Incidence of Geometric Measure Theory in Continuun Mechanics

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Non-linear elasticity of simple bodies

- Consider a general standard energy of simple bodies with polyconvex integrand

$$\mathcal{E}(u;\mathcal{B}) := \int_{\mathcal{B}} e(x, Du(x))\ \mathrm{d}x$$

- Ball's 1976/1977 theorem (ARMA) foresees minimizers in $W^{1,p}$ with $p>3$. A 1994 refinement by Müller-Tang-Yan (AIHP-ANL) includes the case $p=3$.
- The lower semicontinuity holds since the minimizers are functions describing configurations that are *free of fractures and holes* (unless we do not foresee the initial presence of a hole that can be enlarged elastically along the deformation, but that hole is not nucleated anew).
- For $p<3$, an additional condition is necessary to ensure that the weak limits of the sequences of the energy entries are linked to a minimizing deformation free of holes and cracks.
- How can we get such a condition?

Deformation graphs and tangent planes

- Polyconvexity means that the energy is a convex function of Du minors. They are the components of a 3-vector

$$M(Du) = M(F) := (\mathbf{e}_1, F\mathbf{e}_1) \wedge (\mathbf{e}_2, F\mathbf{e}_2) \wedge (\mathbf{e}_3, F\mathbf{e}_3) \in \Lambda_3(\mathbb{R}^3, \tilde{\mathbb{R}}^3)$$

- $M(F)$ describes, at a point x, *the tangent hyperplane to the deformation graph*, namely the collection of points $(x, u(x))$, which is a 3D-surface in a six-dimensional space.
- Once u is fixed, e.g. in $W^{1,1}$, and $|M(Du)|$ is in L^1, we can define a functional G_u on the class of 3-forms ω as

$$G_u(\omega) := \int_{\mathcal{B}} \omega \cdot M(Du)\,\mathrm{d}x$$

- It admits *a notion of boundary* ∂G_u defined as a functional on 2-forms such that

$$\partial G_u(\bar{\omega}) = G_u(d\bar{\omega})$$

- G_u is a *Cartesian current* over the graph of u.

Then… ?

- The condition $\partial G_u(\bar{\omega}) = 0$ assures the the graph of u is *free of boundaries*: the graph of u is free of vertical parts, namely u is not multi-valued in any part of its domain.
- Through G_u we can characterize a subclass of $W^{1,p}$, with summability p>1, the one of so-called *weak diffeomorphisms*. It includes physically significant minimizers of the nonlinear elastic energy in the large-strain regime, namely those that are a.e. orientation preserving and avoid interpenetration of matter (Giaquinta-Modica-Souček, 1989-1990, ARMA).
- The proof is based on the convergence of current sequences; each element may include incompatible strain, while compatibility is recovered at the limit.
- So, a general (functional) current can be intended as *an inner work*, in a generalized sense, for the natural inclusion of incompatible strains.

Do we really need a lower summability?

- An example is the case in which *stress constraints* are introduced. A functional that is the counterpart of the complementary energy can be defined:

$$\mathcal{E}(T) := \int_{\mathcal{T}} W(\boldsymbol{T}(z))\, \mathrm{d}\|T\|(z)$$

- The resulting Piola-Kirchhoff stress is a measure. It may describe localized actions along single dislocations and/or dislocation walls, which can be nucleated once the stress reaches the admissibility threshold.
- The picture includes the *blow-up of a point onto a ball, or a line onto a surface or a cylinder*, or the opposite processes in a precise sense. These effects cannot be described by means of techniques based, e.g., on functions with bounded variation. They describe swelling at finite strain and imply considering every material point as a piece of matter with a certain non-zero diameter (Giaquinta-Mariano-Modica, 2015, PRSE).
- Pertinent numerical analyses are still missing.
- Currents intended in general sense play here a significant role.

Something on rectifiable currents

- Let U be an open set in *n*-dimensional real space. Take *k = 0,…,n*.
- For *k*≥1 consider a k-rectifiable subset $\mathcal{K}$ of U. A rectifiable current T is a functional defined by
  $$T(\omega) := \int_{\mathcal{K}} (\omega \cdot \xi)\theta \, \mathrm{d}\mathcal{H}^k(x) \,, \qquad \forall \omega \in \mathcal{D}^k(U) \,, \qquad \text{with} \qquad \xi(x) \in \Lambda_k \mathbb{R}^n \,;$$
- θ is the current multiplicity and can be $\mathbb{R}$, $\mathbb{Z}^m$, $m \in \mathbb{N}^+$, $\mathbb{R}^m$ -valued…
- Mass of a rectifiable current: $\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(U) \,, \ \|\omega\| \leq 1\}$
- Boundary: $\partial T(\eta) := T(d\eta) \qquad \forall \eta \in \mathcal{D}^{k-1}(U)$
- Weak convergence: $T_h \rightharpoonup T$ if $\lim_{h\to\infty} T_h(\omega) = T(\omega) \,, \forall \, \omega \in \mathcal{D}^k(U)$.
- If $T_h \rightharpoonup T$, by lower semicontinuity we have $\mathbf{M}(T) \leq \liminf_{h\to\infty} \mathbf{M}(T_h)$.

