Cosserat media in dynamics

Géry de Saxcé

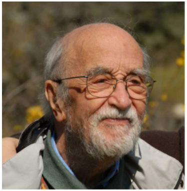
LaMcube UMR CNRS 9013 University of Lille

GAMM 2024 Madrid

" The shoes are a tool to walk ;
the mathematics, a tool to think.
One can walk without shoes,
but one goes less far."

Jean-Marie Souriau

Grammaire de la Nature (2007)



State of the Art

Method of Virtual powers
(Lagrange, 1788)

Generalized continua
(Eugène and François Cosserat, 1909)

Affine mechanics (Souriau, 1997)

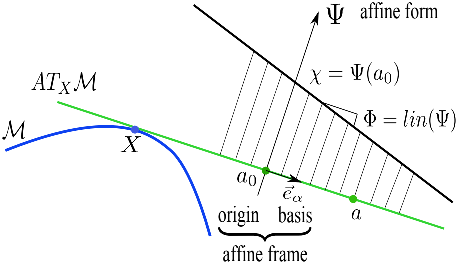
Affine tensors
(Tulczyjew, Urbañski, Grabowski, 1988)

Affine connections
(Élie Cartan, 1923)

Affine tensors and Torsors

The torsor as an affine tensor

- the space-time $\mathcal{M}$ is a differential manifold
- $AT_{\boldsymbol{X}}\mathcal{M}$ is the affine space associated to the tangent vector space $T_{\boldsymbol{X}}\mathcal{M}$
- $A^*T_{\boldsymbol{X}}\mathcal{M}$ is the vector space of the affine functions $\boldsymbol{\Psi}$ on $AT_{\boldsymbol{X}}\mathcal{M}$ (called affine forms)
- The **torsor** is a bilinear and skew-symmetric function on this space
  $\boldsymbol{\tau}(\boldsymbol{\Psi}, \hat{\boldsymbol{\Psi}}) = -\boldsymbol{\tau}(\hat{\boldsymbol{\Psi}}, \boldsymbol{\Psi})$
  It is real or vector-valued



A fragrance of affine tensor calculus...

Affine tensors [Tulczyjew, Urbański & Grabowski 1988]

The simplest ones

- **Points** (1-contravariant) : $\boldsymbol{a} = \boldsymbol{a}_0 + V^\alpha \vec{\boldsymbol{e}}_\alpha$ of components V^α in the affine frame $(\boldsymbol{a}_0, (\vec{\boldsymbol{e}}_\alpha))$
- **Affine forms** (1-covariant) : $\boldsymbol{\Psi} = \chi 1 + \Phi_\alpha \boldsymbol{e}^\alpha$ of components (χ, Φ_α)

