Multicontact field theories: application to Navier's equations

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Outline

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Goals

- Generalize both the multisymplectic and the contact frameworks to introduce the so-called **multicontact structures**.
- Develop the **Lagrangian formulation** for field theories in the multicontact setting.
- Contactify the **steady Navier–Cauchy equations**.

Outline

Idea

We want to develop a geometric framework to describe non-conservative field theories generalizing the notion of **cocontact manifold** and compatible with the k-**contact** and k-**cocontact** formalisms.

This new geometric framework has to lead to the **Herglotz–Euler–Lagrange equations**:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^i_\mu}\right) = \frac{\partial L}{\partial y^i} + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\,.$$

In order to find this new structure, we first consider the fiber bundle $J^1\pi$ of $\pi : E \to M$.

We also consider $\Lambda^{m-1}(\mathrm{T}^*M)$ which, based on the Herglotz's variational principle for fields, is the natural structure to define the new variables s^μ that represent the dependence of the Lagrangian on the action.

Idea

In these fiber bundles we have several natural forms: the **Poincaré–Cartan** m-**form** associated with a Lagrangian function L in $J^1\pi$, the **tautological form** associated to $\Lambda^{m-1}(\mathrm{T}^*M)$, and a **volume form** on M.

We want to obtain a new form defined in an appropriate extension of the jet bundle, whose coordinate expression reads

$$\Theta_{\mathcal{L}} = -\frac{\partial L}{\partial y^i_\mu}\mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu + \left(\frac{\partial L}{\partial y^i_\mu}y^i_\mu - L\right)\mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu\,,$$

for a Lagrangian function L defined in that jet bundle extension. The new variables s^μ must give account for the “non-conservation”.

This form will be used to characterize the field equations so that we reach the **Herglotz–Euler–Lagrange equations**.

Let P be a manifold with $\dim P = m + N$ and $N \geq m \geq 1$, and two forms $\Theta, \omega \in \Omega^m(P)$ with constant rank. These forms play different roles: one of them, ω, is a "reference form", while the other, Θ, is the one that gives the structure that we want to introduce, properly said.

First, given a regular distribution $\mathcal{D} \subset \mathrm{T}P$, consider $\Gamma(\mathcal{D})$, the set of sections of $\mathcal{D}$. For every $k \in \mathbb{N}$, define

$$\mathcal{A}^k(\mathcal{D}) := \{\alpha \in \Omega^k(P) \mid \iota_Z \alpha = 0 \ , \ \forall Z \in \Gamma(\mathcal{D})\} \, ;$$

that is, the set of differential k-forms on P vanishing by the vector fields of $\Gamma(\mathcal{D})$.

At a point $\mathrm{p} \in P$, the point-wise version is

$$\mathcal{A}^k_{\mathrm{p}}(\mathcal{D}) := \left\{\alpha \in \Lambda^k \mathrm{T}^*_{\mathrm{p}} P \mid \iota_v \alpha = 0 \ , \ \forall v \in \mathcal{D}_{\mathrm{p}}\right\} .$$

Lemma

If $\mathcal{D}$ is an involutive distribution and $\alpha \in \mathcal{A}^k(\mathcal{D})$, we have

$$\iota_X \iota_Y \mathrm{d}\alpha = 0\,,$$

for every $X, Y \in \Gamma(\mathcal{D})$.

For a form $\alpha \in \Omega^k(P)$, with $k > 1$, the '1-$\ker$ of α' will be simply denoted as $\ker \alpha$; that is, $\ker \alpha = \{Z \in \mathfrak{X}(P) \mid \iota_Z \alpha = 0\}$. With this in mind, the above definition of $\mathcal{A}^k(\mathcal{D})$ can be written as

$$\mathcal{A}^k(\mathcal{D}) = \left\{\alpha \in \Omega^k(P) \mid \Gamma(\mathcal{D}) \subset \ker \alpha\right\}.$$

The Reeb distribution

For a pair (Θ, ω) we define:

Definition

The **Reeb distribution** associated to the pair (Θ, ω) is the distribution $\mathcal{D}^{\mathfrak{R}} \subset \mathrm{T}P$ defined, at every point $\mathrm{p} \in P$, as

$$\mathcal{D}^{\mathfrak{R}}_{\mathrm{p}} = \left\{ v \in (\ker \omega)|_{\mathrm{p}} \;\middle|\; \iota_v \mathrm{d}\Theta_{\mathrm{p}} \in \mathcal{A}^m_{\mathrm{p}}(\ker \omega) \right\},$$

and $\mathcal{D}^{\mathfrak{R}} = \bigcup_{\mathrm{p} \in P} \mathcal{D}^{\mathfrak{R}}_{\mathrm{p}}$. The set of sections of the Reeb distribution is denoted by $\mathfrak{R} := \Gamma(\mathcal{D}^{\mathfrak{R}})$, and its elements $R \in \mathfrak{R}$ are called **Reeb vector fields**. Then, if $\ker \omega$ has constant rank,

$$\mathfrak{R} = \left\{ R \in \Gamma(\ker \omega) \mid \iota_R \mathrm{d}\Theta \in \mathcal{A}^m(\ker \omega) \right\}.$$

Note that $\ker \omega \cap \ker \mathrm{d}\Theta \subset \mathfrak{R}$.

Lemma

If ω is a closed form and has constant rank, then $\mathfrak{R}$ is involutive.

Definition

The pair (Θ, ω) is a **premulticontact structure** if ω is a closed form and, for $0 \leq k \leq N - m$, we have that:

(1) $\operatorname{rank} \ker \omega = N$.

(2) $\operatorname{rank} \mathcal{D}^{\mathfrak{R}} = m + k$.

(3) $\operatorname{rank} (\ker \omega \cap \ker \Theta \cap \ker \mathrm{d}\Theta) = k$.

