Multiscale Analysis of Effective Properties in Remodelling Composite Media

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International Workshop *Geometrical aspects of material modelling*

Based on

- A. Ramírez-Torres, S. Di Stefano, A. Grillo, R. Rodríguez-Ramos, J. Merodio, R. Penta (2018) *An asymptotic homogenization approach to the microstructural evolution of heterogeneous media.* International Journal of Non-Linear Mechanics, 106:245–257

- A. Giammarini, A. Ramírez-Torres, A. Grillo (2024) *Effective elasto-(visco)plastic coefficients of a bi-phasic composite material with scale-dependent size effects.* Mathematical Methods in the Applied Sciences (MMAS) (Accepted)

- A. Ramírez-Torres, A. Roque-Piedra, A. Giammarini, A. Grillo, R. Rodríguez-Ramos (2024) *Analytical expressions for the effective coefficients of fibre-reinforced composite materials under the influence of inelastic distortions.* Zeitschrift fuer Angewandte Mathematik und Mechanik (ZAMM) (Under review)

Introduction

Motivations

- Our underlying goal is to explore questions related to biology, as biological systems can actively modify their internal structure in response to internal and external factors (e.g. of chemo-mechanical origin).
    - Examples of internal transformations comprise, for instance, damage, growth, and remodelling.
    - They typically induce variations of the mechanical properties of the media in which they take place, e.g., by varying their stiffness.
        - To migrate, the cell body must modify its shape and stiffness to interact with the surrounding tissue structures [1].

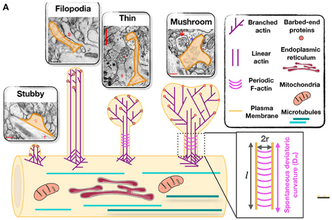

Figure: Dendritic spine stiffness and morphology correlate with actin filament activity [2]. Figure taken from [3].

[1] P. Friedl and K.Wolf. In: *Nat. Rev. Cancer* 5 (2003), pp. 362–374.

[2] Smith BA et al. In: *Biophys J.* 92 (2007), pp. 1419–30.

[3] Haleh Alimohamadi et al. In: *Front. Physiol.* 12 (2021).

Introduction

Motivations

- A common feature of damage, growth, and remodelling is their being intrinsically multiscale phenomena.
- This multiscale nature combines with the complex architecture of biological tissues, which consists of several constituents differing in shape, functionality, and mechanical properties.

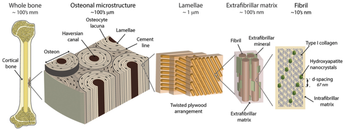

Figure: Bone's multi-scale composite structure [4].

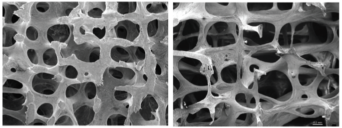

Figure: Normal (left) and osteoporotic (right) bone [5].

⚠ *Our aim is to investigate the impact of microstructural changes on the macroscopic mechanical properties of a composite medium.*

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[4] Elizabeth A. Zimmermann et al. In: *Scientific Reports* 6.1 (2016).

[5] Alain Goriely. Springer Berlin Heidelberg, 2017.

Kinematics

⚠ *The evolution of the microstructure is described by introducing a kinematic descriptor, virtually independent of the changes in the shape of the body.*

- Deformation gradient tensor

$$\boldsymbol{F}_\eta(X,t) := \boldsymbol{I} + \operatorname{Grad} \boldsymbol{u}_\eta(X,t)$$

- Bilby-Kröner-Lee (BKL) decomposition of the deformation gradient tensor [6]

$$\boldsymbol{F}_\eta(X,t) = \boldsymbol{F}_{e\eta}(X,t)\boldsymbol{K}_\eta(X,t)$$

with

— $\boldsymbol{K}_\eta$ - distortions of inelastic nature occurring at the internal structure

— $\boldsymbol{F}_{e\eta}$ - elastic contribution to the visible deformation

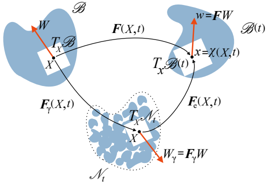


Figure: Schematic of the structural rearrangement [7].

Inelastic processes that do not play a role in promoting growth [8]

$$J_{\boldsymbol{K}\eta} = 1 \quad \text{or rephrased as} \quad \boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0$$

[6] Milan Mićunović. *Thermomechanics of Viscoplasticity*. Springer New York, 2009.

[7] Salvatore Di Stefano et al. In: *International Journal of Non-Linear Mechanics* 106 (2018), pp. 174–187.

[8] M. E. Gurtin and L. Anand. In: *International Journal of Plasticity* 21 (2005), pp. 2297–2318.

Governing equations

– **Governing equations**

$$\boxed{\mathrm{Div}(\boldsymbol{P}_\eta) = \boldsymbol{0}, \text{ with } \boldsymbol{P}_\eta = \left(\frac{\partial \hat{\Psi}_{\nu\eta}}{\partial \boldsymbol{F}_{\mathrm{e}\eta}} \circ (\boldsymbol{F}_{\mathrm{e}\eta})\right) \boldsymbol{K}_\eta^{-\mathrm{T}}}$$

$$\boldsymbol{Y}_\eta = \boldsymbol{Z}_\eta$$

⚠ *While $\boldsymbol{Y}_\eta$ can be derived from constitutive principles, the external non-conventional force $\boldsymbol{Z}_\eta$ can be assigned according to the specific phenomenon under investigation* [9,10].

⚠ *Within this framework (theory of order zero in $\boldsymbol{K}_\eta$), we do not need to work with the balance of generalised forces at this point. We can come back to it at a later stage.*

– **Interface conditions & Isochoricity constraint**

$$\boxed{\boldsymbol{u}_{\mathrm{m}} = \boldsymbol{u}_{\mathrm{f}}}$$

$$\boxed{\boldsymbol{P}_{\mathrm{m}}\boldsymbol{N} = \boldsymbol{P}_{\mathrm{f}}\boldsymbol{N}}$$

$$\boxed{\boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0}$$

[9] Antonio DiCarlo and Sara Quiligotti. In: *Mechanics Research Communications* 29.6 (2002), pp. 449–456.

[10] Alfio Grillo and Salvatore Di Stefano. In: *Mathematics and Mechanics of Complex Systems* 11 (2023), pp. 57–86.

Constitutive assumptions

⚠ *Motivated by our scope of providing analytical expressions, we choose De Saint-Venant energy density*

$$\hat{\Psi}_{\nu\eta} \circ (\boldsymbol{F}_{\mathrm{e}\eta}) = \check{\Psi}_{\nu\eta} \circ (\boldsymbol{E}_{\mathrm{e}\eta}) = \tfrac{1}{2}\boldsymbol{E}_{\mathrm{e}\eta} : \mathscr{C}_{\nu\eta} : \boldsymbol{E}_{\mathrm{e}\eta}, \quad \text{with } \boldsymbol{E}_{\mathrm{e}\eta} := \tfrac{1}{2}[(\boldsymbol{F}_{\mathrm{e}\eta})^{\mathrm{T}}\boldsymbol{F}_{\mathrm{e}\eta} - \boldsymbol{I}]$$

⚠ *We opt to work under the assumption of infinitesimal elastic deformations and retain non-linear traits of the solid constituents through the remodelling tensor* $\boldsymbol{K}_\eta$.

⚠ *This choice remains pertinent in biological scenarios, such as in bones* [11].

$$\boldsymbol{P}_\eta^{\mathrm{lin}} = \mathscr{G}_\eta : \mathrm{Grad}\,\boldsymbol{u}_\eta - \mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta}$$

$$\mathscr{G}_\eta := \mathscr{C}_\eta - \boldsymbol{I}\,\underline{\otimes}\left(\mathscr{C}_\eta : \boldsymbol{E}_{\boldsymbol{K}_\eta}\right)$$

$$\mathscr{C}_\eta := \left((\boldsymbol{K}_\eta)^{-1}\underline{\otimes}(\boldsymbol{K}_\eta)^{-1}\right) : \mathscr{C}_{\nu\eta} : \left((\boldsymbol{K}_\eta)^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta)^{-\mathrm{T}}\right)$$

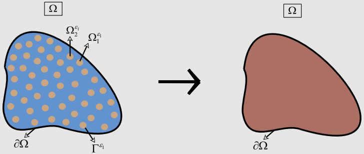
[11] Louna Z, Goda I, and Ganghoffer JF. In: *Continuum Mechanics and Thermodynamics* 31 (2019), pp. 1339–1367.

Asymptotic Homogenisation

General notes

The technique has been successfully used to investigate various physical systems.