Affine tensors of the Mechanics

- **Torsors** (2-contravariant) : $\boldsymbol{\tau}(\boldsymbol{\Psi}, \hat{\boldsymbol{\Psi}})$ of components $(T^\alpha, J^{\alpha\beta})$
  $= \boldsymbol{\tau}(\chi 1 + \Phi_\alpha \boldsymbol{e}^\alpha, \hat{\chi} 1 + \hat{\Phi}_\beta \boldsymbol{e}^\beta)$
  $= \chi\hat{\chi}\, \boldsymbol{\tau}(1,1) + \chi\, \hat{\Phi}_\beta \boldsymbol{\tau}(1, \boldsymbol{e}^\beta) + \Phi_\alpha \hat{\chi}\, \boldsymbol{\tau}(\boldsymbol{e}^\alpha, 1) + \Phi_\alpha \hat{\Phi}_\beta \boldsymbol{\tau}(\boldsymbol{e}^\alpha, \boldsymbol{e}^\beta)$
  Denoting $T^{00} = \boldsymbol{\tau}(1,1) = 0$, $T^\alpha = \boldsymbol{\tau}(1, \boldsymbol{e}^\alpha)$, $J^{\alpha\beta} = \boldsymbol{\tau}(\boldsymbol{e}^\alpha, \boldsymbol{e}^\beta) = -J^{\beta\alpha}$
  As $\hat{\boldsymbol{\Phi}}_\beta = \hat{\boldsymbol{\Phi}}(\vec{\boldsymbol{e}}_\beta) = \hat{\boldsymbol{\Psi}}(\vec{\boldsymbol{e}}_\beta) = \vec{\boldsymbol{e}}_\beta(\hat{\boldsymbol{\Psi}})$, $\quad \chi = \boldsymbol{\Psi}(\boldsymbol{a}_0) = \boldsymbol{a}_0(\boldsymbol{\Psi})$
  then $\chi\, \hat{\Phi}_\beta = \boldsymbol{a}_0(\boldsymbol{\Psi})\, \vec{\boldsymbol{e}}_\beta(\hat{\boldsymbol{\Psi}}) = (\boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_\beta)(\boldsymbol{\Psi}, \hat{\boldsymbol{\Psi}})$
  Decomposition $\boxed{\boldsymbol{\tau} = T^\beta\, (\boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_\beta - \vec{\boldsymbol{e}}_\beta \otimes \boldsymbol{a}_0) + J^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta}$
- **Co-torsors** (2-covariant)
- **Momentum Tensors** (1-covariant and 1-contravariant)

Transformation laws of affine tensors (1/2)

Change of affine frames $(\boldsymbol{a}_0, (\vec{\boldsymbol{e}}_\alpha)) \longrightarrow (\boldsymbol{a}'_0, (\vec{\boldsymbol{e}}'_\beta))$ given by

$$\overrightarrow{\boldsymbol{a}'_0 \boldsymbol{a}_0} = C'^\beta \, \vec{\boldsymbol{e}}'_\beta, \qquad \vec{\boldsymbol{e}}'_\beta = P^\alpha_\beta \, \vec{\boldsymbol{e}}_\alpha$$

In matrix form $C' = \begin{pmatrix} C'^1 \\ \vdots \\ C'^4 \end{pmatrix}, \qquad P = \begin{pmatrix} P^1_1 & \dots & P^1_4 \\ \vdots & \ddots & \vdots \\ P^4_1 & \dots & P^4_4 \end{pmatrix}, \qquad C = -P \, C'$

Laws of transformations of affine tensors

- point $\boldsymbol{a}$ of components V^α : $\qquad V' = C' + P^{-1} V$
- affine form $\boldsymbol{\Psi}$ of components (χ, Φ_α) : $\qquad \chi' = \chi + \Phi \, C, \qquad \Phi' = \Phi \, P$
- torsor $\boldsymbol{\tau}$ of components $(T^\alpha, J^{\alpha\beta})$:

$$\boxed{T' = P^{-1} T, \qquad J' = P^{-1}(J + C T^T - T C^T) \, P^{-T}}$$

Transformation laws of affine tensors (1/2)

Affine transformations

Combining two changes of affine frames, we observe that
the set of couples (C, P) is the affine group $\mathbb{GA}(4) = \mathbb{R}^4 \rtimes \mathbb{GL}(4)$

Trick : using the linear representation on $\mathbb{R}^5$ of the affine group of $\mathbb{R}^4$

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix},$$

and introducing

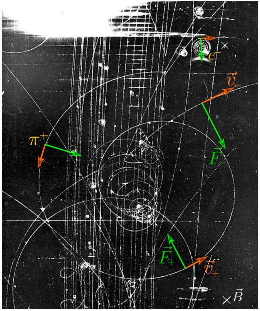
$$\tilde{V} = \begin{pmatrix} 1 \\ V \end{pmatrix}, \qquad \tilde{\Psi} = (\chi,\ \Phi), \qquad \tilde{\tau} = \begin{pmatrix} 0 & T^T \\ -T & J \end{pmatrix}$$

the laws of transformation of affine tensors are reduced to :

$$\tilde{V}' = \tilde{P}^{-1}\tilde{V}, \qquad \tilde{\Psi}' = \tilde{\Psi}\,\tilde{P}, \qquad \boxed{\tilde{\tau}' = \tilde{P}^{-1}\tilde{\tau}\,\tilde{P}^{-T}}$$

