)$ is a vector field on S if the domain of every maximal integral curve of X is open in $\mathbb{R}$.*

Proof. See [6]. ■

5 Orbits of family $\mathfrak{X}(S)$ of vector fields on S

For $X_1, \ldots, X_n$ in the Lie algebra $\mathfrak{X}(S)$ of all vector fields on a locally Euclidean differential space S, consider a piece-wise smooth integral curve c in S, originating at $x_0 \in S$, given by a sequence of steps. First, we follow the integral curve of X_1 through x_0 for time τ_1; next we follow the integral curve of X_2 though $x_1 = \varphi_{\tau_1}^{X}(x_0)$ for time τ_2; and so on. For each $i = 1, \ldots, n$ let J_i be $[0, \tau_i] \subseteq \mathbb{R}$ if $\tau_i > 0$ or $[\tau_i, 0]$ if $\tau_i < 0$. Note that $\tau_i < 0$ means that the integral curve of X_i is followed in the negative time direction. For every i, J_i is contained in the domain $I_{x_{i-1}}$ of the maximal integral curve of X_i starting at x_{i-1}. In other words, for $t = \tau_1 + ... + \tau_{n-1} + \tau_n$,

$$c(t) = c(\tau_1 + \tau_2 + ... + \tau_{n-1} + \tau_n) = \varphi_{\tau_n}^{X_n} \circ \varphi_{\tau_{n-1}}^{X_{n-1}} \circ .... \circ \ \varphi_{\tau_1}^{X_1}(x_0).$$

Definition 19 *The* orbit through x_0 *of the family* $\mathfrak{X}(S)$ *of vector fields on* S *is the set* M *of points* x *in* S *that can be joined to* x_0 *by a piece-wise smooth integral curve of vector fields in* $\mathfrak{X}(S)$*;*

$$M = \{\varphi_{t_n}^{X_n} \circ \varphi_{t_{n-1}}^{X_{n-1}} \circ .... \circ \varphi_{t_1}^{X_1}(x_0) \mid X_1, ..., X_n \in \mathfrak{X}(S) \ t_1, ..., t_n \in \mathbb{R}, \ \ n \in \mathbb{N}\} \ .$$

Theorem 20 *Orbits* M *of the family* $\mathfrak{X}(S)$ *of vector fields on a subcartesian space* S *are submanifolds of* S*. In the manifold topology of* M*, the differential structure on* M *induced by its inclusion in* S *coincides with its manifold differential structure.*

Proof. See [17], or the proof of Theorem 3.4.5 in [18]. ■

Notation 21 *We denote by* $\mathfrak{M}(S)$ *the family of orbits of* $\mathfrak{X}(S)$*.*

By Theorem 20, every orbit M of $\mathfrak{X}(S)$ is a manifold. Moreover, the manifold structure of M is its differential structure induced by the inclusion of M in S. Hence, M is a submanifold of the differential space S. The orbits of $\mathfrak{X}(S)$, give a partition $\mathfrak{M}(S)$ of S by connected smooth manifolds. Since the notion of a vector field on a subcartesian space S is intrinsically defined in terms of its differential structure, it follows that every locally Euclidean differential space has a natural partition by connected smooth manifolds. In particular, every subset S of $\mathbb{R}^n$ has natural partition by connected smooth manifolds.

Proposition 22 *Let* X *be a derivation of* $C^\infty(S)$*. If, for each* $M \in \mathfrak{M}(S)$ *and each* $x \in M$*, the maximal integral curve of* X *originating at* $x \in M$ *is contained in* M*, then* $X \in \mathfrak{X}(S)$*, that is,* X *is a derivation of* $C^\infty(S)$ *that generates local one parameter groups of local diffeomorphisms of* S*.*

Proof. See [6]. ■

Proposition 23 (Frontier Condition) *For $M, M' \in \mathfrak{M}(S)$, if $M' \cap \overline{M} \neq \emptyset$, then either $M' = M$ or $M' \subset \overline{M} \backslash M$.*