Soft and hard tissues [12,13], *Tissue engineering* [14]



- Homogenised characterisation of the system.
- *Ciphers* the information at the micro-scale (e.g. geometry, functionality and mechanical properties) in the so-called *effective coefficients*.

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[12] Raimondo Penta and Alf Gerisch. In: *Computation and Visualization in Science* 17 (2015), pp. 185–201.

[13] William J. Parnell et al. In: *Biomechanics and Modeling in Mechanobiology* 11.6 (2011), pp. 883–901.

[14] Z. Fang et al. In: *Applied Bionics and Biomechanics* 2.1 (2005), pp. 17–29.

Asymptotic Homogenisation

Separation of scales and formal expansion

- We identify two well-separated length scales

  ℓ – associated with the internal structure of the composite
  L – characteristic of the composite as a whole

  and introduce the smallness parameter

$$\varepsilon = \frac{\ell}{L} \ll 1$$

- A given quantity $\Phi_\eta(X,t)$ is written in a multiscale fashion as

$$\Phi_\eta(X,t) = \Phi_\eta^\varepsilon(\tilde{X},\tilde{Y},t)$$

  with $\tilde{X} := L^{-1}X$ being the *macroscopic* variable, and $\tilde{Y} := \ell^{-1}X = \varepsilon^{-1}\tilde{X}$ the *microscopic* variable.

- Thus,

$$\frac{\partial \Phi_\eta(X,t)}{\partial X_A} = \frac{1}{L}\left(\frac{\partial \Phi_\eta^\varepsilon(\tilde{X},\tilde{Y},t)}{\partial \tilde{X}_A} + \varepsilon^{-1}\frac{\partial \Phi_\eta^\varepsilon(\tilde{X},\tilde{Y},t)}{\partial \tilde{Y}_A}\right)$$

Asymptotic Homogenisation

Periodicity and macroscopic uniformity

- *Local periodicity*: There exists a family of vectors $\boldsymbol{r}(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 \boldsymbol{E}_1 + \alpha_2 \boldsymbol{E}_2 + \alpha_3 \boldsymbol{E}_3$, such that

$$\Phi^{\varepsilon}_{\eta}(X, Y_b, t) = \Phi^{\varepsilon}_{\eta}(X, Y_b + \boldsymbol{r}(\alpha_1, \alpha_2, \alpha_3), t)$$

- *Macroscopic uniformity* [13]

$$\partial_{X_A} \left(\int_{\mathscr{Y}_\eta} \Phi^{\varepsilon}_{\eta}(X, Y, t) \mathrm{d}Y \right) = \int_{\mathscr{Y}_\eta} \partial_{X_A} \Phi^{\varepsilon}_{\eta}(X, Y, t) \mathrm{d}Y$$

⚠ *Allows to choose the elementary cell ($\mathscr{Y}$) independently of the macroscopic variable X, so that it is representative of the composite's micro-structure.*

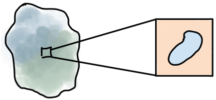

Figure: Schematic of the macroscopic uniformity assumption

[15] M. H. Holmes. *Introduction to Perturbation Methods*. Springer, New York, 2013.

Asymptotic Homogenisation

Multiscale governing equations

Governing equations

$$\boxed{\frac{1}{L}(\mathrm{Div}_X + \varepsilon^{-1}\mathrm{Div}_Y)(\boldsymbol{P}_\eta^{\varepsilon\mathrm{lin}}) = \boldsymbol{0}}$$

$$\boldsymbol{Y}_\eta^{\varepsilon\mathrm{lin}} = \boldsymbol{Z}_\eta^{\varepsilon\mathrm{lin}}$$

Interface conditions

$$\boxed{\boldsymbol{u}_\mathrm{m}^\varepsilon = \boldsymbol{u}_\mathrm{f}^\varepsilon}$$

$$\boxed{\boldsymbol{P}_\mathrm{m}^{\varepsilon\mathrm{lin}}\boldsymbol{N} = \boldsymbol{P}_\mathrm{f}^{\varepsilon\mathrm{lin}}\boldsymbol{N}}$$

Isochoricity constraint

$$\boxed{(\boldsymbol{K}_\eta^\varepsilon)^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta^\varepsilon = 0}$$

where

$$\boldsymbol{P}_\eta^{\varepsilon\mathrm{lin}} = \frac{1}{L}\mathscr{G}_\eta^\varepsilon : (\mathrm{Grad}_X + \varepsilon^{-1}\mathrm{Grad}_Y)\,\boldsymbol{u}_\eta^\varepsilon - \mathscr{C}_\eta^\varepsilon : \boldsymbol{E}^\varepsilon_{\boldsymbol{K}_\eta}$$

$$\mathscr{G}_\eta^\varepsilon = \mathscr{C}_\eta^\varepsilon - \boldsymbol{I}\underline{\otimes}(\mathscr{C}_\eta^\varepsilon : \boldsymbol{E}^\varepsilon_{\boldsymbol{K}_\eta})$$

$$\mathscr{C}_\eta^\varepsilon = \Big((\boldsymbol{K}_\eta^\varepsilon)^{-1}\underline{\otimes}(\boldsymbol{K}_\eta^\varepsilon)^{-1}\Big) : \mathscr{C}^\varepsilon_{\nu\eta} : \Big((\boldsymbol{K}_\eta^\varepsilon)^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta^\varepsilon)^{-\mathrm{T}}\Big)$$

⚠ *Writing of formal expansions for the unknowns and substitution into the model equations*

$$\boldsymbol{u}_\eta^\varepsilon(X,Y,t) = \sum_{k=0}^{+\infty}\varepsilon^k\boldsymbol{u}_\eta^{(k)}(X,Y,t) \quad \text{and} \quad \boldsymbol{K}_\eta^\varepsilon(X,Y,t) = \sum_{k=0}^{+\infty}\varepsilon^k\boldsymbol{K}_\eta^{(k)}(X,Y,t)$$

Asymptotic Homogenisation

Cell problems

The first cell problem

$$\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_Y\, \boldsymbol{u}_\eta^{(0)}) = \boldsymbol{0}$$
$$\boldsymbol{u}_{\mathrm{m}}^{(0)} = \boldsymbol{u}_{\mathrm{f}}^{(0)}$$
$$\frac{1}{L}(\mathscr{G}_{\mathrm{m}}^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_{\mathrm{m}}^{(0)})\boldsymbol{N} = \frac{1}{L}(\mathscr{G}_{\mathrm{f}}^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_{\mathrm{f}}^{(0)})\boldsymbol{N}$$

where

$$\mathscr{G}_\eta^{(0)} := \mathscr{C}_\eta^{(0)} - \boldsymbol{I}\underline{\otimes}(\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)})$$
$$\mathscr{C}_\eta^{(0)} := ((\boldsymbol{K}_\eta^{(0)})^{-1}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-1}) : \mathscr{C}_{\nu\eta} : ((\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}}\underline{\otimes}(\boldsymbol{K}_\eta^{(0)})^{-\mathrm{T}})$$
$$\boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)} := \frac{1}{2}[(\boldsymbol{K}_\eta^{(0)})^{\mathrm{T}}\boldsymbol{K}_\eta^{(0)} - \boldsymbol{I}]$$

⚠ *Under strong ellipticity criteria, we can write* $\boldsymbol{u}_\eta^{(0)}(X, Y, t) = \boldsymbol{u}^{(0)}(X, t)$ [16].