Pointwise object

material
particle



object which can be
thought as pointwise
if it is seen from
a long way off

Galilean structure of the space-time

- Event $X = \begin{pmatrix} t \\ x \end{pmatrix} \in \mathbb{R}^4$
- The **Galilean transformations** $X' \mapsto X$ are the transformations preserving **inertial motions**, **durations**, **distances** and **volume orientations**, then affine of the form $X = P\,X' + C$ with

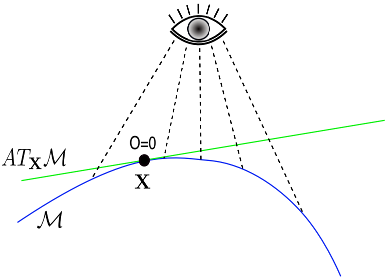
$$P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

where $u \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation

- Their set is the **Galilei group**, $\mathbb{GAL}$
  a Lie group of dimension 10 ,
  a Lie subgroup of $\mathbb{GA}(4)$
- The Galilean geometry is not Riemannian

Proper frame

- [Élie Cartan 1923 Sur les variétés à connexion affine...]
  "The affine space at $\boldsymbol{X}$ could be seen as the manifold itself
  that would be perceived in an affine manner by an observer located at $\boldsymbol{X}$"



- A frame of reference in which an object is **at rest**
  at the origin $\boldsymbol{O}$ of $AT_{\boldsymbol{X}}\mathcal{M}$ is called a **proper frame** attached to this object

The torsor of a pointwise object

For an object in a **proper frame** $\tilde{\tau}' = \begin{pmatrix} 0 & T'^T \\ -T' & J' \end{pmatrix} = \begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & -j(l_0) \end{pmatrix}$

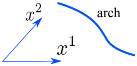
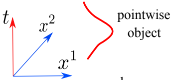
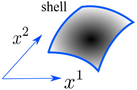
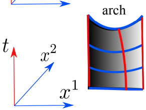
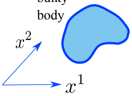
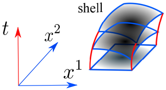
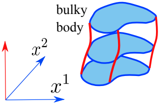
- Apply a **Galilean boost** $\tilde{P} = \begin{pmatrix} 1 & 0 \\ C & P \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & v & 1_{\mathbb{R}^3} \end{pmatrix}$
- The transformation law of torsors gives $\tilde{\tau} = \tilde{P}\,\tilde{\tau}'\,\tilde{P}^T = \begin{pmatrix} 0 & m & p^T \\ -m & 0 & -q^T \\ -p & q & -j(l) \end{pmatrix}$

  where the invariant (mass) m and

  - the linear momentum $p = m\,v$,
  - the quantity of position (or passage) $q = m\,x$
  - the angular momentum $l = l_0 + x \times m\,v$

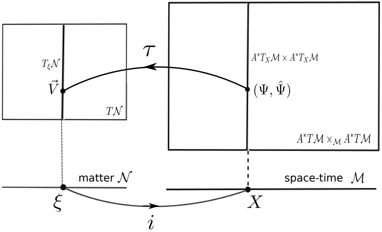
Cosserat media of arbitrary dimensions

Matter manifold $\mathcal{N}$

$dim(\mathcal{N})$	*Statics* $dim(\mathcal{M}) = 3$	*Dynamics* $dim(\mathcal{M}) = 4$
1		
2		
3		
4		

The torsor of a Cosserat medium of arbitrary dimension

- The **matter manifold** $\mathcal{N}$ is the set of material particles ξ
  We describe the motion of the matter by an embedding $i : \xi \mapsto X$
- A **torsor** at X is valued in the tangent vector space to $\mathcal{N}$ at ξ