(4) $\mathcal{A}^{m-1}(\ker \omega) = \{\iota_R \Theta \mid R \in \mathfrak{R}\}$,

Then, the triple (P, Θ, ω) is said to be a **premulticontact manifold** and Θ is called a **premulticontact form** on P. The distribution $\mathcal{C} \equiv \ker \omega \cap \ker \Theta \cap \ker \mathrm{d}\Theta$ is called the **characteristic distribution** of (P, Θ, ω).

If $k = 0$, the pair (Θ, ω) is a **multicontact structure**, (P, Θ, ω) is a **multicontact manifold** and, in this situation, Θ is said to be a **multicontact form** on P.

The dissipation form σ_Θ

Lemma

The characteristic distribution of a (pre)multicontact manifold (P, Θ, ω) is involutive and

$$\ker\omega \cap \ker\Theta \cap \ker \mathrm{d}\Theta = \mathcal{D}^{\mathfrak{R}} \cap \ker\Theta\,.$$

Associated to a (pre)multicontact structure, we have the following one-form:

Proposition

Given a (pre)multicontact manifold (P, Θ, ω), there exists a unique 1-form $\sigma_\Theta \in \Omega^1(P)$ verifying that

$$\sigma_\Theta \wedge \iota_R\Theta = \iota_R \mathrm{d}\Theta\,, \ \text{ for every } R \in \mathfrak{R}\,.$$

Definition

The 1-form σ_Θ is called the **dissipation form**.

The operator $\overline{\mathrm{d}}$

Using this dissipation form we can define the following operator, which will be used later to set the field equations in a (pre)multicontact manifold.

Definition

Let $\sigma_\Theta \in \Omega^1(P)$ be the dissipation form. We define the operator

$$\begin{aligned} \overline{\mathrm{d}} : \Omega^k(P) &\longrightarrow \Omega^{k+1}(P) \\ \beta &\longmapsto \overline{\mathrm{d}}\beta = \mathrm{d}\beta + \sigma_\Theta \wedge \beta \,. \end{aligned}$$

We have that $\overline{\mathrm{d}}^2 = 0$ if, and only if, $\mathrm{d}\sigma_\Theta = 0$. In this case, it induces a *Lichnerowicz–Jacobi cohomology*. One consequence is that, locally, there exists a function such that $\sigma_\Theta = \mathrm{d}f$ and $\overline{\mathrm{d}}\beta = e^{-f}\mathrm{d}(e^f\beta)$. In this case, we say that the pair (Θ, ω) is a **closed multicontact structure**. This is also the condition required in order to consider variational higher-order contact Lagrangian field theories.

Adapted coordinates

A premulticontact manifold (P, Θ, ω) has three associated distributions: $\ker \omega$, the Reeb distribution $\mathcal{D}^{\mathfrak{R}}$, and the characteristic distribution $\mathcal{C}$. They are all involutive and are nested: $\mathcal{C} \subset \mathcal{D}^{\mathfrak{R}} \subset \ker \omega$. We can use these facts to obtain adapted coordinates.

Theorem

Around every point $\mathrm{p} \in P$ *of a premulticontact manifold* (P, Θ, ω)*, there exists a local chart of* **adapted coordinates** $(U; x^1, \dots, x^m, u^1 \dots, u^{N-m-k}, s^1 \dots, s^m, w^1, \dots, w^k)$ *such that*

$$\ker \omega|_U = \left\langle \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^{N-m-k}}, \frac{\partial}{\partial s^1}, \dots, \frac{\partial}{\partial s^m}, \frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^k} \right\rangle,$$

$$\mathcal{D}^{\mathfrak{R}}|_U = \left\langle \frac{\partial}{\partial s^1}, \dots, \frac{\partial}{\partial s^m}, \frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^k} \right\rangle,$$

$$\mathcal{C}|_U = \left\langle \frac{\partial}{\partial w^1}, \dots, \frac{\partial}{\partial w^k} \right\rangle.$$

For multicontact manifolds, since $\mathcal{C} = \{0\}$, there are no coordinates (w^j).

Local Reeb vector fields

On these charts, the coordinates (x^μ) can be chosen in such a way that the form ω reads $\omega|_U = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$, and so we shall do henceforth.
Then we denote $\mathrm{d}^{m-1}x_\mu = \iota\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^m x$.
Taking into account these results, we can give a local characterization of the Reeb vector fields.

Proposition

If (P, Θ, ω) is a multicontact manifold, in the above chart of coordinates, there exists a unique local basis $\{R_\mu\}$ of $\mathfrak{R}$ such that

$$\iota_{R_\mu}\Theta = \mathrm{d}^{m-1}x_\mu\,.$$

In addition, $[R_\mu, R_\nu] = 0$.

Definition

The above vector fields $R_\mu \in \mathfrak{R}$ are the **local Reeb vector fields** of the multicontact manifold (P, Θ, ω) in the chart $U \subset P$.

Local Reeb vector fields

There exist local functions $\Gamma_\mu \in \mathscr{C}^\infty(U)$ associated with the basis $\{R_\mu\}$, which are given by

$$\iota_{R_\mu} \mathrm{d}\Theta = \Gamma_\mu\, \omega\,, \quad \forall \mu\,,$$

because $\iota_{R_\mu} \mathrm{d}\Theta \in \mathcal{A}^m_{\mathrm{p}}(\ker\omega) = \langle \omega \rangle$. As a consequence, the dissipation form can be locally expressed as

$$\sigma_\Theta = \Gamma_\mu \mathrm{d}x^\mu\,,$$

because $\sigma_\Theta \wedge \mathrm{d}^{m-1}x_\mu = \Gamma_\mu\, \omega = \Gamma_\mu\, \mathrm{d}^m x$, for every μ.