Proof. See [6]. ■

Proposition 24 (Whitney's Condition A) *Consider a differential subspace S of $\mathbb{R}^n$. Let $y \in M' \subseteq \overline{M} \backslash M$, where $M, M' \in \mathfrak{M}(S)$, and let $m = \dim M$. If x_i is a sequence of points in M such that $x_i \to y \in M'$, and $T_{x_i} M$ converges to some m-plane $E \subseteq T_y S \subseteq T_y \mathbb{R}^n$ then $T_y M' \subseteq E$.*

Proof. See [6]. ■

In general the partition $\mathfrak{M}(S)$ need not be locally finite. If $\mathfrak{M}(S)$ is locally finite, then it is a stratification of S.

6 The cotangent bundle

6.1 Covectors

Definition 25 Covectors *at $x \in S$ are differentials $\mathrm{d}f_{|x}$ at x of functions $f \in \mathcal{C}^\infty(S)$. The space of covectors at $x \in S$ is called the* cotangent space *of S at x, and it is denoted $T_x^* S$. Thus,*

$$T_x^* S = \{\mathrm{d}f_{|x} \mid f \in \mathcal{C}^\infty(S)\}.$$

The cotangent bundle space *of S is the space of*

$$T^* S = \coprod_{x \in S} T_x^* S = \{\mathrm{d}f_{|x} \mid x \in S,\ f \in \mathcal{C}^\infty(S)\}$$

of all covectors at all points of S.The cotangent bundle projection *is the map $\pi_S : T^* S \to S$ assigning to each $\mathrm{d}f_{|x} \in T^* S$ the point $x \in S$ at which the differential $\mathrm{d}f$ is evaluated,*

$$\pi_S(\mathrm{d}f_{|x}) = x.$$

For $V \subseteq S$,

$$T_V^* S = \coprod_{x \in V} T_x^* S = \pi_S^{-1}(V) \subseteq T^* S.$$

The definition of the differential of a function can be re-interpreted as the evaluation *function on the fibre product $T^* S \times_S TS$,*

$$\langle \cdot \mid \cdot \rangle : T^* S \times_S TS \to \mathbb{R} : (\mathrm{d}f_{|x}, \mathrm{v}_x) \mapsto \left\langle \mathrm{d}f_{|x} \mid \mathrm{v}_x \right\rangle = \mathrm{v}_x f.$$

Since $\mathcal{C}^\infty(S)$ is closed under the operations of addition and multiplication by constants, and the derivation in T_xS are linear, it follows that T_x^*S is closed under addition and multiplication by constants. For every $\mathrm{d}f_{1|x}$ and $\mathrm{d}f_{2|x}$ in T_x^*S and every $a_1, a_2 \in \mathbb{R}$,

$$a_1\mathrm{d}f_{1|x} + a_2\mathrm{d}f_{2|x} = \mathrm{d}(a_1f_1 + a_2f_2)_{|x} \in T_x^*S.$$

Hence, T_x^*S is a vector space for every $x \in S$.

Proposition 26 *If the dimension of TS is locally finite, then the dimension of T^*S is locally finite and* $\dim T_x^*S = \dim T_xS$ *for every $x \in S$.*

Proof. [19]. ■

Let $\varphi : S \to R$ be a smooth map of differential spaces. It gives rise to the derived map $T\varphi : TS \to TR$, see Definition 7. For every $x \in S$, the restriction of $T\varphi$ to T_xS is a linear map $T\varphi_x : T_xS \to T_yR$, where $y = \varphi(x)$. Moreover, for every $\mathrm{v}_x \in T_xS$ and $f \in \mathcal{C}^\infty(R)$, $T\varphi_x(\mathrm{v}_x) \in T_yR$ and $\varphi^*h = f \circ \varphi \in \mathcal{C}^\infty(S)$ because $\varphi : S \to R$ is smooth. Hence,

$$\left\langle \mathrm{d}(\varphi^*f)_{|x} \mid \mathrm{v} \right\rangle = \mathrm{v}(\varphi^*f) = T\varphi_x(\mathrm{v})f = \left\langle \mathrm{d}f_{|y} \mid T\varphi_x(\mathrm{v}) \right\rangle$$