A key tool: compactness

- Write $\mathcal{R}_k(U)$ for the space of integer multiplicity (i.m.) rectifiable currents

Theorem 1 (Federer-Fleming, 1960, Ann. Math.) *If a sequence* $\{T_h\} \subset \mathcal{R}_k(U)$ *satisfies* $\sup_h \mathbf{M}(T_h) < \infty$ *and* $\sup_h \mathbf{M}((\partial T_h) \llcorner U) < \infty$, *there exists* $T \in \mathcal{R}_k(U)$ *and a subsequence of* $\{T_h\}$ *(not relabeled) such that* $T_h \rightharpoonup T$ *weakly in* $\mathcal{D}_k(U)$. *As a consequence, if* $T \in \mathcal{R}_k(U)$ *satisfies* $\mathbf{M}((\partial T) \llcorner U) < \infty$, *the boundary rectifiability theorem states that* $\partial T \in \mathcal{R}_{k-1}(U)$.

Where do they have incidence?

- They are essential in the *varifold-based representation of cracks*.
  - In this case, curvature varifolds play a pivotal role. They are Radon measures that admit a weak notion of curvature – another tool from geometric measure theory.

- They matter in the analysis of *equilibrium states admitting irrecoverable large strains* as a coalescence of internal slips: plasticity.
  - In this case, also size bounded currents with $\mathbb{R}^m$ -valued multiplicity play a role.
  - The size **S**(T) is the H^k-measure of the set where $||T||$ is positive.
  - A lower semicontinuity result holds true for the size (Mucci, 2005, Calc. Var.).
  - These currents admit a closure result for the pertinent space, even when they are referred to metric spaces (Ambrosio-Kirchheim, 2000, Acta Math.)

- In both cases, namely fracture and plasticity, SBV-maps come into play.

Slips S with dislocation Γ at crystal scale

- To account for slips the deformation can be taken as an orientation preserving SBV map so that at crystal scale λ we have (Reina-Conti, 2014, JMPS)

$$Du_\lambda = \nabla u_\lambda \,\mathcal{L}^3 + b \otimes \bar{\nu}\, \mathcal{H}^2 \llcorner S \;, \qquad \det \nabla u_\lambda > 0 \;, \qquad \mathrm{curl}(\nabla u_\lambda \,\mathcal{L}^3) = -b \otimes \tau \mathcal{H}^1 \llcorner \Gamma$$

$$F_\lambda := Du_\lambda \;, \quad F_\lambda = F_\lambda^e F_\lambda^p \;, \quad F_\lambda^e = \nabla u_\lambda \;, \quad F_\lambda^p = I\,\mathcal{L}^3 + (\nabla u_\lambda)^{-1} b \otimes \bar{\nu}\,\mathcal{H}^2 \llcorner S$$

- Consider distinct finite dislocations in 2D space, each with core in a ball of radius (say) c λ, and take a coercive energy that is sum of out-of-core and in-core contributions with a second-gradient regularization. It can be proven that in the space of 2x2 tensor valued measure over the plane (Reina-Schlömerkemper-Conti, 2016, JMPS)

$$F_\lambda^p \stackrel{*}{\rightharpoonup} F^p$$

Needs for weak summability

- Around Γ the deformation gradient ∇u fails to be L^2: its norm diverges logarithmically.
- Energies with growth less than quadratic enter into play; Ball's results cannot be applied directly.
- For fields that are locally gradients of Sobolev functions with distributional Jacobian determinant and cofactors represented by functions, we have (Müller-Palombaro, 2008, Calc. Var.)