Proposition

If (P, Θ, ω) is a premulticontact manifold, there exist local vector fields $\{R_\mu\}$ of $\mathfrak{R}$ such that $\mathfrak{R} = \langle R_\mu \rangle + \mathcal{C}$ and $\iota_{R_\mu}\Theta = \mathrm{d}^{m-1}x_\mu$. They are unique up to a term in the characteristic distribution. Moreover $[R_\mu, R_\nu] \in \Gamma(\mathcal{C})$.

Using adapted coordinates, the local Reeb vector fields read $R_\mu = \dfrac{\partial}{\partial s^\mu}$.

Bundle structures

Associated to a (pre)multicontact structure (Θ, ω) in a manifold P there are the two involutive distributions: $\ker\omega$ and the Reeb distribution $\mathcal{D}^{\mathcal{R}}$, with $\mathcal{D}^{\mathcal{R}} \subset \ker\omega$. We can consider the corresponding quotient sets, $M \equiv P/\ker\omega$ and $\mathcal{E} \equiv P/\mathcal{D}^{\mathcal{R}}$.

From now on we assume we will assume that the quotients M and $\mathcal{E}$ are smooth manifolds.

We have the natural projections

$$\begin{array}{cccc} \tau : & P & \longrightarrow & M \\ & (x^\mu, u^I, s^\mu, w^r) & \longmapsto & (x^\mu)\,, \\ \varsigma : & P & \longrightarrow & \mathcal{E} \\ & (x^\mu, u^I, s^\mu, w^r) & \longmapsto & (x^\mu, u^I)\,, \\ \varepsilon : & \mathcal{E} & \longrightarrow & M \\ & (x^\mu, u^I) & \longmapsto & (x^\mu)\,. \end{array}$$

Furthermore, the form ω is obviously τ-projectable to a form $\omega_{M} \in \Omega^m(M)$, which is a volume form in M.

Proposition

Every (pre)multicontact manifold (P, Θ, ω) *is locally diffeomorphic to a fiber bundle* $\tau\colon P \longrightarrow M$*, where* M *is an orientable manifold with volume form* ω_M*, and* $\omega = \tau^*\omega_M$*.*

From now on, we assume this as the canonical model for (pre)multicontact manifolds since, in addition, this is the situation which is interesting in field theories.

Thus, we consider a fiber bundle $\tau\colon P \to M$, with $\dim M = m$, $\dim P = m + N$, and such that M is an orientable manifold with volume form $\omega_M \in \Omega^m(M)$.

Let $\omega = \tau^*\omega_M \in \Omega^m(P)$. We always take local coordinates (x^μ, z^A) in P $(1 \leq \mu \leq m,\ 1 \leq A \leq N)$, adapted to the bundle structure, and such that $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$.

The τ-vertical bundle is defined as

$$V(\tau) = \bigcup_{\mathrm{p}\in P} V(\tau_{\mathrm{p}}) = \bigcup_{\mathrm{p}\in P} \{\mathrm{v} \in \mathrm{T_p}P \mid \mathrm{T_p}\tau(\mathrm{v}) = 0\}\,.$$

Let $\mathfrak{X}^{V(\tau)}(P)$ denote the $\mathscr{C}^\infty(P)$-module of τ-vertical vector fields and $V(\tau)$ the corresponding τ-**vertical distribution**.

A form $\alpha \in \Omega^k(P)$ is τ-**semibasic** if $\iota_Y\alpha = 0$, for every $Y \in \mathfrak{X}^{V(\tau)}(P)$.

Let $\mathcal{A}^k(V(\tau))$ denote the $\mathscr{C}^\infty(P)$-module of τ-semibasic k-forms and $\mathcal{A}^k_{\mathrm{p}}(V(\tau))$ the corresponding fiber at $\mathrm{p} \in P$. We have that

$$\Gamma(\ker\omega) = \mathfrak{X}^{V(\tau)}(P)\,.$$

Taking this forms $\omega \in \Omega^m(P)$ and $\Theta \in \Omega^m(P)$, the definition of (pre)multicontact structure adapted to this context (where condition (1) holds automatically) is:

Multicontact bundles

Definition

The pair (Θ, ω) is a **multicontact bundle structure** and (P, Θ, ω) is said to be a **multicontact bundle** if:

(1) $\operatorname{rank} \mathcal{D}^{\mathfrak{R}} = m$.

(2) $\ker \omega \cap \ker \Theta \cap \ker \mathrm{d}\Theta = \{0\}$.

(3) $\mathcal{A}^{m-1}(\ker \omega) = \{\iota_R \Theta \mid R \in \mathfrak{R}\}$.

The pair (Θ, ω) is a **premulticontact bundle structure** and (P, Θ, ω) is said to be a **premulticontact bundle** if, for $0 < k \leq N - m$, we have that:

(1) $\operatorname{rank} \mathcal{D}^{\mathfrak{R}} = m + k$.

(2) $\operatorname{rank} (\ker \omega \cap \ker \Theta \cap \ker \mathrm{d}\Theta) = k$.

(3) $\mathcal{A}^{m-1}(\ker \omega) = \{\iota_R \Theta \mid R \in \mathfrak{R}\}$,

In classical field theories we will be specially interested in the situation in which $P = \mathcal{E} \times \Lambda^{m-1}(\mathrm{T}^* M)$, where $\mathcal{E} \longrightarrow M$ is a (pre)multisymplectic bundle and, in particular, a jet bundle or a bundle of forms.

Multicontact structures of variational type

We are going to restrict the kind of (pre)multicontact structures we are interested in.

This is motivated by the following fact: If (P, Θ, ω) is a (pre)multicontact manifold, we can introduce a system of PDEs associated with the (pre)multicontact structure.

We want these equations, when expressed in coordinates, to coincide with those derived from the variational Herglotz principle for fields, which are also those obtained in the k-(co)contact formulation of non-conservative field theories.

In the particular case when $P \longrightarrow M$ are certain kinds of fiber bundles, we can formulate Lagrangian and Hamiltonian descriptions for these systems and the PDEs associated with the "variational" multicontact structure are the corresponding Euler–Lagrange (Herglotz) equations and the Hamilton–de Donder–Weyl (Herglotz) equations (we will maintain the usual terminology of the Lagrangian and the Hamiltonian formalisms of multisymplectic field theories).