This equation describes a covector $\mathrm{d}(\varphi^*f)_{|x} \in T_x^*S$ acting on an arbitrary vector $\mathrm{v} \in T_xS$ in terms of a covector $\mathrm{d}f_{|y} \in T_y^*R$ acting on $T\varphi(\mathrm{v})$, where $y = \varphi(x)$. We can rewrite this equation in the form

$$\mathrm{d}(\varphi^*f)_{|x} = \mathrm{d}f_{|\varphi(x)} \circ T\varphi.$$

Definition 27 *A smooth map $\varphi : S \to R$ gives rise to the* cotangent map *of φ given by*

$$T^*\varphi : T^*_{\varphi(S)}R \to T^*S : \mathrm{d}f_{|\varphi(x)} \mapsto T^*\varphi(\mathrm{d}f_{|\varphi(x)}) = \mathrm{d}f_{|\varphi(x)} \circ T\varphi = \mathrm{d}(f \circ \varphi)_x = \mathrm{d}(\varphi^*f)_{|x}.$$

It follows from the definition that the following diagram commutes

$$\begin{array}{cccc} & T^*_{|\varphi(S)}R & \xrightarrow{T^*\varphi} & T^*S & \\ \pi_R & \downarrow & & \downarrow & \pi_S \\ & \varphi(S) & \xleftarrow[\varphi]{} & S & \end{array}.$$

In particular, if $\varphi : S \to R$ is a diffeomorphism of differential spaces, then $\varphi(S) = R$ and $T^*\varphi : T^*R \to T^*S$ is a bijection, and $T^*\varphi_{|T_y^*R} : T_y^*R \to T^*_{\varphi^{-1}(y)}S$ is a vector space isomorphism for every $y \in R$.

Lemma 28 *Let V be a differential subspace of S with the inclusioon map $\iota_V : V \to S$. If V is open in S, then*
(i) $T^*V = T_V^*S = \pi_S^{-1}(V) \subseteq T^*S$ *and*
(ii) $T^*\iota_V : T_V^*S \to T^*V : \mathrm{d}\, f_{|x} \mapsto \mathrm{d}\, f_{|x}$ *is the identity map.*

Proof. [19] ∎

In Definition 7 we specified the differential structure $C^\infty(TS)$ of the tangent bundle of a differential space S as generated by the family of functions

$$\mathcal{F}(TS) = \{f \circ \tau_S,\ \mathrm{d}f \mid f \in C^\infty(S)\},$$

where $\tau_S : TS \to S$ is the tangent bundle projection. We may attempt a similar approach to the differential structure $C^\infty(T^*S)$ of the cotangent bundle. For every differential space S, the evaluation map allows for interpretations of derivations $X \in \mathrm{Der}\,\mathcal{C}^\infty(S)$ as functions

$$X^{\mathrm{ev}} : T^*S \to \mathbb{R} : \mathrm{d}f_{|x} \mapsto X^{\mathrm{ev}}(\mathrm{d}f_{|x}) = \big\langle \mathrm{d}f_{|x} \mid X(x)\big\rangle = (Xf)(x).$$

Proposition 29 *If the tangent bundle TS of a locally Euclidean differential space S is spanned by derivations in $\mathrm{Der}\,\mathcal{C}^\infty(S)$, then the family*

$$\mathcal{F}(T^*S) = \{f \circ \pi_S,\ X^{\mathrm{ev}} \mid f \in C^\infty(S),\ \text{and } X \in \mathrm{Der}\,\mathcal{C}^\infty(S)\},$$

of functions on T^*S separates points in TS and generates a locally Hausdorff, locally Euclidean differential structure $C^\infty(T^*S)$ of T^*S.

Among locally Euclidean differential spaces only regular spaces satisfy the condition that TS is locally spanned by derivations in $\mathrm{Der}C^\infty(S)$, [7], [**?**].