⚠ *Within this framework (zero-grade theory in* $\boldsymbol{K}_\eta$*) together with the above, the evolution law for the inelastic distortions will not provide further cell problems accompanying the ones stemming from the balance of linear momentum as these will become an identity.*

[16] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

Asymptotic Homogenisation

Second cell problem

The second cell problem related to the balance of linear momentum is

$$\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_Y \boldsymbol{u}_\eta^{(1)}) = -\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_X \boldsymbol{u}^{(0)} - L\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)})$$

⚠ *The existence of a solution requires the average of the hand-right-sides to be zero (e.g. $\mathscr{C}_{\nu\eta}$ and $\boldsymbol{K}_\eta^{(0)}$ are assumed to be Y-constant for each $\eta = \mathrm{m}, \mathrm{f}$).*

Owing to its linear nature in $\boldsymbol{u}_\eta^{(1)}$, we represent $\boldsymbol{u}_\eta^{(1)}$ by the *ansatz* [17]

$$\boldsymbol{u}_\eta^{(1)}(X, Y, t) = \boldsymbol{\xi}_\eta(X, Y, t) : \mathrm{Grad}_X \boldsymbol{u}^{(0)}(X, t) + \boldsymbol{\omega}_\eta(X, Y, t)$$

The third-order tensor field $\boldsymbol{\xi}_\eta$ is solution of

$$\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{G}_\eta^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi}_\eta) = -\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{G}_\eta^{(0)})$$

$$\boldsymbol{\xi}_\mathrm{m} = \boldsymbol{\xi}_\mathrm{f}$$

$$\frac{1}{L}(\mathscr{G}_\mathrm{m}^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi}_\mathrm{m} - \mathscr{G}_\mathrm{f}^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi}_\mathrm{f})\boldsymbol{N} = -\frac{1}{L}(\mathscr{G}_\mathrm{m}^{(0)} - \mathscr{G}_\mathrm{f}^{(0)})\boldsymbol{N}$$

The vector field $\boldsymbol{\omega}_\eta$ is solution of

$$\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{G}_\eta^{(0)} : \mathrm{Grad}_Y \boldsymbol{\omega}_\eta) = \frac{1}{L^2}\mathrm{Div}_Y(\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)})$$

$$\boldsymbol{\omega}_\mathrm{m} = \boldsymbol{\omega}_\mathrm{f}$$

$$\frac{1}{L}(\mathscr{G}_\mathrm{m}^{(0)} : \mathrm{Grad}_Y \boldsymbol{\omega}_\mathrm{m} - \mathscr{G}_\mathrm{f}^{(0)} : \mathrm{Grad}_Y \boldsymbol{\omega}_\mathrm{f})\boldsymbol{N} = \frac{1}{L}(\mathscr{C}_\mathrm{m}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\mathrm{f}}^{(0)} - \mathscr{C}_\mathrm{f}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\mathrm{m}}^{(0)})\boldsymbol{N}$$

[17] Ariel Ramírez-Torres et al. In: *International Journal of Non-Linear Mechanics* 106 (2018), pp. 245–257.

Asymptotic Homogenisation technique

Second cell problem

⚠ *In the absence of inelastic distortions, i.e.* $\boldsymbol{K}_\eta = \boldsymbol{I}$, *the cell problem reduces to the classical one in homogenisation theory* [18], *i.e.*

$$\frac{1}{L^2}\mathrm{Div}_Y(\mathscr{C}_{\nu\eta} + \mathscr{C}_{\nu\eta} : \mathrm{TGrad}_Y\boldsymbol{\xi}_\eta) = \boldsymbol{0}$$

$$\boldsymbol{\xi}_\mathrm{m} = \boldsymbol{\xi}_\mathrm{f}$$

$$\frac{1}{L}(\mathscr{C}_{\nu\mathrm{m}} : \mathrm{TGrad}_Y\boldsymbol{\xi}_\mathrm{m} - \mathscr{C}_{\nu\mathrm{f}} : \mathrm{TGrad}_Y\boldsymbol{\xi}_\mathrm{f}) = \boxed{-\frac{1}{L}(\mathscr{C}_{\nu\mathrm{m}} - \mathscr{C}_{\nu\mathrm{f}})}$$

⚠ *Nontrivial solutions are not driven by prescribed tractions or displacements, but due to the interface loadings which appear in the stress jump conditions.*

[18] D. Cioranescu and P. Donato. *An introduction to homogenization.* Oxford: Oxford University Press, 1999.

Asymptotic Homogenisation technique

Homogenised problem

- The homogenised balance of linear momentum reads

$$\mathrm{Div}_X \left(\mathscr{G}_{\mathrm{eff}} : \mathrm{Grad}_X \boldsymbol{u}^{(0)} + \boldsymbol{D}_{\mathrm{eff}} \right) = \boldsymbol{0}$$

⚠ *Concerning the original balance of linear momentum, the contribution of the elastic and inelastic properties at the lower scales, alongside the geometrical features of the internal structure, are encoded in $\mathscr{G}_{\mathrm{eff}}$ and $\boldsymbol{D}_{\mathrm{eff}}$.*

- The effective coefficients are

$$\mathscr{G}_{\mathrm{eff}} := \langle \mathscr{G}^{(0)} + \mathscr{G}^{(0)} : \mathrm{TGrad}_Y \boldsymbol{\xi} \rangle$$

$$\boldsymbol{D}_{\mathrm{eff}} := \langle \mathscr{G}^{(0)} : \mathrm{Grad}_Y \boldsymbol{\omega} - \mathscr{C}^{(0)} : \boxed{\boldsymbol{E}_{\boldsymbol{K}}^{(0)}} \rangle$$

The *average* over the cell $\mathscr{Y} = \mathscr{Y}_{\mathrm{m}} \sqcup \mathscr{Y}_{\mathrm{f}}$ of $\Phi^{\varepsilon}(X, Y, t) = \Phi^{\varepsilon}_{\mathrm{m}}(X, Y, t)\Upsilon_{\mathrm{m}}(Y) + \Phi^{\varepsilon}_{\mathrm{f}}(X, Y, t)\Upsilon_{\mathrm{f}}(Y)$, with $\Upsilon_{\eta}(Y)$ being the indicator function of $\mathscr{Y}_{\eta}$, is

$$\langle \Phi \rangle (X, t) := \frac{1}{|\mathscr{Y}|} \int_{\mathscr{Y}} \Phi^{\varepsilon}(X, Y, t) \, \mathrm{d}V(Y) = \frac{1}{|\mathscr{Y}|} \sum_{\eta \in \{\mathrm{m}, \mathrm{f}\}} \int_{\mathscr{Y}_{\eta}} \Phi^{\varepsilon}_{\eta}(X, Y, t) \, \mathrm{d}V(Y)$$

Asymptotic Homogenisation
Challenges

One of our key motivations is to circumvent the computational complexities involved in determining the effective properties of such active composites which requires addressing the interactions across different scales of the *cell* and *homogenised problems* [19].

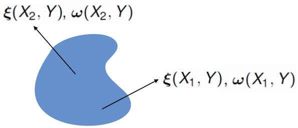

Figure: Macro-scale dependence

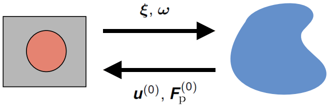

Figure: Transfer of information (properties, unknowns, etc.)

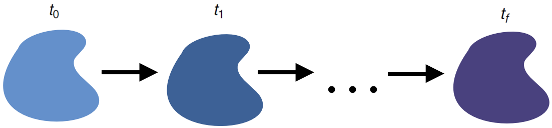

Figure: Time-dependent way.

[19] Ariel Ramírez-Torres et al. In: *International Journal of Non-Linear Mechanics* 106 (2018), pp. 245–257.

Benchmark problem I

Multilayered elasto-plastic composite

For the statement of the flow rule, we rely on the framework established in [20,21]

$$\boldsymbol{Y}_\eta = \boldsymbol{Z}_\eta \quad \Leftrightarrow \quad \begin{cases} \mathrm{DevSym}\{\sigma_\eta \tau_\eta \dot{\boldsymbol{K}}_\eta \boldsymbol{K}_\eta^{-1} - J_{\boldsymbol{K}_\eta} \boldsymbol{\Sigma}_{\nu\eta}\} = \boldsymbol{O} \\ \boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0 \\ \dot{\boldsymbol{K}}_\eta \boldsymbol{K}_\eta^{-1} - \boldsymbol{K}_\eta^{-\mathrm{T}} \dot{\boldsymbol{K}}_\eta^{\mathrm{T}} = \boldsymbol{O} \end{cases}$$

where

- σ_η is a constant having physical units of stress and representing the *initial yield stress* of the material
- τ_η is a *characteristic time scale* of the remodeling distortions
- $\boldsymbol{\Sigma}_{\nu\eta} := J_{\boldsymbol{K}_\eta}^{-1} \boldsymbol{K}_\eta^{-\mathrm{T}} \boldsymbol{F}_\eta^{\mathrm{T}} \boldsymbol{P}_\eta \boldsymbol{K}_\eta^{\mathrm{T}}$ is the Mandel stress tensor associated with the natural state

⚠ *The* DevSym *operator extracts only 5 linearly independent scalar equations. Thus, we solve explicitly the kinematic constraints of isochoricity (in differential form) and of null spin of the remodelling distortions.*

[20] M. E. Gurtin and L. Anand. In: *International Journal of Plasticity* 21 (2005), pp. 2297–2318.