Multicontact structures of variational type

Definition

If (P, Θ, ω) is a (pre)multicontact manifold such that

$$i(X)i(Y)\Theta = 0\,, \qquad \text{for every } X, Y \in \Gamma(\ker\omega)\,, \tag{1}$$

then (Θ, ω) is said to be a **variational (pre)multicontact structure** and (P, Θ, ω) is a **variational (pre)multicontact manifold**.

The terminology comes from the above comment and from the fact that this condition (1) is precisely what is imposed to the multisymplectic potential forms in the multisymplectic formulation of field theories in order to ensure that the theory is variational and, hence, it comes from a Lagrangian (in these cases, $\ker\omega$ is just the vertical distribution on the corresponding bundles).

Darboux coordinates

Now, from previous results, we can state a Darboux-like theorem for this class of (pre)multicontact manifolds:

Theorem

If (P, Θ, ω) *is a variational (pre)multicontact manifold, then there exist local charts of adapted coordinates* $(U; x^\mu, u^I, s^\mu, w^r)$ *(*$1 \leq \mu \leq m$*,* $1 \leq I \leq N - m - k$*,* $1 \leq r \leq k$*) in* P *such that the local expression of the (pre)multicontact form* Θ *is*

$$\Theta|_U = H(x^\nu, u^I, s^\nu)\, \mathrm{d}^m x + f_I^\mu(x^\nu, u^J)\, \mathrm{d}u^I \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu\,.$$

Furthermore, in these coordinates,

$$\sigma_\Theta|_U = \Gamma_\mu\, \mathrm{d}x^\mu = \frac{\partial H}{\partial s^\mu}\, \mathrm{d}x^\mu\,.$$

In most physical models of field theory, (x^μ) are spacetime coordinates, (u^I) are coordinates related to the physical fields, (w^r) are gauge variables, and (s^μ) are the 'contact variables' related to 'damping' or 'dissipative' phenomena and also to the variational action.

(Pre)multicontact variational systems: field equations

The equations for variational multicontact and premulticontact bundles can be stated using different geometric elements as follows:

Definition

Let (P, Θ, ω) be a variational (pre)multicontact bundle.

(1) The **(pre)multicontact field equations for sections** $\psi\colon M \longrightarrow P$ are

$$\iota_{\psi_{(m)}}(\Theta \circ \psi) = 0\,, \qquad \iota_{\psi_{(m)}}(\bar{\mathrm{d}}\Theta \circ \psi) = 0\,. \tag{2}$$

(2) The **(pre)multicontact field equations for τ-transverse, locally decomposable multivector fields $\mathbf{X} \in \mathfrak{X}^m(P)$** are

$$\iota_{\mathbf{X}}\Theta = 0\,, \qquad \iota_{\mathbf{X}}\bar{\mathrm{d}}\Theta = 0\,, \tag{3}$$

where the condition of τ-transversality is $\iota_{\mathbf{X}}\omega = 1$.

(3) The **(pre)multicontact field equations for Ehresmann connections** ∇ on $P \longrightarrow M$ are

$$\iota_{\nabla}\Theta = (m-1)\Theta\,, \qquad \iota_{\nabla}\bar{\mathrm{d}}\Theta = (m-1)\bar{\mathrm{d}}\Theta\,. \tag{4}$$

The relations among all these field equations are given by the following results:

Theorem

If $\mathbf{X} \in \mathfrak{X}^m(P)$ *is a representative of a class of* τ*-transverse and integrable* m*-multivector fields* $\{\mathbf{X}\} \subset \mathfrak{X}^m(P)$ *satisfying the (pre)multicontact field equations for multivector fields* (3)*, then the integral sections of* $\mathbf{X}$ *are solutions to the (pre)multicontact field equations for sections* (2)*.*

Conversely, if $\psi \colon M \longrightarrow P$ *is a solution to the (pre)multicontact field equations for sections* (2)*, then there exist a tubular neighborhood* $U \subset P$ *of* $\operatorname{Im}\psi$ *and a* τ*-transverse and integrable multivector field* $\mathbf{X} \in \mathfrak{X}^m(U)$ *such that:*

(1) ψ *is an integral section of* $\mathbf{X}$*.*

(2) $\mathbf{X}$ *is a solution to the (pre)multicontact field equations for multivector fields* (3) *on* $\operatorname{Im}\psi$*.*

(Pre)multicontact variational systems: field equations

Theorem

The (integrable) Ehresmann connections ∇ which are the solutions to the (pre)multicontact field equations for Ehresmann connections (4) *are locally associated with classes of (integrable) τ-transverse, locally decomposable multivector fields $\{\mathbf{X}\} \subset \mathfrak{X}^m(P)$ which are solutions to the (pre)multicontact field equations for multivector fields* (3)*, and conversely.*

As a last result, the field equations for sections can be expressed in an equivalent way which is analogous to what is commonly used to write such equations in the multisymplectic formulation of classical field theories (see Saunders):

Proposition

The (pre)multicontact field equations for sections (2) *are equivalent to*

$$\psi^*\Theta = 0\,, \qquad \psi^*\iota_Y\overline{\mathrm{d}}\Theta = 0\,, \qquad \textit{for every}\;\; Y \in \mathfrak{X}(P)\,. \tag{5}$$

(Pre)multicontact variational systems: field equations

Definition

A variational (pre)multicontact bundle (P, Θ, ω) along with some of the field equations (2), (3) or (4) is said to be a **(pre)multicontact system**.