$$\frac{3}{2} < p < 2.$$

- In terms of currents we find existence for

$$1 < p < 2.$$

Measure-valued decomposition

- Let Ω be the reference configuration: a fit region. The slip set S is 2-rectifiable.
- Under the suggestion of the weak-$*$ convergence result we select

$$F = F^e F^p \,, \quad \text{with} \quad F^p = a(x) I \mathcal{L}^3 + \hat{F}(\bar{S}, \bar{\Gamma}) \quad \text{and} \quad F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|) \,,$$

  where $C^{-1} \leq a(x) \leq C \,, \quad \forall x \in \Omega \,, C > 1 \,.$

- $\hat{F}(\bar{S}, \bar{\Gamma})$ is *a tensor-valued measure* depending on the currents constructed over S and Γ; its support is a 2-rectifiable set and its curl is supported by a 1-rectifiable set.
- Confinement condition: $\mathrm{spt}\tilde{F}(\bar{S}) \subset \Omega \,, \qquad \tilde{F} = \tilde{F}(\bar{S}) = \Theta_S \otimes \nu_S \, \mathcal{H}^2 \llcorner S$

$$\mathrm{spt}(\mathrm{curl} F(\bar{S})) \subset \Omega \qquad \mathrm{curl}\tilde{F}(\bar{S}) = \Theta_\Gamma \otimes \tau_\Gamma \, \mathcal{H}^1 \llcorner \Gamma \,.$$

- Tangency condition: $\Theta_S(x) \bullet \nu_S(x) = 0$ at $\mathcal{H}^2$-a.e. $x \in S$

Energies

- Looking at lattice scale (i.m. rectifiable currents)

$$\mathcal{F}_{p,s}(u) := \int_\Omega \Big(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \Big) \, \mathrm{d}x + |\mathrm{curl} F^p|(\Omega)$$

- Looking at continuum scale (size bounded currents)

$$\widetilde{\mathcal{F}}_{p,s}(u) := \int_\Omega \Big(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \Big) \, \mathrm{d}x + |\mathrm{curl} F^p|(\Omega) + \mathbf{S}(\overline{\Gamma})$$

Competitors in the second case: $\widetilde{\mathscr{A}}_{M,C,c_1,\mathscr{K},c}(\Omega)$

- $\varphi \in SBV(\Omega, \tilde{\mathbb{R}}^3)$ satisfies $||\varphi||_\infty \leq M$ with M>1.
- $\hat{F}(\bar{S}, \bar{\Gamma}) \cdot \zeta = \bar{S}(\bar{\omega}^2_\zeta)$, $\zeta \in C^\infty_c(\Omega, \mathbb{R}^{3\times 3})$, for some generalized slip surface with $\bar{\Gamma} = \partial \bar{S}$ a bounded size current such that $\mathbf{M}(\bar{S}) \leq c\mathbf{M}(\bar{\Gamma})^2$ and $\mathbf{S}(\bar{S}) \leq c\mathbf{S}(\bar{\Gamma})^2$.
- $\mathrm{spt}\bar{S} \subset \mathcal{K}$, with $\mathcal{K}$ a compact set, and the constant C>1 bounding $a(x)$ is fixed.
- $F^e \in L^1(\Omega, \mathbb{R}^{3\times 3}; |F^p|)$, with $\det F^e > 0$ a.e. in Ω and $M(F^e) \in L^1(\Omega, \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)) \simeq L^1(\Omega, \mathbb{R}^{19})$
- $\mathbf{M}(\partial G_\varphi \llcorner V \times \mathbb{R}^3) \leq c_1 |\hat{F}(\bar{S}, \bar{\Gamma})|(V)$ for each open set $V \subset \Omega$ with c_1>0 .
- The non-penetration condition holds:

$$\int_\Omega f(x, \varphi(x)) \det \nabla\varphi(x)\, \mathrm{d}x \leq \int_{\mathbb{R}^3} \sup_{x\in\Omega} f(x,y)\, \mathrm{d}y \quad \forall f \in C^\infty_c(\Omega \times \mathbb{R}^3), \quad f \geq 0 .$$

Energy minima

- Define $$\widetilde{\mathscr{A}^{\mathfrak{s}}_{\gamma}} := \{\varphi \in \widetilde{\mathscr{A}^{\mathfrak{s}}_{M,C,c_1,\mathscr{K},c}}(\Omega) \mid \widetilde{\mathscr{F}}_{p,s}(\varphi) < \infty\,, \quad \mathrm{Tr}(\varphi) = \gamma\}\,.$$