6.2 Differential structure of T^*S for $S \subseteq \mathbb{R}^n$

Let $S \subseteq \mathbb{R}^n$ be a differential subspace of $\mathbb{R}^n$, and $\iota_S : S \to \mathbb{R}^n$ the inclusion map. Then T^*S is the quotient of $T^*_S\mathbb{R}^n$ by the equivalence relation

$$\mathrm{d}f_x \sim \mathrm{d}f\,' \iff x = x' \text{ and } \langle \mathrm{d}f \ \mid \mathrm{v}\rangle = \langle \mathrm{d}f' \ \mid \mathrm{v}\rangle\ \ \forall\, \mathrm{v} \in T_xS.$$

Definition 30 *For every $x \in S \subseteq \mathbb{R}^n$, the subspace of $T_x\mathbb{R}^n$ normal to T_xS is*

$$T_x^\perp S = \{\ \mathrm{u} \in T_x\mathbb{R}^n \mid (\ \mathrm{v} \mid\ \mathrm{u}) = 0\ \ \forall\ \ \mathrm{v} \in T_xS\}.$$

Clearly, $T_xS \oplus T_x^\perp S = T_x\mathbb{R}^n$. The normal bundle of S is

$$T^\perp S = \coprod\nolimits_{x \in S} T_x^\perp S = \{\ \mathrm{u} \in T_S\mathbb{R}^n \mid (\ \mathrm{v} \mid\ \mathrm{u}) = 0\ \ \forall\ \ \mathrm{v} \in T_{\boldsymbol{\tau}_W(\mathrm{u})}S\}.$$

For every $x \in S$, the annihilator of $T_x^\perp S$ is

$$AT_x^\perp S = \{\mathrm{d}F_x \in T_x^*\mathbb{R}^n \mid \langle \mathrm{d}F_x \mid \mathrm{v}\rangle = 0\ \ \forall\ \ \mathrm{v} \in T_x^\perp S\},$$

and the the annihilator of $T^\perp S$ is the disjoint union of the annihilators of $T_x^\perp S$,

$$AT^\perp S = \coprod_{x \in S} AT_x^\perp S.$$

The direct sum decomposition

$$T_S^*\mathbb{R}^n = ATS \oplus AT^{\perp}S,$$

ensures that, there exists a map

$$\delta : T^*S \to AT^{\perp}S : \mathrm{d}f_{|x} \mapsto \mathrm{d}F_{|x}$$

where $\mathrm{d}F_{|\boldsymbol{x}} \in AT^{\perp}_{\boldsymbol{x}}S$ is the unique element of the intersection of $(T^*\iota_S)^{-1}(\mathrm{d}f_{|\boldsymbol{x}})$ and $AT^{\perp}_{\boldsymbol{x}}S$. Also, there is a map

$$\gamma = T^*\iota_W \circ \iota_{AT^{\perp}W} = T^*\iota_{W|AT^{\perp}W} : AT^{\perp}W \to T^*W : \mathrm{d}F_{|\boldsymbol{x}} \mapsto \mathrm{d}(\iota_W^* F)_{|\boldsymbol{x}},$$

where $\mathrm{d}F_{|\boldsymbol{x}} \in AT^{\perp}_{x}W$.

Proposition 31 *The map $\delta : T^*S \to AT^{\perp}_{\boldsymbol{x}}S$ is a bijection and $\gamma = \delta^{-1}$*

Proof. [19]. ■

Definition 32 *(i) Since $T_S^*\mathbb{R}^n \subseteq T^*\mathbb{R}^n$, we endow $T_S^*\mathbb{R}^n$ with the differential structure $C^\infty(T_S^*\,\mathbb{R}^n)$ of a differential subspace of $T^*\mathbb{R}^n$.*
(ii) Since $AT^{\perp}S \subseteq T_S^*\mathbb{R}^n \subseteq T^*\mathbb{R}^n$, we endow $AT^{\perp}S$ with the differential structure $C^\infty(AT^{\perp}S)$ of a differential subspace of $T_S^*\mathbb{R}^n$.
(iii) We endow T^*S with the unique differential structure given by the pull-back to T^*S of functions in $C^\infty(AT^{\perp}S)$ by the bijection