Remark

In the premulticontact case, in general, for the premulticontact system (P, Θ, ω)*, the field equations for sections* $\psi\colon M \longrightarrow P$*, multivector fields* $\mathbf{X} \in \mathfrak{X}^m(P)$*, and Ehresmann connections* ∇ *on* P *are not compatible on* P *and a constraint algorithm must be implemented in order to find a submanifold* $P_f \hookrightarrow P$ *(when it exists) where there are integrable distributions whose associated multivector fields* $\mathbf{X}$ *and Ehresmann connections* ∇ *are solutions to the premulticontact field equations on* P_f *and are tangent to* P_f*.*

In this situation note that the constraint algorithm and the final solutions are independent of the Reeb vector fields selected for the premulticontact system, as a consequence of the construction of σ_Θ*.*

(Pre)multicontact variational systems: field equations

Summarizing,

- we have introduced different ways of setting the field equations in classical field theories: for sections, fields, and connections.
- The equations for sections, written in its two equivalent forms (2) and (5), give straightforwardly the system of PDEs to be solved to describe the behaviour of the system.
- On the other hand, the equations for multivector fields and connections give a more geometrical interpretation of the solutions (as distributions) that often make it easier to study and characterize qualitative properties of such solutions.
- In particular, these geometric characterizations are the most suitable in order to apply the constraint algorithms in the case of premulticontact theories.
- Note that one can write these equations for a general (pre)multicontact system although, if the structure is not variational, the resulting equations may not correspond to those of the Herglotz principle for fields.

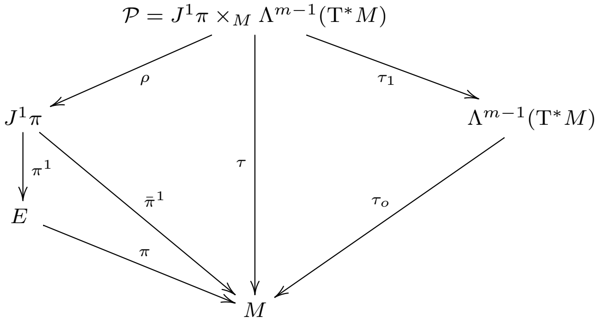
Outline

Geometric elements

Let $\pi : E \to M$ be a fiber bundle over the spacetime M, where $\dim M = m$, $\dim E = m + n$, and hence $\dim J^1\pi = m + n + mn$. In the Lagrangian setting, consider the bundle

$$\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M) \simeq J^1\pi \times \mathbb{R}^m \,,$$

whose natural projections are presented in the next diagram:



If (x^μ, y^i) are natural coordinates in E, then the induced natural coordinates in $\mathcal{P}$ are $(x^\mu, y^i, y^i_\mu, s^\mu)$ where, taking $\{\mathrm{d}^{m-1}x_\mu\}$ as the local basis of $\Lambda^{m-1}(\mathrm{T}^*M)$, we have that $\xi = s^\mu\, \mathrm{d}^{m-1}x_\mu$, for every $\xi \in \Lambda^{m-1}(\mathrm{T}^*M)$.

Note that, since $\Lambda^{m-1}(\mathrm{T}^*M)$ is a bundle of forms over M, it is endowed with a canonical structure $\theta \in \Omega^{m-1}(\Lambda^{m-1}(\mathrm{T}^*M))$, the **tautological form**, which is defined as follows: for every $\boldsymbol{\xi} \equiv (x, \xi) \in \Lambda^{m-1}(\mathrm{T}^*M)$ and $X^1_{\boldsymbol{\xi}}, \dots, X^{m-1}_{\boldsymbol{\xi}} \in \mathrm{T}_{\boldsymbol{\xi}}(\Lambda^{m-1}(\mathrm{T}^*M))$,

$$\theta_{\boldsymbol{\xi}}(X^1_{\boldsymbol{\xi}}, \cdots, X^{m-1}_{\boldsymbol{\xi}}) := \xi\Big(\mathrm{T}_{\boldsymbol{\xi}}\tau_o(X^1_{\boldsymbol{\xi}}), \dots, \mathrm{T}_{\boldsymbol{\xi}}\tau_o(X^{m-1}_{\boldsymbol{\xi}})\Big)\,.$$

Its local expression in natural coordinates is $\theta = s^\mu\, \mathrm{d}^{m-1}x_\mu$.

The canonical action form $\overline{S}$

Definition

The **canonical action form** is the differential form $\overline{S} \in \Omega^{m-1}(\mathcal{P})$ defined as

$$\overline{S} := \tau_1^* \theta \,,$$

or, what is equivalent, at every point $\mathrm{p} \in \mathcal{P}$,

$$\overline{S}_{\mathrm{p}}(X_{\mathrm{p}}^1, \cdots, X_{\mathrm{p}}^{m-1}) := \tau_1(\mathrm{p})_{\tau(\mathrm{p})}(\mathrm{T}_{\mathrm{p}}\tau(X_{\mathrm{p}}^1), \ldots, \mathrm{T}_{\mathrm{p}}\tau(X_{\mathrm{p}}^{m-1}))\,,$$

for every $X_{\mathrm{p}}^1, \ldots, X_{\mathrm{p}}^{m-1} \in \mathrm{T}_{\mathrm{p}}\mathcal{P}$.

Note that every section $\boldsymbol{\psi} \colon M \to \mathcal{P}$ of τ defines the $(m-1)$-form $\tau_1 \circ \boldsymbol{\psi} \in \Lambda^{m-1}(\mathrm{T}^* M)$ and then $\boldsymbol{\psi}^* \overline{S} = \tau_1 \circ \boldsymbol{\psi}$. It is also immediate to check that $\overline{S}$ is a τ-semibasic form, whose expression in coordinates is

$$\overline{S} = s^\mu \, \mathrm{d}^{m-1} x_\mu \,.$$

The terminology is justified because this form $\overline{S}$ is closely related to the action of the system: in fact, $\mathrm{d}\overline{S}$ is the **Lagrangian action** that appears in the *action functional*.