- Then we defined a bundle map over $\mathcal{N}$
  $i^*(A^* T\mathcal{M} \times_{\mathcal{M}} A^* T\mathcal{M}) \to T\mathcal{N}$
- In an affine frame $(\boldsymbol{a}_0, (\vec{\boldsymbol{e}}_\alpha))$ of $AT_{\boldsymbol{X}}\mathcal{M}$
  and a basis $({}_\gamma\vec{\boldsymbol{\eta}})$ of $T_{\boldsymbol{\xi}}\mathcal{N}$, it is decomposed into

$$\boxed{\boldsymbol{\tau} = {}^\gamma\boldsymbol{\tau}\ {}_\gamma\vec{\boldsymbol{\eta}}, \qquad {}^\gamma\boldsymbol{\tau} = {}^\gamma T^\beta \left(\boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_\beta - \vec{\boldsymbol{e}}_\beta \otimes \boldsymbol{a}_0\right) + {}^\gamma J^{\alpha\beta}\, \vec{\boldsymbol{e}}_\alpha \otimes \vec{\boldsymbol{e}}_\beta}$$

Dynamics of a bulky body

- As $\dim(\mathcal{N}) = \dim(\mathcal{M}) = 4$, we can choose for easiness $\xi^\alpha = X^\alpha$, then ${}^\gamma T^\beta = T^{\beta\gamma}, \quad {}^\gamma J^{\alpha\beta} = J^{\alpha\beta\gamma}$
- In a **proper frame** attached to the elementary reference volume :
  - If $J^{\alpha\beta\gamma} \neq 0$, we modelize a **3D Cosserat medium**
  - If $J^{\alpha\beta\gamma} = 0$, we modelize a **3D Cauchy medium** (as supposed in the sequel)

  Moreover, in this proper frame, $T' = \begin{pmatrix} \rho & 0 \\ 0 & -\sigma \end{pmatrix}$

  where ρ is the mass density and σ is Cauchy's stress tensor
- Applying a **Galilean boost** of velocity v, the transformation law of torsors gives

$$\boxed{T = \begin{pmatrix} \rho & p^T \\ p & -\sigma_\star \end{pmatrix} = \begin{pmatrix} \rho & \rho\, v^T \\ \rho\, v & \rho\, v\, v^T - \sigma \end{pmatrix}}$$

where $\sigma_\star$ are the dynamical stresses.
For these reasons, the tensor of components $T^{\beta\gamma}$
is called **stress-mass tensor**

1D Cosserat media

beam, arch or
string if solid

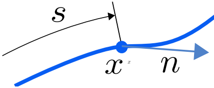
flow in a pipe
or jet if fluid

Modelization of the matter motion

- The motion of a 1D material body can be described by the embedding of a matter manifold $\mathcal{N}$ into the space-time $\mathcal{M}$

$$i : \xi = \begin{pmatrix} t \\ s \end{pmatrix} \mapsto X = \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ \psi(t, s) \end{pmatrix}$$

where t is the time and s is the arclength



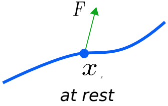
- Its tangent map U is given by a 4-by-2 matrix

$$dX = \begin{pmatrix} dt \\ dx \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v - v_t\, n & n \end{pmatrix} \begin{pmatrix} dt \\ ds \end{pmatrix} = U\, d\xi$$

where n is the unit tangent vector to the curve and $v_t = v \cdot n$

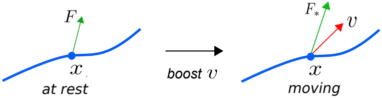
The tensor of force-mass

- The components ${}^{\gamma}T^{\beta}$ can be stored in a 2-by-4 matrix. In a **proper frame** attached to an elementary segment of 1D material body, it is reduced to



$$T' = \begin{pmatrix} \rho_l & 0 \\ 0 & -F^T \end{pmatrix}$$

where ρ_l is the mass density, F is the statical force density acting through the cross-section (both per unit arclength)