$$\gamma = T^*\iota_S \circ \iota_{AT^{\perp}S} : AT^{\perp}S \to T^*S.$$

It is easy to see that in these differential structures the bijections $\delta : T^*S \to AT^{\perp}S$ and $\gamma : AT^{\perp}S \to T^*S$ are diffeomorphims. Moreover, the inclusion maps $\iota_{T_S^*\mathbb{R}^n} : T_S^*\mathbb{R}^n \to T^*\mathbb{R}^n$, $\iota_{AT^{\perp}S} : AT^{\perp}S \to T^*\mathbb{R}^n$, and the projections maps $T^*\iota_S : T_S^*\mathbb{R}^n$, $\pi_S : T^*S \to S$ and $\hat{\pi}_S : T_S^*\mathbb{R}^n \to S$ are smooth.

Proposition 33 *(i) A function $h : T^*S \to \mathbb{R}$ is in $C^\infty(T^*S)$ if $h \circ T^*\iota_S : T_S^*\mathbb{R}^n \to \mathbb{R}$ is in $C^*(T_S^*\mathbb{R}^n)$.*
(ii) The evaluation map

$$T^*S \times_S TS \to \mathbb{R} : (\mathrm{d}f_{|x}, \mathrm{v}) \mapsto \big\langle \mathrm{d}f_{|x} \mid \mathrm{v} \big\rangle$$

is smooth.
(iii) For every $X \in \operatorname{Der} \mathcal{C}^\infty(S)$, the function

$$X^{\mathrm{ev}} : T^*S \to \mathbb{R} : \mathrm{d}f_x \mapsto X^{\mathrm{ev}}(\mathrm{d}f_x) = \langle \mathrm{d}f_x \mid X(x) \rangle = (Xf)(x)$$

is smooth.

Proof. [19]. ■

6.3 Differential structure of T^*S of locally Euclidean space S

Definition 34 *Differential structure $C^\infty(T^*S)$ of a locally Euclidean space S is determined by the condition that, for every $x \in S$, there exists an open neighbourhood V of x in S, locally diffeomorphic to a differential subspace W of $\mathbb{R}^n$, where n depends on x, and that T^*V is diffeomorphic to T^*W.*

Proposition 35 *Differential structure $C^\infty(T^*S)$ specified in Definition 34 satisfies the conditions of Definition 1.*

Proof. [19]. ∎

Corollary 36 *If TS is spanned by derivations X in $\mathrm{Der}C^\infty(S)$, then the family of functions then the family*

$$\mathcal{F}(T^*S) = \{f \circ \pi_S,\ X^{\mathrm{ev}} \mid f \in C^\infty(S),\ \textit{and } X \in \mathrm{Der}\,\mathcal{C}^\infty(S)\},$$

generates the differential structure given in Definition 34.

7 Sections and forms

Definition 37 *A* section *of the cotangent bundle T^*S of S is a smooth map $\vartheta : S \to T^*S$ such that $\pi \circ \vartheta = \mathrm{id}_S$, where $\pi : T^*S \to S$ is the cotangent bundle projection and $\mathrm{id}_S : S \to S : x \mapsto x$ is the identity mapping of S to itself. We denote by $\mathrm{Sec}(T^*S)$ the space of sections of the cotangent bundle of S.*

It is easy to see that, for every $f \in \mathcal{C}^\infty(S)$, the map $\mathrm{d}f : S \to T^*S : x \mapsto \mathrm{d}f_{|x}$, is a section of T^*S. Therefore, every covector $\mathrm{d}f_{|x} \in T^*S$ is in the range of a section $\mathrm{d}f : S \to T^*S$.