[21] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

University of Glasgow

Benchmark problem I

Multilayered elasto-plastic composite

The homogenised flow rule reads [22]

$$
\begin{cases}
\operatorname{DevSym}\left\{\gamma^{\mathrm{eff}}\overline{\boldsymbol{K}^{(0)}}^{\cdot}(\boldsymbol{K}^{(0)})^{-1}-\sum_{\eta=1,2}\langle\boldsymbol{\Sigma}^{(0)}_{\nu\eta\mathrm{lin}}\rangle_\eta\right\}=\boldsymbol{O}\\
(\boldsymbol{K}^{(0)})^{-\mathrm{T}}:\overline{\boldsymbol{K}^{(0)}}^{\cdot}=0\\
\overline{\boldsymbol{K}^{(0)}}^{\cdot}(\boldsymbol{K}^{(0)})^{-1}-(\boldsymbol{K}^{(0)})^{-\mathrm{T}}\overline{(\boldsymbol{K}^{(0)})^{\mathrm{T}}}^{\cdot}=\boldsymbol{O}
\end{cases}
$$

where $\gamma^{\mathrm{eff}}:=\sum_{\eta=1,2}\varphi_\eta\sigma_\eta\tau_\eta$ denotes the *effective viscosity* and

$$
\begin{aligned}
\sum_{\eta=1,2}\langle\boldsymbol{\Sigma}^{(0)}_{\nu\eta\mathrm{lin}}\rangle_\eta=&\frac{2}{L_0}\frac{1}{\det\boldsymbol{K}^{(0)}}(\boldsymbol{K}^{0})^{-\mathrm{T}}\left\{\mathscr{G}_{\mathrm{eff}}:\mathrm{Grad}_X\boldsymbol{u}^{(0)}+\boldsymbol{D}_{\mathrm{eff}}+\frac{1}{2}\left[\sum_{\eta=1,2}\varphi_\eta L_0\left(\mathscr{C}^{(0)}_\eta:\boldsymbol{E}^{(0)}_{\boldsymbol{K}}\right)\right.\right.\\
&\left.\left.-\sum_{\eta=1,2}\mathscr{C}^{(0)}_\eta:\left(\varphi_\eta\mathbb{I}_4+\langle\mathrm{TGrad}_Y\boldsymbol{\xi}_\eta\rangle_\eta\right):\mathrm{Grad}_X\boldsymbol{u}^{(0)}-\sum_{\eta=1,2}\mathscr{C}^{(0)}_\eta:\langle\mathrm{Grad}_Y\boldsymbol{\omega}_\eta\rangle_\eta\right]\right\}(\boldsymbol{K}^{(0)})^{\mathrm{T}}
\end{aligned}
$$

⚠ *Interaction between remodelling and displacement in the (averaged) Mandel stress tensor.*

⚠ *The micro-structural fields $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$ concur to determine the macroscopic remodelling distortions.*

[22] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

Benchmark problem I

Multilayered elasto-plastic composite

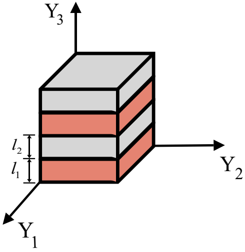

Figure: Multilayered structure of the composite material.

We restrict our investigation to the case of isotropic constituents

$$\mathscr{C}_{\nu\eta} = \lambda_\eta \boldsymbol{I} \otimes \boldsymbol{I} + 2\mu_\eta \frac{\boldsymbol{I} \underline{\otimes} \boldsymbol{I} + \boldsymbol{I} \overline{\otimes} \boldsymbol{I}}{2}$$

Furthermore, we set

$$[\boldsymbol{K}^{(0)}]_{11} = [\boldsymbol{K}^{(0)}]_{22} = 1/\sqrt{p} \quad \text{and} \quad [\boldsymbol{K}^{(0)}]_{33} = p$$

Then, since

$$\mathscr{C}_\eta^{(0)} = ((\boldsymbol{K}^{(0)})^{-1} \underline{\otimes} (\boldsymbol{K}^{(0)})^{-1}) : \mathscr{C}_{\nu\eta} : ((\boldsymbol{K}^{(0)})^{-\mathrm{T}} \underline{\otimes} (\boldsymbol{K}^{(0)})^{-\mathrm{T}})$$

we have that

$$\begin{aligned}
&[\mathscr{C}_\eta^{(0)}]_{1111} \equiv [\mathscr{C}_\eta^{(0)}]_{2222} = (\lambda_\eta + 2\mu_\eta)p^2, && [\mathscr{C}_\eta^{(0)}]_{3333} = (\lambda_\eta + 2\mu_\eta)p^{-4} \\
&[\mathscr{C}_\eta^{(0)}]_{1122} = \lambda_\eta p^2, && [\mathscr{C}_\eta^{(0)}]_{1313} \equiv [\mathscr{C}_\eta^{(0)}]_{2323} = \mu_\eta p^{-1} \\
&[\mathscr{C}_\eta^{(0)}]_{1133} \equiv [\mathscr{C}_\eta^{(0)}]_{2233} = \lambda_\eta p^{-1}, && [\mathscr{C}_\eta^{(0)}]_{1212} = \mu_\eta p^2
\end{aligned}$$

Benchmark problem I

Multilayered elasto-plastic composite

We simulate a stretch test and require ($\partial\mathscr{S}$ – boundary of the lower surface of the body)

$$\begin{aligned}
&[\boldsymbol{u}]_1(X,t) = [\boldsymbol{u}]_2(X,t) = 0, && \forall X \in \partial\mathscr{S} \times [0, L_0]\\
&[\boldsymbol{u}]_3(X,t) = 0, && \forall X \in \partial\mathscr{S}\\
&[\boldsymbol{u}]_3(X,t) = u_{\mathrm{L}} T^{-1} t, && \text{at } X_3 = L_0
\end{aligned}$$

The homogenised equations are

$$\frac{\partial}{\partial X_3}\left\{[\mathscr{G}_{\mathrm{eff}}]_{3333}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_3} + [\boldsymbol{D}_{\mathrm{eff}}]_{33}\right\} = 0$$

$$\gamma^{\mathrm{eff}}\frac{\dot{p}}{p} - [\langle \mathrm{DevSym}\boldsymbol{\Sigma}^{(0)}_{\nu\eta\mathrm{lin}}\rangle]_{33} = 0$$

with

$$[\mathscr{G}_{\mathrm{eff}}]_{3333} = \sum_{\eta=1,2}\left\{\left([\mathscr{C}_\eta^{(0)}]_{3333} - [\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}\right)\left(\varphi_\eta + \left\langle \frac{\partial[\boldsymbol{\xi}_\eta]_{333}}{\partial Y_3}\right\rangle_\eta\right)\right\}$$

$$[\boldsymbol{D}_{\mathrm{eff}}]_{33} = \sum_{\eta=1,2}\left\{\left([\mathscr{C}_\eta^{(0)}]_{3333} - [\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}\right)\left\langle \frac{\partial[\boldsymbol{\omega}_\eta]_{3}}{\partial Y_3}\right\rangle_\eta - \varphi_\eta L_0 [\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}\right\}$$

$$\gamma^{\mathrm{eff}} = Y_\Gamma \sigma_1 \tau_1 + (1 - Y_\Gamma)\sigma_2\tau_2$$

Benchmark problem I

Multilayered elasto-plastic composite

In accordance, the cell problems become [23]

$$\mathcal{Q}_\eta \frac{\partial^2 [\boldsymbol{\xi}_\eta]_{333}}{\partial Y_3 \partial Y_3} = 0 \quad \text{and} \quad \mathcal{Q}_\eta \frac{\partial^2 [\boldsymbol{\omega}_\eta]_3}{\partial Y_3 \partial Y_3} = 0$$

with $\mathcal{Q}_\eta := [\mathscr{C}_\eta^{(0)}]_{3333} - [\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}$.