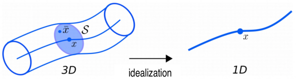


- Applying a **Galilean boost** v, the transformation law of tensor gives

$$\boxed{T = \begin{pmatrix} \rho_l & p^T \\ p_t & -F_\star^T \end{pmatrix} = \begin{pmatrix} \rho_l & \rho_l v^T \\ \rho_l v_t & (\rho_l v_t v - F)^T \end{pmatrix}}$$

It is called **force-mass tensor** in which appears the linear momentum per unit arclength p and the dynamical force $F_\star$

3D **slender body**
idealized by an 1D body
(approximation)



- We define the **projection** Π of $T_{\boldsymbol{X}}\mathcal{M}$ into $T_{\boldsymbol{\xi}}\mathcal{N}$ as where $\boldsymbol{X} = i(\boldsymbol{\xi})$

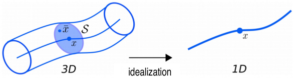
$$d\xi = \begin{pmatrix} dt \\ ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & n^T \end{pmatrix} \begin{pmatrix} dt \\ dx \end{pmatrix} = \Pi\, dX \quad \text{such that} \quad \Pi\, U = 1_{\mathbb{R}^2}$$

- The 3D body is described by the
  stress-mass tensor $\bar{T} = \begin{pmatrix} \rho & \rho\,\bar{v}^T \\ \rho\,\bar{v} & \rho\,\bar{v}\,\bar{v}^T - \sigma \end{pmatrix}$
- Combining $\bar{T}$ with Π and integrating on the cross-section $\mathcal{S}$,

  we obtain the force-mass tensor of the 1D body $\boxed{T = \int_{\mathcal{S}} \Pi\, \bar{T}\, d\mathcal{S}}$

  where
  $\rho_l = \int_{\mathcal{S}} \rho\, d\mathcal{S}, \quad \rho_l v = \int_{\mathcal{S}} \rho\, \bar{v}\, d\mathcal{S}, \quad F = \int_{\mathcal{S}} (\sigma\, n - \rho\,(v_t - \bar{v}_t)(v - \bar{v}))\, d\mathcal{S}$
  remarking a dynamical contribution to F

3D-to-1D reduction (2/2)



- The torsor of the 3D body (considered as a **Cauchy medium**) is such that $\bar{J}^{\alpha\beta\gamma} = 0$ at points $\bar{x}$ of the cross-section. Applying a spatial translation $\bar{x}$, the transformation law gives at the mass-center of the cross-section

$$\bar{J}^{i0\rho} = \bar{x}^i \, T^{0\rho}, \qquad \bar{J}^{ij\rho} = \bar{x}^i \, \bar{T}^{j\rho} - \bar{x}^j \, \bar{T}^{i\rho}$$

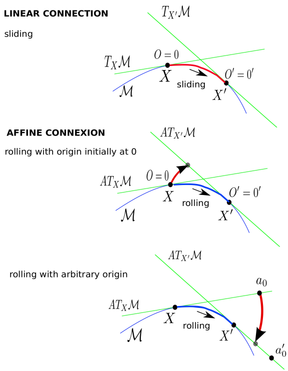
- In the same spirit, we calculate $\boxed{{}^{\gamma}J^{\alpha\beta} = \int_{\mathcal{S}} {}^{\gamma}\Pi_{\rho} \, \bar{J}^{\alpha\beta\rho} \, d\mathcal{S}}$

  that is, putting $q^i = {}^0J^{i0}, \quad l^i_\star = {}^1J^{i0}, \quad l^i = {}^0J^{kl}, \quad M^i_\star = {}^1J^{kl}$

  we obtain in matrix form for the 1D **Cosserat medium**
  - the position quantity $q = \int_{\mathcal{S}} \rho \, \bar{x} \, d\mathcal{S}$
  - the angular momentum $l = \int_{\mathcal{S}} \bar{x} \times \rho \, \bar{v} \, d\mathcal{S}$,
  - the tangent angular momentum $l_\star = \int_{\mathcal{S}} \rho \, \bar{v}_t \bar{x} \, d\mathcal{S}$
  - the dynamical moment $M_\star = \int_{\mathcal{S}} \bar{x} \times (\rho \bar{v}_t \bar{v} - (\sigma \, n)) \, d\mathcal{S}$

- If x is the mass-center, then $q = \rho_l x$

[Élie Cartan 1923 Sur les variétés à connexion affine...]