Holonomic sections

Definition

Let $\boldsymbol{\psi}\colon M \to \mathcal{P}$ be a section of the projection τ. Then $\boldsymbol{\psi}$ is a **holonomic section** in $\mathcal{P}$ if the section $\psi := \rho \circ \boldsymbol{\psi}\colon M \longrightarrow J^1\pi$ is holonomic in $J^1\pi$.

We also say that $\boldsymbol{\psi}$ is the **canonical prolongation** of ψ to $\mathcal{P}$.

Then, we can write $\boldsymbol{\psi} = (\psi, s) = (j^1\phi, s)$, where $s\colon M \longrightarrow \Lambda^{m-1}(\mathrm{T}^*M)$ is a section of the projection $\tau_0 : \Lambda^{m-1}(\mathrm{T}^*M) \longrightarrow M$.

SOPDEs

Definition

An m-multivector field $\mathbf{\Gamma} \in \mathfrak{X}^m(\mathcal{P})$ is a **second-order partial differential equation** (or SOPDE) in $\mathcal{P}$ if

(1) it is τ-transverse,

(2) it is integrable,

(3) the multivector field $\mathbf{X} := \Lambda^m \mathrm{T}\rho \circ \mathbf{\Gamma}$, which is obviously integrable and $\bar{\pi}^1$-transverse, is a SOPDE in $J^1\pi$.

An Ehresmann connection ∇ in $\mathcal{P}$ is a **second-order partial differential equation** (or SOPDE) in $\mathcal{P}$ if

(1) it is integrable,

(2) the natural restriction of ∇ to $J^1\pi$ is a SOPDE in $J^1\pi$.

Local expressions of SOPDEs

The local expression of a SOPDE multivector field in $\mathcal{P}$ verifying the transversality condition $\iota_{\mathbf{\Gamma}}\omega = 1$ is

$$\mathbf{\Gamma} = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu} \frac{\partial}{\partial y^i_\nu} + g^\nu_\mu \frac{\partial}{\partial s^\nu} \right) .$$

On the other hand, the local expression of a SOPDE connection is

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu} \frac{\partial}{\partial y^i_\nu} + g^\nu_\mu \frac{\partial}{\partial s^\nu} \right) .$$

As usual, multivector fields and connections in $\mathcal{P}$ which have these local expressions but are not integrable are called **semi-holonomic**.

A straightforward consequence of the above definitions is that $\mathbf{\Gamma} \in \mathfrak{X}^m(\mathcal{P})$ and ∇ are SOPDEs in $\mathcal{P}$ if, and only if, their integral sections are holonomic in $\mathcal{P}$.

Canonical endomorphism

Since $\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)$, the canonical endomorphism J of $J^1\pi$ can be extended to $\mathcal{P}$ in a natural way and has the same coordinate expression.

Denoting this extension with the same notation J, in natural coordinates

$$\mathrm{J} = \left(\mathrm{d}y^i - y^i_\mu \mathrm{d}x^\mu\right) \otimes \frac{\partial}{\partial y^i_\nu} \otimes \frac{\partial}{\partial x^\nu}\,.$$

Now we can state the Lagrangian formalism of field theories with dissipation in the multicontact setting.

Lagrangian density, function, form, and energy

A **Lagrangian density** is a τ-semibasic form $\mathcal{L} \in \Omega^m(\mathcal{P})$. If ω_M is the volume form in M, we have that $\mathcal{L} = L\,\tau^*\omega_M$, where $L \in \mathscr{C}^\infty(\mathcal{P})$ is the **Lagrangian function** associated to $\mathcal{L}$.

Definition

The **Lagrangian form** associated to $\mathcal{L}$ is the form

$$\Theta_{\mathcal{L}} = -\iota_{\mathrm{J}}\mathrm{d}\mathcal{L} - \mathcal{L} + \mathrm{d}\overline{S} \in \Omega^m(\mathcal{P}) \,,$$

and then $\overline{\mathrm{d}}\Theta_{\mathcal{L}} = \mathrm{d}\Theta_{\mathcal{L}} + \sigma_{\Theta_{\mathcal{L}}} \wedge \Theta_{\mathcal{L}}$.

In natural coordinates, the expression of the form $\Theta_{\mathcal{L}}$ is

$$\Theta_{\mathcal{L}} = -\frac{\partial L}{\partial y^i_\mu}\mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu + \left(\frac{\partial L}{\partial y^i_\mu} y^i_\mu - L\right)\mathrm{d}^m x + \mathrm{d}s^\mu \wedge \mathrm{d}^{m-1}x_\mu \,,$$

and the local function $E_{\mathcal{L}} := \dfrac{\partial L}{\partial y^i_\mu} y^i_\mu - L$ is called the **Lagrangian energy** associated with L. Therefore, $\sigma_{\Theta_{\mathcal{L}}} = \dfrac{\partial E_{\mathcal{L}}}{\partial s^\mu}\,\mathrm{d}x^\mu$.

Alternative definition of the Lagrangian form

The (pre)multicontact form $\Theta_{\mathcal{L}}$ in $\mathcal{P}$ can also be obtained in an equivalent way which is based on using the multisymplectic formalism for Lagrangian field theories:

Taking the restriction of the Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ to the fibers of the projection τ_1 (considering L with s^μ 'freezed'), since $\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)$, these fibers are identified with $J^1\pi$, and hence this restricted function is $L_s \in \mathscr{C}^\infty(J^1\pi)$.

We can construct the Poincaré–Cartan m-form $\Theta_{\mathcal{L}_s} \in \Omega^m(J^1\pi)$ associated with the Lagrangian density $\mathcal{L}_s = L_s\,\bar{\pi}^{1*}\omega$, which has local expression

$$\Theta_{\mathcal{L}_s} = \frac{\partial L_s}{\partial y^i_\mu}\,\mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu - \left(\frac{\partial L_s}{\partial y^i_\mu} y^i_\mu - L_s\right)\mathrm{d}^m x\,.$$

Proposition

The Lagrangian form associated with $\mathcal{L}$ is $\Theta_{\mathcal{L}} = -\rho^*\Theta_{\mathcal{L}_s} + \mathrm{d}\overline{S}$.