General solutions

$$[\boldsymbol{\xi}_\eta]_{333}(X_3, Y_3, t) = [\boldsymbol{\mathfrak{X}}_\eta]_{3333}(X_3, t) Y_3 + [\boldsymbol{\Xi}_\eta]_{333}(X_3, t)$$

$$[\boldsymbol{\omega}_\eta]_3(X_3, Y_3, t) = [\boldsymbol{\mathfrak{W}}_\eta]_{33}(X_3, t) Y_3 + [\boldsymbol{\Omega}_\eta]_3(X_3, t)$$

⚠ $\mathcal{Q}_\eta = \frac{\lambda_\eta}{2p^4}\left[2p^3 - 3\left(1 + \frac{2}{3}\frac{\mu_\eta}{\lambda_\eta}\right)p^2 + 3\left(1 + 2\frac{\mu_\eta}{\lambda_\eta}\right)\right] > 0$ *if* $\lambda_\eta > 0$ *and* $\frac{\mu_\eta}{\lambda_\eta} \in]0, (\mu_\eta/\lambda_\eta)_{\mathrm{cr}}[$, *with* $(\mu_\eta/\lambda_\eta)_{\mathrm{cr}} > 0$

⚠ $\left(\frac{\mu_\eta}{\lambda_\eta}\right)_{\mathrm{cr}} \approx 2.376$

[23] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

Benchmark problem I

Multilayered elasto-plastic composite

⚠ *Periodicity, no jump and solvability ($[\boldsymbol{\xi}_\eta]_{333}$ and $[\boldsymbol{\omega}_\eta]_3$ have null average) conditions are used.*

- Integration constants for $[\boldsymbol{\xi}_\eta]_{333}$

$$[\boldsymbol{\mathfrak{X}}_1]_{3333} = (1 - Y_\Gamma)\frac{\mathcal{Q}_2 - \mathcal{Q}_1}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}, \qquad [\boldsymbol{\Xi}_1]_{333} = -\frac{Y_\Gamma(1-Y_\Gamma)}{2}\frac{\mathcal{Q}_2 - \mathcal{Q}_1}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}$$

$$[\boldsymbol{\mathfrak{X}}_2]_{3333} = -Y_\Gamma\frac{\mathcal{Q}_2 - \mathcal{Q}_1}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}, \qquad [\boldsymbol{\Xi}_2]_{333} = \frac{Y_\Gamma(1+Y_\Gamma)}{2}\frac{\mathcal{Q}_2 - \mathcal{Q}_1}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}$$

- Integration constants for $[\boldsymbol{\omega}_\eta]_3$

$$[\boldsymbol{\mathfrak{W}}_1]_{33} = L_0(1-Y_\Gamma)\frac{[\mathscr{C}_1^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33} - [\mathscr{C}_2^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}, \qquad [\boldsymbol{\Omega}_1]_3 = -L_0\frac{Y_\Gamma(1-Y_\Gamma)}{2}\frac{[\mathscr{C}_1^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33} - [\mathscr{C}_2^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}$$

$$[\boldsymbol{\mathfrak{W}}_2]_{33} = -L_0 Y_\Gamma\frac{[\mathscr{C}_1^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33} - [\mathscr{C}_2^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}, \qquad [\boldsymbol{\Omega}_2]_3 = L_0\frac{Y_\Gamma(1+Y_\Gamma)}{2}\frac{[\mathscr{C}_1^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33} - [\mathscr{C}_2^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}}{(1-Y_\Gamma)\mathcal{Q}_1 + Y_\Gamma \mathcal{Q}_2}$$

[24] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

Benchmark problem I

Multilayered elasto-plastic composite

The homogenised equations are

$$\frac{\partial}{\partial X_3}\left\{[\mathscr{G}_{\text{eff}}]_{3333}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_3}+[\boldsymbol{D}_{\text{eff}}]_{33}\right\}=0$$

$$\gamma^{\text{eff}}\frac{\dot{p}}{p}-[\langle\text{DevSym}\boldsymbol{\Sigma}^{(0)}_{\nu\eta\text{lin}}\rangle]_{33}=0$$

with

$$[\mathscr{G}_{\text{eff}}]_{3333}=\frac{\mathscr{Q}_1\mathscr{Q}_2}{(1-Y_\Gamma)\mathscr{Q}_1+Y_\Gamma\mathscr{Q}_2}$$

$$[\boldsymbol{D}_{\text{eff}}]_{33}=-L_0\frac{(1-Y_\Gamma)\mathscr{Q}_1[\mathscr{C}_2^{(0)}:\boldsymbol{E}^{(0)}_{\boldsymbol{K}}]_{33}+Y_\Gamma\mathscr{Q}_2[\mathscr{C}_1^{(0)}:\boldsymbol{E}^{(0)}_{\boldsymbol{K}}]_{33}}{(1-Y_\Gamma)\mathscr{Q}_1+Y_\Gamma\mathscr{Q}_2}$$

$$\gamma^{\text{eff}}=Y_\Gamma\sigma_1\tau_1+(1-Y_\Gamma)\sigma_2\tau_2$$

⚠ *It is through the averaged Mandel stress tensor that the micro-structural fields $\boldsymbol{\xi}_\eta$ and $\boldsymbol{\omega}_\eta$ concur to determine the macroscopic remodeling distortions.*

⚠ *The coupling of the homogenised equations is given through the dependence of $[\langle\text{DevSym}\boldsymbol{\Sigma}^{(0)}_{\nu\eta\text{lin}}\rangle]_{33}$ on $[\boldsymbol{u}^{(0)}]_3$ and of the effective coefficients on $\boldsymbol{K}^{(0)}$.*

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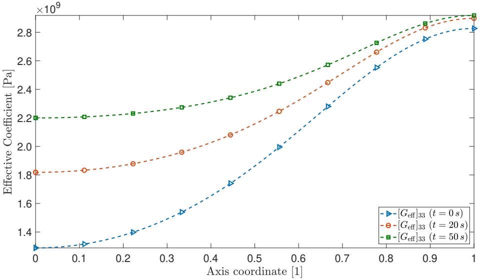
[25] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

Benchmark problem I

Multilayered elasto-plastic composite

⚠ *Initial condition for the homogenised flow rule*

$$p_{\text{in}}(X_3) = \alpha + \beta \cos(\pi X_3)$$



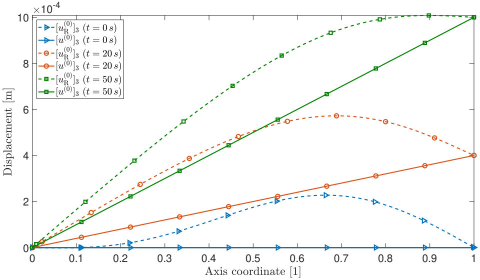


Figure: Spatial distribution of $[\mathscr{G}_{\text{eff}}]_{3333}$ (left) and of the displacement field (right) with $\alpha = 1.1$ and $\beta = 0.1$.

⚠ *In the absence of inelastic distortions, the effective coefficients are time and space independent. The cell and homogenised problems are decoupled.*

Benchmark problem I

Multilayered elasto-plastic composite

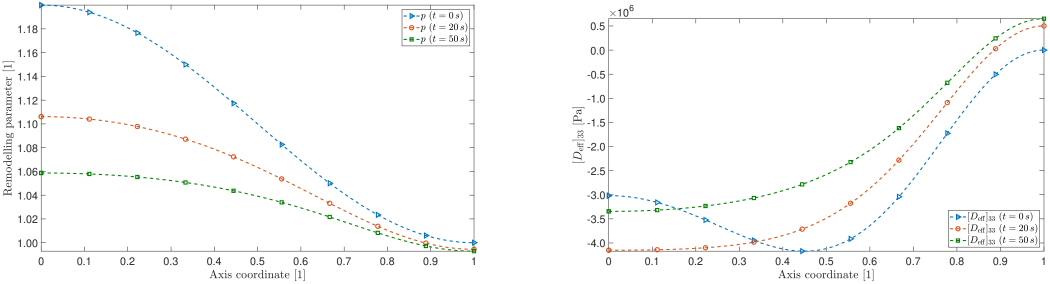


Figure: Spatial distribution of the remodelling parameter p (left) and of $[\boldsymbol{D}_{\mathrm{eff}}]_{33}$ (right).

Benchmark problem II

Fibre-reinforced elasto-plastic composite

For the statement of the flow rule, we rely on the framework established in [26]

$$\boldsymbol{Y}_\eta = \boldsymbol{Z}_\eta \quad \Leftrightarrow \quad \begin{cases} 2\mathfrak{b}_{\nu\eta}\mathrm{sym}\left(\left((\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta\right)(\boldsymbol{C}_\eta)^{-1}\right) = (\boldsymbol{C}_\eta)^{-1}\mathrm{dev}(\boldsymbol{\Sigma}_\eta + \boldsymbol{Z}_\eta) \\ \boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = 0 \\ 2\mathfrak{c}_{\nu\eta}\mathrm{skew}\left(\left((\boldsymbol{K}_\eta)^{-1}\dot{\boldsymbol{K}}_\eta\right)(\boldsymbol{C}_\eta)^{-1}\right) = \boldsymbol{O} \end{cases}$$

where $\mathfrak{b}_{\nu\eta}$, $\mathfrak{c}_{\nu\eta} \geq 0$ are material parameters with physical units of stress per time.