To define a covariant derivative $\tilde{\nabla}$ of an affine tensor,
we consider an infinitesimal affine transformation :

$$dX \mapsto (dC, dP) = (\Gamma_A(dX), \Gamma(dX))$$

- $\Gamma = (\Gamma^\alpha_{\mu\beta} dX^\mu)$ is the **gravitation**
- $\Gamma_A = (\Gamma^\alpha_{A\mu} dX^\mu)$ represent the **infinitesimal motion of the observer** $\Gamma_A(dX) = dX - \nabla_{dX} C$

[Kobayashi & Nomizu 1963 Foundation of Differential Geometry]

Covariant divergence of the torsor

The **affine covariant divergence** of the torsor is decomposed into

$$\tilde{\boldsymbol{Div}}\, \boldsymbol{\tau} = {}_{\gamma}\tilde{\nabla}\, {}^{\gamma}T^{\beta}\, (\boldsymbol{a}_0 \otimes \vec{\boldsymbol{e}}_{\beta} - \vec{\boldsymbol{e}}_{\beta} \otimes \boldsymbol{a}_0) + {}_{\gamma}\tilde{\nabla}\, {}^{\gamma}J^{\alpha\beta}\, \vec{\boldsymbol{e}}_{\alpha} \otimes \vec{\boldsymbol{e}}_{\beta}$$

with

- $${}_{\gamma}\tilde{\nabla}\, {}^{\gamma}T^{\beta} = \frac{\partial({}^{\gamma}T^{\beta})}{\partial({}^{\gamma}\xi)} + {}^{\gamma}_{\gamma\rho}\Gamma^{\rho}\, T^{\beta} + {}^{\gamma}T^{\rho}\, {}_{\gamma}U^{\sigma}\, \Gamma^{\beta}_{\sigma\rho}$$
- $$\begin{aligned}{}_{\gamma}\tilde{\nabla}\, {}^{\gamma}J^{\alpha\beta} = \frac{\partial({}^{\gamma}J^{\alpha\beta})}{\partial({}^{\gamma}\xi)} &+ {}^{\gamma}J^{\rho\beta}\, {}_{\gamma}U^{\sigma}\, \Gamma^{\alpha}_{\sigma\rho} + {}^{\gamma}J^{\alpha\rho}\, {}_{\gamma}U^{\sigma}\, \Gamma^{\beta}_{\sigma\rho} + {}^{\gamma}_{\gamma\rho}\Gamma^{\rho}\, J^{\alpha\beta} \\ &+ {}_{\gamma}U^{\sigma}\, \Gamma^{\alpha}_{A\sigma}\, {}^{\gamma}T^{\beta} - {}^{\gamma}T^{\alpha}\, {}_{\gamma}U^{\sigma}\, \Gamma^{\beta}_{A\sigma}\end{aligned}$$

Gravitation in relativity :
40 Christoffel symbols

Galilean gravitation :
6 non vanishing Christoffel symbols

- $g^{i} = -\Gamma^{i}_{00}$ is the classical **gravity**
- $\Omega^{i}_{j} = -\Omega^{j}_{i} = \Gamma^{i}_{0j}$ is the **spinning**, responsible of Coriolis' force

Dynamics of 0D and 3D Cosserat media

To be compatible with Galilei principle of relativity,
the physical laws must be covariant