The Legendre maps

Definition

The **Legendre map** associated with the Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ is the map $\mathcal{F}L\colon \mathcal{P} \longrightarrow \mathcal{P}^*$ given by $\mathcal{F}L(y^i, y^i_\mu, s^\mu) = \left(y^i, \dfrac{\partial L}{\partial y^i_\mu}, s^\mu\right)$.

Proposition

For a Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$, the following conditions are equivalent:

(1) *The Legendre map $\mathcal{F}L$ is a local diffeomorphism.*

(2) *The Hessian matrix $(W^{\mu\nu}_{ij}) = \left(\dfrac{\partial^2 L}{\partial y^i_\mu \partial y^j_\nu}\right)$ is regular everywhere.*

(3) *The Lagrangian form $\Theta_\mathcal{L}$ is a multicontact form in $\mathcal{P}$ and $(\Theta_\mathcal{L}, \omega)$ is a multicontact structure.*

Definition

A Lagrangian function $L \in \mathscr{C}^\infty(\mathcal{P})$ is said to be **regular** if the above equivalent conditions hold. Otherwise L is a **singular** Lagrangian. In particular, L is said to be **hyperregular** if $\mathcal{F}L$ is a global diffeomorphism.

As we have seen, L is regular in $\mathcal{P}$ if, and only if, L_s is regular in $J^1\pi$, for every $s \in \Lambda^{m-1}(\mathrm{T}^*M)$.

Remark

Note that non-regular Lagrangians can induce premulticontact structures but also structures which are neither multicontact nor premulticontact. For example, the Lagrangian $L = \sum_{i=1}^{n} y^i_\mu s^\mu$ *yields a structure* $(\Theta_{\mathcal{L}}, \omega)$ *which has no Reeb distribution associated to it.*

Definition

The premulticontact bundle $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ is called a **(pre)multicontact Lagrangian system**.

Local expressions

Given a multicontact Lagrangian system $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$, the Reeb vector fields $(R_{\mathcal{L}})_\mu \in \mathfrak{R}_{\mathcal{L}} \subset \mathfrak{X}(\mathcal{P})$ are the only solutions to $\iota_{(R_{\mathcal{L}})_\mu}\Theta = \mathrm{d}^{m-1}x_\mu$.

Since L is regular, there exists the inverse $(W^{ij}_{\mu\nu})$ of the Hessian matrix, namely $W^{ij}_{\mu\nu}\dfrac{\partial^2 L}{\partial y^j_\nu \partial y^k_\gamma} = \delta^i_k \delta^\gamma_\mu$, and thus,

$$(R_{\mathcal{L}})_\mu = \frac{\partial}{\partial s^\mu} - W^{ji}_{\gamma\nu}\frac{\partial^2 \mathcal{L}}{\partial s^\mu \partial y^j_\gamma}\frac{\partial}{\partial y^i_\nu}\,.$$

Therefore, we have that

$$\sigma_{\Theta_{\mathcal{L}}} = -\frac{\partial L}{\partial s^\mu}\,\mathrm{d}x^\mu\,.$$

If $(\mathcal{P}, \Theta_{\mathcal{L}})$ is a premulticontact Lagrangian system, the Reeb vector fields are not uniquely determined from the equation $\iota_{(R_{\mathcal{L}})_\mu}\Theta = \mathrm{d}^{m-1}x_\mu$.

In general, the natural coordinates in $\mathcal{P}$ are not adapted coordinates for the (pre)multicontact structure $(\Theta_{\mathcal{L}}, \omega)$.

Multicontact Lagrangian field equations

Let $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ be a (pre)multicontact Lagrangian system.

(1) The **(pre)multicontact Lagrangian equations** for holonomic sections $\boldsymbol{\psi}\colon M \longrightarrow \mathcal{P}$ are

$$\iota_{\boldsymbol{\psi}^{(m)}}(\Theta_{\mathcal{L}} \circ \boldsymbol{\psi}) = 0\,, \qquad \iota_{\boldsymbol{\psi}^{(m)}}(\overline{\mathrm{d}}\Theta_{\mathcal{L}} \circ \boldsymbol{\psi}) = 0\,. \tag{6}$$

or equivalently

$$\boldsymbol{\psi}^*\Theta_{\mathcal{L}} = 0\,, \qquad \boldsymbol{\psi}^*\iota_Y\overline{\mathrm{d}}\Theta_{\mathcal{L}} = 0\,, \qquad \text{for every } \; Y \in \mathfrak{X}(\mathcal{P})\,. \tag{7}$$

(2) The **(pre)multicontact Lagrangian equations for τ-transverse, locally decomposable multivector fields $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^m(\mathcal{P})$** are

$$\iota_{\mathbf{X}_{\mathcal{L}}}\Theta_{\mathcal{L}} = 0\,, \qquad \iota_{\mathbf{X}_{\mathcal{L}}}\overline{\mathrm{d}}\Theta_{\mathcal{L}} = 0\,, \tag{8}$$

where the condition of τ-transversality is $\iota_{\mathbf{X}_{\mathcal{L}}}\omega = 1$.

An m-multivector field solution to these equations is called a **Lagrangian multivector field**.

(3) The **(pre)multicontact Lagrangian equations for Ehresmann connections $\nabla_{\mathcal{L}}$** on $\mathcal{P} \longrightarrow M$ are

$$\iota_{\nabla_{\mathcal{L}}}\Theta_{\mathcal{L}} = (m-1)\Theta_{\mathcal{L}}\,, \qquad \iota_{\nabla_{\mathcal{L}}}\overline{\mathrm{d}}\Theta_{\mathcal{L}} = (m-1)\overline{\mathrm{d}}\Theta_{\mathcal{L}}\,. \tag{9}$$

An Ehresmann connection solution to these equations is called a **Lagrangian connection**.