The homogenised evolution law [27]

$$2\mathrm{sym}\left(\left\langle \mathfrak{b}_\nu^\varepsilon \left((\boldsymbol{K}^{(0)})^{-1}\dot{\boldsymbol{K}}^{(0)}\right)\left[\boldsymbol{I} - \frac{2}{L}\mathrm{sym}(\boldsymbol{\Pi})\right]\right\rangle\right) = \frac{1}{L}\mathrm{dev}\left(\left\langle\left(\mathscr{G}^{(0)} - \boldsymbol{I}\overline{\otimes}(\mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)})\right) : \boldsymbol{\Pi}\right\rangle\right)$$
$$+ \mathrm{dev}\left(\left\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\right\rangle\right) + \frac{1}{L}\left\langle\left[\boldsymbol{I}\underline{\overline{\otimes}}\mathrm{dev}(\mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)} - \boldsymbol{Z}^{(0)})\right] : \boldsymbol{\Pi}\right\rangle + \mathrm{dev}\left(\left\langle \boldsymbol{Z}^{(0)}\right\rangle\right)$$

with

$$\boldsymbol{\Pi}_\eta(X, Y, t) := [\boldsymbol{I}\underline{\otimes}\boldsymbol{I} + \mathrm{TGrad}_Y\boldsymbol{\xi}_\eta(X, Y, t)] : \mathrm{Grad}_X\boldsymbol{u}^{(0)}(X, t) + \mathrm{Grad}_Y\boldsymbol{\omega}_\eta(X, Y, t)$$

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[26] Alfio Grillo and Salvatore Di Stefano. In: *Mathematics and Mechanics of Complex Systems* 11 (2023), pp. 57–86.

[27] A. Ramírez-Torres et al. In: *Zeitschrift fuer Angewandte Mathematik und Mechanik (ZAMM)* Under review (2024).

Benchmark problem II

Uniaxially fiber-reinforced composite

As an initial step

$$
\begin{aligned}
[\boldsymbol{u}^{\varepsilon}_{\eta}]_1 &= [\boldsymbol{u}^{\varepsilon}_{\eta}]_2 \equiv 0 \\
[\boldsymbol{u}^{\varepsilon}_{\eta}]_3 &= [\boldsymbol{u}^{\varepsilon}_{\eta}]_3(X_1, X_2, Y_1, Y_2, t) \\
&= [\boldsymbol{u}^{(0)}]_3 + \varepsilon([\boldsymbol{\xi}_{\eta}]_{33D} \frac{\partial [\boldsymbol{u}^{(0)}]_3}{\partial X_D} + [\boldsymbol{\omega}_{\eta}]_3)
\end{aligned}
$$

The elasticity tensor is prescribed as

$$
\mathscr{C}^{\varepsilon}_{\nu\eta}(X, Y) \equiv \mathscr{C}_{\nu\eta}(Y_1, Y_2) = \begin{cases} 3\kappa_{\mathrm{m}}\mathscr{K} + 2\mu_{\mathrm{m}}\mathscr{M}, & \text{in } \mathscr{Y}_{\mathrm{m}} \\ 3\kappa_{\mathrm{f}}\mathscr{K} + 2\mu_{\mathrm{f}}\mathscr{M}, & \text{in } \mathscr{Y}_{\mathrm{f}} \end{cases}
$$

and consider

$$
[\boldsymbol{K}^{(0)}]_{11} = [\boldsymbol{K}^{(0)}]_{22} = 1/\sqrt{p} \quad \text{and} \quad [\boldsymbol{K}^{(0)}]_{33} = p
$$

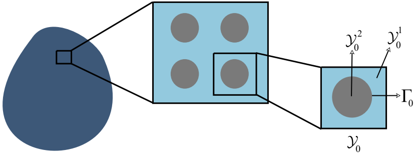

Figure: The elastoplastic composite under study possesses a uniaxially, fibre-reinforced structure.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

⚠ *The assumptions made so far imply that the only non-null components of the second order tensor* $\mathbf{\Pi}_\eta$ *are* $[\mathbf{\Pi}_\eta]_{31}$ *and* $[\mathbf{\Pi}_\eta]_{32}$.

⚠ *As per the constraints on* $\boldsymbol{K}_\eta^{(0)}$,

$$\frac{3}{2L}\frac{\dot{p}_\eta}{p_\eta}[\mathbf{\Pi}_\eta]_{31} = 0 \quad \textit{and} \quad \frac{3}{2L}\frac{\dot{p}_\eta}{p_\eta}[\mathbf{\Pi}_\eta]_{32} = 0$$

Since $\dot{p}_\eta = 0$ *would imply no evolution in the inelastic distortions, it is necessary that*

$$[\mathbf{\Pi}_\eta]_{31} = 0 \quad \textit{and} \quad [\mathbf{\Pi}_\eta]_{32} = 0.$$

⚠ *The homogenised flow rule becomes*

$$-\mathfrak{b}_\nu \frac{\dot{p}}{p} - \frac{1}{3}\left([\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\rangle]_{11} - [\langle \mathscr{C}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}\rangle]_{33}\right) = [\mathrm{dev}\left(\left\langle \boldsymbol{Z}^{(0)}\right\rangle\right)]_{11}$$

where for solvability issues (resulting from the specific form of $\boldsymbol{K}^{(0)}$*), we require that*

$$[\mathrm{dev}\left(\left\langle \boldsymbol{Z}^{(0)}\right\rangle\right)]_{11} = [\mathrm{dev}\left(\left\langle \boldsymbol{Z}^{(0)}\right\rangle\right)]_{22} \quad \textit{and} \quad [\mathrm{dev}\left(\left\langle \boldsymbol{Z}^{(0)}\right\rangle\right)]_{33} = -2[\mathrm{dev}\left(\left\langle \boldsymbol{Z}^{(0)}\right\rangle\right)]_{11}$$

[28] A. Ramírez-Torres et al. In: *Zeitschrift fuer Angewandte Mathematik und Mechanik (ZAMM)* Under review (2024).

Benchmark problem II

Fibre-reinforced elasto-plastic composite

The homogenised equations are

$$\frac{\partial}{\partial X_B}\left\{[\mathscr{G}_{\text{eff}}]_{3B3D}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_D}+[\boldsymbol{D}_{\text{eff}}]_{3B}\right\}=0$$

$$\frac{\dot{p}}{p}=-\frac{1}{\mathfrak{b}_\nu}\left(\frac{2}{3}\sigma_{\mathrm{T}}+[\operatorname{dev}\left(\langle\boldsymbol{Z}^{(0)}\rangle\right)]_{11}\right)$$

with

$$\sigma_{\mathrm{T}}:=\tfrac{1}{2}([\langle\mathscr{C}^{(0)}:\boldsymbol{E}_{\boldsymbol{K}}^{(0)}\rangle]_{11}-[\langle\mathscr{C}^{(0)}:\boldsymbol{E}_{\boldsymbol{K}}^{(0)}\rangle]_{33})$$

$$[\mathscr{G}_{\text{eff}}]_{3B3D}=\left\langle[\mathscr{G}^{(0)}]_{3B3D}+[\mathscr{G}^{(0)}]_{3B3Q}\frac{\partial\xi_{33D}}{\partial Y_Q}\right\rangle$$

$$[\boldsymbol{D}_{\text{eff}}]_{3B}=\left\langle[\mathscr{G}^{(0)}]_{3B3Q}\frac{\partial\omega_3}{\partial Y_Q}\right\rangle$$

⚠ *σ_{T} is the only surviving contribution stemming from Mandel stress tensor.*

⚠ *p is driven by the imbalance between σ_{T} and additional interactions encoded in* $[\operatorname{dev}\left(\langle\boldsymbol{Z}^{(0)}\rangle\right)]_{11}$.

[29] A. Ramírez-Torres et al. In: *Zeitschrift fuer Angewandte Mathematik und Mechanik (ZAMM)* Under review (2024).

Benchmark problem II

Fibre-reinforced elasto-plastic composite

As per our prior considerations, the cell problem associated with $[\boldsymbol{\xi}_\eta]_{33D}$, with $D = 1, 2$, is given by

$$\frac{1}{L^2}\sum_{B=1}^{2}\left\{[\mathscr{G}_\eta^{(0)}]_{3B3B}\frac{\partial}{\partial Y_B}\left(\frac{\partial[\boldsymbol{\xi}_\eta]_{33D}}{\partial Y_B}\right)\right\} = 0$$

$$[\boldsymbol{\xi}_{\mathrm{m}}]_{33D} = [\boldsymbol{\xi}_{\mathrm{f}}]_{33D}$$

$$\frac{1}{L}\sum_{B=1}^{2}\left\{[\mathscr{G}_{\mathrm{m}}^{(0)}]_{3B3B}\frac{\partial[\boldsymbol{\xi}_{\mathrm{m}}]_{33D}}{\partial Y_B} - [\mathscr{G}_{\mathrm{f}}^{(0)}]_{3B3B}\frac{\partial[\boldsymbol{\xi}_{\mathrm{f}}]_{33D}}{\partial Y_B}\right\} N_B = \frac{1}{L}\left\{[\mathscr{G}_{\mathrm{f}}^{(0)}]_{3B3D} - [\mathscr{G}_{\mathrm{m}}^{(0)}]_{3B3D}\right\} N_B$$

⚠ *Even though we are dealing with inelastic distortions that make the cell problem depends on the macroscopic variable, its solution can be found by invoking complex variable methods* [30,31].