Theorem 2 (Mariano-Mucci, 2023, ARMA) *Take $M, C, c_1, c > 0$, with $C > 1$, $\mathcal{K} \subset \Omega$ a compact set, and $p > 1$, $s > 0$. If for some $\gamma \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ the class $\widetilde{\mathcal{A}^{\mathfrak{s}}_{\gamma}}$ is non-empty, the functional $\varphi \mapsto \widetilde{\mathcal{F}}_{p,s}(\varphi)$ attains a minimum in $\widetilde{\mathcal{A}^{\mathfrak{s}}_{\gamma}}$, i.e., there exists $\varphi_0 \in \widetilde{\mathcal{A}^{\mathfrak{s}}_{\gamma}}$ such that*

$$\widetilde{\mathcal{F}}_{p,s}(\varphi_0) = \inf\{\widetilde{\mathcal{F}}_{p,s}(\varphi) \mid \varphi \in \widetilde{\mathcal{A}^{\mathfrak{s}}_{\gamma}}\}\,.$$

The case of fractures

- Cracking is a process that can be tackled in variational way (Francfort-Marigo, 1998, JMPS).
- The view rests on De Giorgi's minimizing movement technique (De Giorgi, 1993, in a book edited by Baiocchi-Lions); it is an energy-driven implicit time discretization; it involves a coercive and lower semicontinuous functional to be minimized when augmented by a power of the distance between two subsequent states.
- We can also consider a (scalar or vector) phase field whose concentration approximates the crack path: minimizers of an energy depending on a phase field and its gradient admit localization under specific conditions.
- What kind of energy?

A problem and ways to solve it

- A first natural attempt is to consider Griffith's energy (it includes a constant surface energy)

$$\mathcal{G} := \int_{\mathcal{B}} e(x, F)\, \mathrm{d}x + \int_{\mathcal{C}} \phi \, \mathrm{d}\mathcal{H}^2(x)$$

- Unknown are deformation and crack path C. However, in 2D-space we can control sequences of curves, while in 3D-space we cannot do it: they can produce bulbs growing at infinity.
- Solution 1: Forget the surface density and consider SBV-type deformation identifying the crack path with the jump set of minimizers of a polyconvex energy (Dal Maso-Toader, 2002, ARMA).
- Solution 2: Maintain the idea of a surface energy and consider sequences of surfaces with bounded curvature.

How can we realize Solution 2?

- The idea is to consider a family of infinitely many reference configurations differing from each other only by the crack path, and to parameterize such a family by varifolds.
- Consider a fiber bundle over the reference configuration, say in n-space, with fiber the Grassmanian of tangent k-planes, namely $\pi : \mathcal{G}_k(\mathcal{B}) \to \mathcal{B}$
- A varifold is a non-negative Radon measures *V* over such a bundle.
- Its mass is the measure of the set where *V* is defined.
- For $\mathcal{H}^k-$measurable, $k-$rectifiable subsets $\mathfrak{b}$ of $\mathcal{B}$ densities $\theta \in L^1\left(\mathfrak{b}, \mathcal{H}^k\right)$ define rectifiable varifolds by

$$V_{\mathfrak{b},\theta}\left(\varphi\right) := \int_{\mathcal{G}_k(\mathcal{B})} \varphi\left(x,\Pi\right)\ dV_{\mathfrak{b},\theta}\left(x,\Pi\right) := \int_{\mathfrak{b}} \theta\left(x\right)\varphi\left(x,\Pi(x)\right)\ d\mathcal{H}^k(x) \quad \text{for any } \varphi \in C^0\left(\mathcal{G}_k(\mathcal{B})\right)$$

A bit more: curvature varifolds

- A special class of varifolds comes into play (Mantegazza, 1996, JDG):

Definition : *A varifold V is called an* ***integer rectifiable curvature k–varifold with boundary*** *if*

(1) *V is an integer, rectifiable k–varifold $V_{\mathfrak{b},\theta}$ associated with the triple $(\mathfrak{b}, \theta, \mathcal{H}^k)$, i.e., V is defined by* (1.2),

(2) *there exists a function $A \in L^1(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*})$, in components $A_j^{\ell i}$, and a vector Radon measure $\partial V \in \mathsf{M}(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^n)$, called the* ***varifold boundary measure****, such that, for every $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$, with values $\varphi(x, \Pi)$, the following relation holds:*

$$\int_{\mathcal{G}_k(\mathcal{B})} \left(\Pi_j^i D_{x_j} \varphi + A_j^{il} D_{\Pi_j^l} \varphi + A_j^{ij} \varphi \right) \, dV(x, \Pi) = - \int_{\mathcal{G}_k(\mathcal{B})} \varphi \, d\partial V^i(x, \Pi) .$$