Multicontact Lagrangian field equations

Proposition

Let $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ be a multicontact (i.e., regular) Lagrangian system. Then:

(1) *The multicontact Lagrangian field equations for multivector fields* (8) *and for Ehresmann connections* (9) *have solutions on $\mathcal{P}$. The solutions are not unique if $m > 1$.*

(2) *The Lagrangian m-multivector fields $\mathbf{X}_{\mathcal{L}}$ solution to equations* (8) *and the corresponding Ehresmann connections $\nabla_{\mathcal{L}}$ in $\mathcal{P}$ which are associated with the classes $\{\mathbf{X}_{\mathcal{L}}\}$ and are solutions to* (9), *are semi-holonomic.*

(3) *In addition, if $\mathbf{X}_{\mathcal{L}}$ and $\nabla_{\mathcal{L}}$ are semi-holonomic and integrable solutions, namely* SOPDE*s, their integral sections are solutions to the multicontact Euler–Lagrange field equations* (6) *or* (7).
In this case, these SOPDE*s $\mathbf{X}_{\mathcal{L}}$ and $\nabla_{\mathcal{L}}$ are called the* ***Euler–Lagrange multivector fields*** *and* ***connections*** *associated with the Lagrangian function L.*

Of course all these equations are the same as those obtained in the k-cocontact formulation of non-conservative field theories and also match those of k-contact formalism when the Lagrangian function does not depend on the spacetime variables x^μ. In coordinates, they read

$$\frac{\partial s^\mu}{\partial x^\mu} = L \circ \boldsymbol{\psi}\,, \qquad (10)$$

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^i_\mu} \circ \boldsymbol{\psi}\right) = \left(\frac{\partial L}{\partial y^i} + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\right) \circ \boldsymbol{\psi}\,,$$

Furthermore, equation (10) relates the canonical action form with the variational formulation through the Lagrangian density. In fact, we have the following.

Corollary

If ψ is a holonomic section such that $\psi^*\Theta_{\mathcal{L}} = 0$, we have that

$$\mathrm{d}(\overline{S} \circ \boldsymbol{\psi}) = \mathcal{L} \circ \boldsymbol{\psi}\,.$$

A note on the singular case

As in the case of premultisymplectic field theories, when L is not regular and $(\mathcal{P}, \Theta_{\mathcal{L}}, \omega)$ is a premulticontact system, the field equations (6), (7), (8), or (9) have no solutions everywhere on $\mathcal{P}$, in general.

In the most favourable situations, these equations have solutions on a submanifold $\mathcal{P}$ which is obtained by applying a suitable constraint algorithm.

Nevertheless, solutions to equations (8) or (9) are not necessarily SOPDEs and, as a consequence, if they are integrable, their integral sections are not necessarily holonomic; so this requirement must be imposed as an additional condition.

Hence, the final objective consists in finding the maximal submanifold $\mathcal{S}_f$ of $\mathcal{P}$ where there are holonomic distributions whose associated Lagrangian multivector fields $\mathbf{X}_{\mathcal{L}}$ and connections $\nabla_{\mathcal{L}}$ are SOPDE solutions to the premulticontact Lagrangian field equations on $\mathcal{S}_f$ and are tangent to $\mathcal{S}_f$.

Outline

Navier–Cauchy equations

The steady Navier–Cauchy equations

$$(\lambda+\mu)\nabla(\nabla\cdot u)+\mu\Delta u=0\,,$$

where λ, μ are Lamé parameters, can be obtained from the Lagrangian function

$$\begin{aligned}L(q^i, v^i_j) = \left(\frac{\lambda}{2}+\mu\right)&\left((v^1_1)^2+(v^2_2)^2+(v^3_3)^2\right)\\ &+\frac{\mu}{2}\left((v^1_2)^2+(v^2_1)^2+(v^3_1)^2+(v^1_3)^2+(v^3_2)^2+(v^2_3)^2\right)\\ &+(\lambda+\mu)(v^1_1v^2_2+v^1_1v^3_3+v^2_2v^2_3)\,,\end{aligned}$$

with $i, j = 1, 2, 3$.
Contactifying this Lagrangian function by adding an extra term, namely

$$L_c(q^i, v^i_j, s^j) = L + \gamma_j s^j\,,$$

where $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3)\in\mathbb{R}^3$, we obtain the following modified Navier–Cauchy equations:

$$(\lambda+\mu)\nabla(\nabla\cdot u)+\mu\Delta u=\boldsymbol{\gamma}(\lambda+\mu)\nabla\cdot u+\mu\boldsymbol{\gamma}\cdot\nabla u\,,$$

where we understand $\boldsymbol{\gamma}\cdot\nabla u=(\gamma_1\cdot\nabla u^1,\gamma_2\cdot\nabla u^2,\gamma_3\cdot\nabla u^3)$.

Further work

- Find out whether this dissipation term (or other possible ones) have a physical meaning.
- Extend this to time-dependent Navier–Cauchy equation.
- Run computer simulations of these equations.

Outline

References

- M. de León, J. Gaset, M. C. Muñoz-Lecanda, X. Rivas and N. Román-Roy, "Multicontact formulation for non-conservative field theories", *J. Phys. A: Math. Theor.* **56**(2):025201, 2023. doi:10.1088/1751-8121/acb575
- J. Gaset, X. Gràcia, M. Muñoz-Lecanda, X. Rivas, N. Román-Roy, "A contact geometry framework for field theories with dissipation", *Ann. Phys.* **414**:168092, 2020. doi:10.1016/j.aop.2020.168092
- J. Gaset, M. Lainz, A. Mas and X. Rivas, "The Herglotz variational principle for dissipative field theories", *Geom. Mech.* **1**(2):153–178, 2024. doi:10.1142/S2972458924500060