[30] N. I. Muskhelishvili. *Some basic problems of the mathematical theory of elasticity.* Dordrecht: Springer, 1977.

[31] Reinaldo Rodríguez-Ramos et al. In: *Mechanics of Materials* 33.4 (2001), pp. 223–235.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

We set $[\boldsymbol{\xi}_\eta]_{331}$ and $[\boldsymbol{\xi}_\eta]_{332}$ to be expressed as

$$[\boldsymbol{\xi}_\eta(X,Y,t)]_{331} = \Re\{\varphi^1_\eta(X,Z,t)\}$$
$$[\boldsymbol{\xi}_\eta(X,Y,t)]_{332} = \Im\{\varphi^2_\eta(X,Z,t)\}$$

with $Z = Y_1 + iY_2$ and

$$\varphi^D_{\mathrm{m}}(X,Z,t) := a^D_0(X,t)Z + \sum_{\ell=1}^{+\infty\, o} a^D_\ell(X,t)Z^\ell \frac{\zeta^{(\ell-1)}(Z)}{(\ell-1)!}$$

$$\varphi^D_{\mathrm{f}}(X,Z,t) := \sum_{\ell=1}^{+\infty\, o} c^D_\ell(X,t)Z^\ell$$

⚠ $\zeta^{(\ell-1)}$ *denotes the* $(\ell-1)$*-th derivative of the quasi-periodic Weierstrass zeta function of periods* $w_1 = 1$ *and* $w_2 = i$.

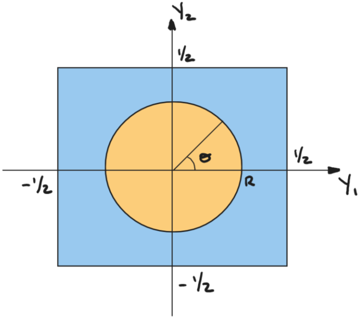

Figure: The unit periodic cell.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

The substitution into the first interface condition yields

$$\Re\left\{\sum_{\ell=1}^{+\infty\, o}\left[a_\ell^1(X,t)Z^{-\ell}-A_\ell^1(X,t)Z^\ell\right]\right\}=\Re\left\{\sum_{\ell=1}^{+\infty\, o}c_\ell^1(X,t)Z^\ell\right\}$$

$$\Im\left\{\sum_{\ell=1}^{+\infty\, o}\left[a_\ell^2(X,t)Z^{-\ell}-A_\ell^2(X,t)Z^\ell\right]\right\}=\Im\left\{\sum_{\ell=1}^{+\infty\, o}c_\ell^2(X,t)Z^\ell\right\}$$

where

$$A_\ell^D(X,t):=\sum_{m=1}^{+\infty\, o}m\Lambda_{\ell m}^D\,a_m^D(X,t)\quad\text{(Obtained from the Laurent series of }\zeta^{(k)}(Z)\text{ about }Z=0)$$

$$\Lambda_{\ell m}^D:=\begin{cases}(\ell!m!)^{-1}(\ell+m-1)!S_{\ell+m}, & \text{if }\ell,m>1,\\ (-1)^{D+1}\pi, & \text{if }\ell=m=1\end{cases}$$

$$S_{\ell+m}:=\sum_{w\in L\setminus\{0\}}w^{-(\ell+m)},\quad L:=\{w=rw_1+sw_2\,|\,w_1,w_2\text{ l.i., }r,s\in\mathbb{Z}\}\quad\text{(Lattice)}$$

Benchmark problem II

Fibre-reinforced elasto-plastic composite

The substitution into the second interface condition yields, for each $\ell = 1, 3, \dots$

$$\frac{1}{L}\overline{b_\ell^D(X,t)} + (-1)^{D+1}\frac{1}{L}\chi(X,t)\left\{\sum_{m=1}^{+\infty\, o}\sqrt{\ell m}\,\Lambda_{\ell m}^D R^{\ell+m} b_m^D(X,t)\right\} = \frac{1}{L}(-1)^{D+1}\chi(X,t)\sqrt{\ell}R^\ell\delta_{\ell 1}$$

where $b_\ell^D := a_\ell^D R^{-\ell}\sqrt{\ell}$ and

$$\chi(X,t) := \frac{[\mathscr{G}_{\mathrm{m}}^{(0)}(X,t)]_{3131} - [\mathscr{G}_{\mathrm{f}}^{(0)}(X,t)]_{3131}}{[\mathscr{G}_{\mathrm{m}}^{(0)}(X,t)]_{3131} + [\mathscr{G}_{\mathrm{f}}^{(0)}(X,t)]_{3131}}$$

$$[\mathscr{G}_\eta^{(0)}]_{3131} \equiv [\mathscr{G}_\eta^{(0)}]_{3232} = [\mathscr{C}_\eta^{(0)}]_{3131} - [\mathscr{C}_\eta^{(0)} : \boldsymbol{E}_{\boldsymbol{K}_\eta}^{(0)}]_{33}$$

Or equivalently

$$\left(\frac{1}{L}\begin{bmatrix}\mathfrak{I} & \mathfrak{O}\\ \mathfrak{O} & \mathfrak{I}\end{bmatrix} + (-1)^{D+1}\frac{1}{L}\chi(X,t)\begin{bmatrix}\Re\{\mathfrak{W}^D\} & -\Im\{\mathfrak{W}^D\}\\ -\Im\{\mathfrak{W}^D\} & -\Re\{\mathfrak{W}^D\}\end{bmatrix}\right)\begin{bmatrix}\Re\{\boldsymbol{b}^D(X,t)\}\\ \Im\{\boldsymbol{b}^D(X,t)\}\end{bmatrix} = \frac{1}{L}\boldsymbol{V}^D(X,t)$$

with $[\mathfrak{W}^D]_{\ell m} = \sqrt{\ell m}\,\Lambda_{\ell m}^D R^{\ell+m}$, $\boldsymbol{b}^D = (b_1^D, b_3^D, \dots)^{\mathrm{T}}$ and $\boldsymbol{V}^D = ((-1)^{D+1}\chi R, 0, \dots)^{\mathrm{T}}$.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

The cell problem associated with $[\boldsymbol{\omega}_\eta]_3$ is

$$\frac{1}{L^2}\sum_{B=1}^{2}\left\{[\mathscr{G}_\eta^{(0)}]_{3B3B}\frac{\partial}{\partial Y_B}\left(\frac{\partial[\boldsymbol{\omega}_\eta]_3}{\partial Y_B}\right)\right\}=0$$

$$[\boldsymbol{\omega}_{\mathrm{m}}]_3=[\boldsymbol{\omega}_{\mathrm{f}}]_3$$

$$\frac{1}{L}\sum_{B=1}^{2}\left\{[\mathscr{G}_{\mathrm{m}}^{(0)}]_{3B3B}\frac{\partial[\boldsymbol{\omega}_{\mathrm{m}}]_3}{\partial Y_B}-[\mathscr{G}_{\mathrm{f}}^{(0)}]_{3B3B}\frac{\partial[\boldsymbol{\omega}_{\mathrm{f}}]_3}{\partial Y_B}\right\}N_B$$
$$=-\frac{1}{L}\sum_{J=1}^{3}\left([\mathscr{C}_{\mathrm{f}}^{(0)}]_{3BJJ}[\boldsymbol{E}_{\boldsymbol{K}_{\mathrm{f}}}^{(0)}]_{JJ}-[\mathscr{C}_{\mathrm{m}}^{(0)}]_{3BJJ}[\boldsymbol{E}_{\boldsymbol{K}_{\mathrm{m}}}^{(0)}]_{JJ}\right)N_B$$

⚠ *Due to the characterisitcs of* $\mathscr{C}_\eta^{(0)}$*, the hand-right hand side of the second interface condition is zero.*

⚠ *Since we are assuming that the average of* $\boldsymbol{\omega}_\eta$ *is zero, by the uniqueness of the solution* $[\boldsymbol{\omega}_\eta]_3\equiv 0$*.*

⚠ *This result is analogous, to what we found in* [32]*, for which the solutions to the cell problems in the direction of 'no changes in material properties' are the trivial ones.*

[32] Giammarini A, Ramírez-Torres A, and Grillo A. In: *Mathematical Methods in the Applied Sciences* (2024), Accepted.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