The space $\mathrm{Sec}(T^*S)$ of sections of T^*S is closed under the operations of addition of sections, multiplication of sections by smooth functions

$$\begin{aligned} \mathrm{Sec}(T^*S) \times \mathrm{Sec}(T^*S) &\to \mathrm{Sec}(T^*S) : (\vartheta_1, \vartheta_2) \mapsto \vartheta_1 + \vartheta_2, \\ \mathcal{C}^\infty(S) \times \mathrm{Sec}(T^*S) &\to \mathrm{Sec}(T^*S)(S) : (f, \vartheta) \mapsto f\vartheta. \end{aligned}$$

Moreover, the differential d may be interpreted as a linear map from $\mathcal{C}^\infty(S)$ to $\mathrm{Sec}(T^*S)$,

$$\mathrm{d} : \mathcal{C}^\infty(S) \to \mathrm{Sec}(T^*S) : f \mapsto \mathrm{d}f.$$

Proposition 38 *For every x in a locally Euclidean differential space S there exists an open neighbourhood V of x in S such that, for every section $\vartheta : S \to T^*S$, the restriction $\vartheta_{|V} : V \to T^*_VS$ can be written in the form*

$$\vartheta_{|V} = \sum_{i=1}^{n} p_i \mathrm{d}q_i,$$

where $p_1, ..., p_n, q_1, ..., q_n \in \mathcal{C}^\infty(V)$ and $n = \dim T_xS$.

Proof. [19]. ■

The notion of a section $\vartheta : S \to T^*S$ of a cotangent bundle of a subcartesian space S may be extended to sections of the of wedge products $\wedge^k T^*S$ of the cotangent bundle.

Definition 39 *Let $\pi : \wedge^k T^*S \to S$ be the projection map of the wedge product of the cotangent bundle T^*S of a locally Eucliean differential space S. A section of $\wedge^k T^*S$ is a smooth map $\sigma : S \to \wedge^k T^*S$ such that $\pi \circ \sigma = \mathrm{id}_S$. The space of sections of $\wedge^k T^*S$ is denoted by* $\mathrm{Sec}(\wedge^k T^*S)$.

If M is a manifold, one can construct smooth local sections of $\wedge^k T^*M$ in terms of wedge products $\vartheta_1 \wedge ... \wedge \vartheta_k$ of smooth sections $\vartheta_1, ..., \vartheta_k$ of T^*M. The algebraic construction of the wedge product of sections of the cotangent bundle T^*S of a subcartesian space S is well defined. However, it does not guarantee that the wedge product of smooth sections is smooth.

Example 40 *Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. It is a stratified space with 5 strata: the origin $M_0 = \{(0, 0)\}$ and four open coordinate half-lines $M_1, ..., M_4$. The tangent bundle of S is*

$$TS = \left(\coprod_{i=1}^{4} TM_i \right) \coprod \left(T_{(0,0)}\mathbb{R}^2 \right).$$

Every derivation $X \in DerC^\infty(S)$ vanishes at $(0, 0)$, [18].
The cotangent bundle of S is

$$T^*S = \left(\coprod_{i=1}^{4} T^*M_i \right) \coprod \left(T^*_{(0,0)}\mathbb{R}^2 \right).$$

Moreover,

$$\wedge^2 T^*S = \left(\coprod_{i=1}^{4} 0_{M_i} \right) \coprod \left(\wedge^2 T^*_{(0,0)}\mathbb{R}^2 \right).$$

Thus, $\wedge^2 T^*S$ has no smooth non-zero section.
At present, we do not know how to resolve successfully problems presented by this example.

Differential multiforms on a manifold are sections of the corresponding wedge products of the corresponding cotangent bundle. Our example shows that a straightforward application of this approach is unlikely to succeed.

Another possibility is to get an appropriate definition of differential forms on a locally Euclidean differential space which would overcome these difficulties. There are several definitions of differential forms on singular spaces which satisfy an analogue of the de Rham theorem, e.g. [15], [8], [10]. However, we do not know which of them solve concrete problems in understanding the structure of singular spaces.

Differential forms on orbit spaces proper actions of Lie groups on manifolds were studied in [4]. The authors write in the introduction:

> "Here, in our search for an intrinsic notion of a differential form, we have been led to see them as multilinear maps on vector fields."

In the paper [19], to be submitted to the special issue of Mathematics and Mechanics of Solids, we use the space $\wedge\mathfrak{X}(S)^*$ of differential forms consisting of multilinear alternating maps on vector fields on S to describe the presymplectic structure and the corresponding reduced symplectic structure of the cotangent bundle of a locally Euclidean differential space S.

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