Principle : the **covariant divergence** of the torsor vanishes $\boxed{\tilde{\boldsymbol{Div}}\, \boldsymbol{\tau} = 0}$

dynamics of a bulky body (3D) : **balance equations**

- of the $\boldsymbol{T}$ components. Euler's equations of fluid : **balance equations**
  - of the mass $\dfrac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0$
  - of the linear momentum $\rho\, \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x}\, v\right] = div\ \sigma + \rho(g - 2\,\Omega \times v)$
- of the $\boldsymbol{J}$ components. $T^{\alpha\beta} = T^{\beta\alpha}$

dynamics of a pointwise object (0D) : **balance equations**

- of the mass $\dot{m} = 0$
- of the linear momentum $\dot{p} = m\,(g - 2\,\Omega \times v)$
- of the position quantity $\dot{q} = p$
- of the angular momentum $\dot{l} + \Omega \times l_0 = x \times m\,(g - 2\,\Omega \times v)$

Dynamics of 1D and 2D Cosserat media

From the principle $\boxed{\tilde{Div}\, \tau = 0}$, we deduce

the **dynamics of 1D Cosserat media** : **balance equations**

- of the mass $\dfrac{\partial \rho_l}{\partial t} + \dfrac{\partial}{\partial s}(\rho_l v_t) = 0$
- of the linear momentum $\rho_l \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial s}\, v_t \right] = \dfrac{\partial F}{\partial s} + \rho_l (g - 2\,\Omega \times v)$
- of the position quantity $\dfrac{\partial q}{\partial t} + \dfrac{\partial l_\star}{\partial s} = \rho_l v$
- of the angular momentum $\dfrac{\partial l}{\partial t} + \Omega \times l + l_\star \times (\Omega \times n) = -\dfrac{\partial M_\star}{\partial s} + n \times F$

the **dynamics of 2D Cosserat media** [de Saxcé & Vallée 2003]

This principle of divergence free torsor is abstract but covers a broad spectrum of applications

To know more... [de Saxcé & Vallée 2016 Galilean Mechanics and Thermodynamics of Continua]

Perspectives

Affine tensors of the Mechanics

- **Torsors** (2-contravariant) : $(\Psi, \hat{\Psi}) \mapsto \boldsymbol{\tau}(\Psi, \hat{\Psi}) \in T_\xi \mathcal{N}$ (skew-symmetric)
- **Co-torsors** (2-covariant) : $(\boldsymbol{a}, \hat{\boldsymbol{a}}) \mapsto \gamma(\boldsymbol{a}, \hat{\boldsymbol{a}}) \in T^*_\xi \mathcal{N}$ (bi-affine, skew-sym.)
  - Describe the kinematics, in duality with torsors
- **Momentum tensors** (mixed 1-1) : $(\vec{\boldsymbol{V}}, \Psi) \mapsto \boldsymbol{\mu}(\vec{\boldsymbol{V}}, \Psi)) \in T_\xi \mathcal{N}$ (bilinear)
  - Decomposition $\boldsymbol{\mu} = {}^\gamma \boldsymbol{\mu}\, {}_\gamma \vec{\boldsymbol{\eta}}, \qquad {}^\gamma \boldsymbol{\mu} = \boldsymbol{e}^\beta \otimes ({}^\gamma \Sigma_\beta\, \boldsymbol{a}_0 + {}^\gamma M^\alpha_\beta\, \vec{\boldsymbol{e}}_\alpha)$
  - Components : $({}^\gamma \Sigma, {}^\gamma M) \in (\mathbb{R}^m)^* \times \mathfrak{gl}(m) \cong (\mathfrak{ga}(m))^* \quad (m = \dim(\mathcal{M}))$
  - The law of transformation is just the **coadjoint representation** of the $\mathbb{GA}(m)$
  - The system of components of $\boldsymbol{\mu}$ can be considered as
    the value of the **momentum map** in symplectic mechanics
- **Strain tensors** (mixed 1-1) : $(\Phi, \boldsymbol{a}) \mapsto \boldsymbol{\omega}(\Phi, \boldsymbol{a}) \in T^*_\xi \mathcal{N}$ (linear and affine)
  - Generalized deformations, in duality with momentum tensors

Thank you very much for your attention !

¡Muchas gracias por su atención !