The homogenised equations

$$\frac{\partial}{\partial X_B}\left\{[\mathscr{G}_{\text{eff}}]_{3B3D}\frac{\partial[\boldsymbol{u}^{(0)}]_3}{\partial X_D}\right\} = 0$$

$$\frac{\dot{p}}{p} = -\frac{1}{\mathfrak{b}_\nu}\left(\frac{2}{3}\sigma_{\mathrm{T}} + [\operatorname{dev}\left(\langle \boldsymbol{Z}^{(0)}\rangle\right)]_{11}\right)$$

with

$$[\mathscr{G}_{\text{eff}}]_{3131} = \left\langle[\mathscr{G}^{(0)}]_{3131}\right\rangle + \left\langle[\mathscr{G}^{(0)}]_{3131}\frac{\partial\xi_{331}}{\partial Y_1}\right\rangle = \frac{1}{|\mathscr{Y}|}[\mathscr{G}^{(0)}_{\mathrm{m}}]_{3131}\left(1 - 2\pi\Re\{a_1^1\}\right)$$

$$[\mathscr{G}_{\text{eff}}]_{3231} = \left\langle[\mathscr{G}^{(0)}]_{3232}\frac{\partial\xi_{331}}{\partial Y_2}\right\rangle = -\frac{2\pi}{|\mathscr{Y}|}[\mathscr{G}^{(0)}_{\mathrm{m}}]_{3131}\Im\{a_1^1\}$$

$$[\mathscr{G}_{\text{eff}}]_{3132} = \left\langle[\mathscr{G}^{(0)}]_{3131}\frac{\partial\xi_{332}}{\partial Y_1}\right\rangle = -\frac{2\pi}{|\mathscr{Y}|}[\mathscr{G}^{(0)}_{\mathrm{m}}]_{3131}\Im\{a_1^2\}$$

$$[\mathscr{G}_{\text{eff}}]_{3232} = \left\langle[\mathscr{G}^{(0)}]_{3232}\right\rangle + \left\langle[\mathscr{G}^{(0)}]_{3232}\frac{\partial\xi_{332}}{\partial Y_2}\right\rangle = \frac{1}{|\mathscr{Y}|}[\mathscr{G}^{(0)}_{\mathrm{m}}]_{3131}\left(1 + 2\pi\Re\{a_1^2\}\right)$$

⚠ *We used the local periodicity property of* $\boldsymbol{\xi}_\eta$ *and the orthogonality properties of* $\sin(\Theta\ell)$ *and* $\cos(\Theta\ell)$ *with respect to the inner product* $\langle f(\Theta), g(\Theta)\rangle = \int_0^{2\pi} f(\Theta)g(\Theta)\mathrm{d}\Theta$.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

⚠ *For* $\ell = m = 1$, *we can express* a_1^D *in closed form and depending on the elastic properties through* $\mathscr{C}_{\nu\eta}$ *and on the remodelling tensor* $\boldsymbol{K}^{(0)}$. *Specifically,*

$$\Re\{a_1^D(X,t)\} = (-1)^{D+1}\frac{\chi(X,t)R^2}{1+\chi(X,t)\pi R^2} \quad \textit{and} \quad \Im\{a_1^D(X,t)\} = 0$$

The relevant effective coefficients are

$$[\mathscr{G}_{\text{eff}}(X,t)]_{3131} = \frac{1}{|\mathscr{Y}|}[\mathscr{G}_{\text{m}}^{(0)}(X,t)]_{3131}\left(\frac{1-\chi(X,t)\pi R^2}{1+\chi(X,t)\pi R^2}\right)$$

$$[\mathscr{G}_{\text{eff}}(X,t)]_{3231} = 0$$

$$[\mathscr{G}_{\text{eff}}(X,t)]_{3132} = 0$$

$$[\mathscr{G}_{\text{eff}}(X,t)]_{3232} = \frac{1}{|\mathscr{Y}|}[\mathscr{G}_{\text{m}}^{(0)}(X,t)]_{3131}\left(\frac{1-\chi(X,t)\pi R^2}{1+\chi(X,t)\pi R^2}\right)$$

with

$$[\mathscr{G}_{\text{m}}^{(0)}]_{3131} = [\mathscr{C}_{\text{m}}^{(0)}]_{3131} - [\mathscr{C}_{\text{m}}^{(0)} : \boldsymbol{E}_{\boldsymbol{K}}^{(0)}]_{33}$$

Benchmark problem II

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⚠ *Initial condition for the homogenised flow rule*

$$p(X_1, X_2, 0) = 1 + \beta \exp\left(-\frac{(X_1 - L/2)^2}{2(\tau_1)^2} - \frac{(X_2 - L/2)^2}{2(\tau_2)^2} \right)$$

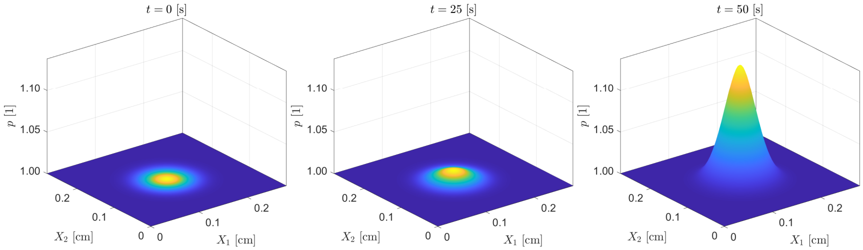

Figure: Distribution of the remodelling parameter p at three distinct time steps. For the initial condition we have set $\beta = 0.001$ and $\tau_1 = \tau_2 = L/10$.

Benchmark problem II

Fibre-reinforced elasto-plastic composite

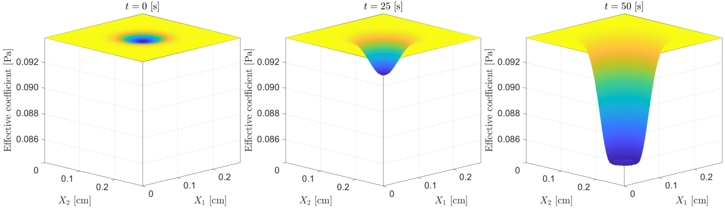

Figure: Distribution of the effective coefficient $[\mathscr{G}_{\text{eff}}]_{3131}$ at three distinct time steps.

In the absence of inelastic distortions, namely, for $p = 1$, the effective coefficient reduces to the constant

$$[\mathscr{G}_{\text{eff}}]_{3131} = \frac{1}{|\mathscr{Y}|}[\mathscr{C}_{\text{m}}^{(0)}]_{3131}\left(\frac{1 - \chi|\mathscr{Y}_{\text{f}}|}{1 + \chi|\mathscr{Y}_{\text{f}}|}\right)$$
$$= 0.093908 \text{ Pa}.$$

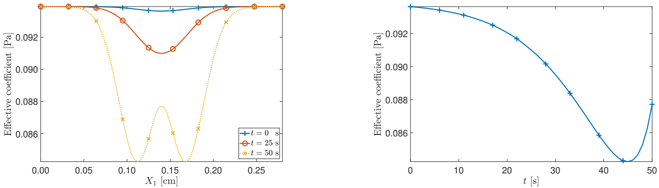

Figure: (Panel on the left) Distribution of the effective coefficient $[\mathscr{G}_{\text{eff}}]_{3131}$ at three distinct time steps with $X_2 = L/2$. (Panel on the right) Time evolution of $[\mathscr{G}_{\text{eff}}]_{3131}$ at the point of $(L/2, L/2)$ in $\mathcal{B}_{\text{h}}$.

⚠ *When the remodelling parameter reaches a particularly high value, it triggers a situation where the material experiences gains in* $[\mathscr{G}_{\text{eff}}]_{3131}$.

Conclusions and further work

- The scheme is valid for a zero-order theory for $\boldsymbol{K}_\eta$ (with some considerations).

- The framework can be extended, with modifications, to describe other specific biological situations (such as growth) where the microstructure and the inelastic distortions play an important role. A first step towards requires reconceiving the constraint in the form [33]

$$\boldsymbol{K}_\eta^{-\mathrm{T}} : \dot{\boldsymbol{K}}_\eta = R_{\mathrm{g}\eta}$$

  where $R_{\mathrm{g}\eta}$ is referred to as growth law.

- The approach offers several challenges
  - Existence and uniqueness of the solutions of the local problems
  - Derivation of suitable evolution laws
  - Development of computational algorithms

[33] Alfio Grillo and Salvatore Di Stefano. In: *Mathematics and Mechanics of Complex Systems* 11 (2023), pp. 57–86.

Thank you for the attention

Acknowledgments

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