Energy

- We account for varifolds and SBV-type deformations with jump set contained in the varifold support.
- An appropriate choice for the energy is

$$\mathcal{F} := \underbrace{\int_{\mathcal{B}} e(x, Du)\,\mathrm{d}x + \bar{a}\mu_V(\mathcal{B})}_{\text{Griffith's term}} + \int_{\mathcal{G}_2(\mathcal{B})} a_1 \|A\|^{\bar{p}}\,\mathrm{d}V + a_2\|\partial V\|$$

- Existence theorems are available under different conditions and with energy variants:
- (a) deformations as discontinuous weak diffeomorphisms (Giaquinta-Mariano-Modica-Mucci, 2010, Physica D);
- (b) deformations as a.e. orientation preserving SBV maps and the inclusion of manifold-valued phase fields (Giaquinta-Mariano-Modica, 2010, DCDSA);
- (c) crack in shells with through-the-thickness microstructure (Mariano-Mucci, 2023, SIAM-JMA).

Criticisms to the varifold-based view

- What is the physics---if any---covered by the circumstance in which the minimizing deformation jump set is strictly contained in the varifold support, so it does not coincide with it?

- Besides the technical need, what is covered and what is excluded by the introduction of a weak curvature term in the energy?

Answers

- If we consider the varifold-based approach in a minimizing movement, the lack of coincidence between deformation jump and varifold support is a way to take memory of the crack at the previous step, that is another option with respect to the introduction of an additional memory variable.
- If we look only at initial and final states, the latter reached by asking energy minimality, the mentioned lack of coincidence refers to circumstances in which at least on a portion of the crack path the macroscopic displacement does not record jumps although material bonds at microscopic scale are broken.
- The crack surface is always rough at microscopic scale. Inserting the varifold weak curvature tensor in the energy is a way to account for such a roughness.

An example by Frank Morgan

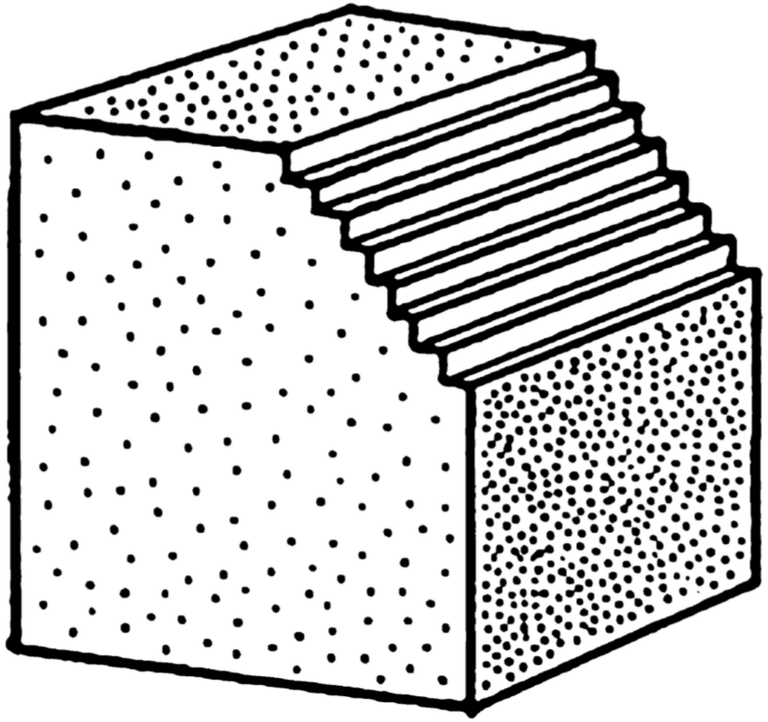

Figure 1. If you cut an edge off a cubical crystal, the exposed surface forms tiny steps, well modeled by a varifold.

Photo Credit

Figure 1 drawn by J. Bredt, from F. Morgan's *Geometric Measure Theory*, courtesy of Frank Morgan.

Can we put together both views?

- Find minimizers with SBV orientation preserving deformations of the energy

$$\mathcal{F} := \int_{\mathcal{B}} \left(|M(F^e(x))|^p + |\det F^e(x)|^{-s} \right) \mathrm{d}x + |\mathrm{curl} F^p|(\mathcal{B}) + \mathbf{S}(\overline{\Gamma}) + \bar{a}\mu_V(\mathcal{B}) + \int_{\mathcal{G}_2(\mathcal{B})} (a_1 \|A\|^{\bar{p}} + g(x, M))\, \mathrm{d}V + a_2 \|\partial V\|$$

- This one could be a reasonable representation of ductile fracture.