

The Multisymplectic Framework of Field Theories

Workshop on Geometric Aspects of Material Modelling

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Structure of the talk

The geometry of calculus of variations

- 1.1 The geometric setting
- 1.2 The Euler-Lagrange equations

Multisymplectic geometry

- 2.1 Basic definitions
- 2.2 Noether's Theorem
- 2.3 Brackets
- 2.4 ...and more!

Examples

- 3.1 Classical Mechanics
- 3.2 Hyperelastic materials

The geometry of calculus of variations

The geometric setting I

What to minimize/maximize? Sections!

Fixed some fibered manifold

$$Y \xrightarrow{\pi_{YX}} X \text{ with coordinates } (x^\mu, y^i) \mapsto x^\mu,$$

we want to find a section

$$\phi : X \rightarrow Y, (x^\mu) \mapsto (x^\mu, y^i = \phi^i(x^\mu))$$

minimizing/maximizing the functional

$$\mathcal{J}[\phi] = \int_X L \left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}, \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu}, \dots \right) d^n x.$$

We will focus on **first order** theories,

$$\mathcal{J}[\phi] = \int_X L \left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu} \right) d^n x.$$

The geometric setting II

We can interpret

$$L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}\right) d^n x$$

as an n -form on the first jet bundle

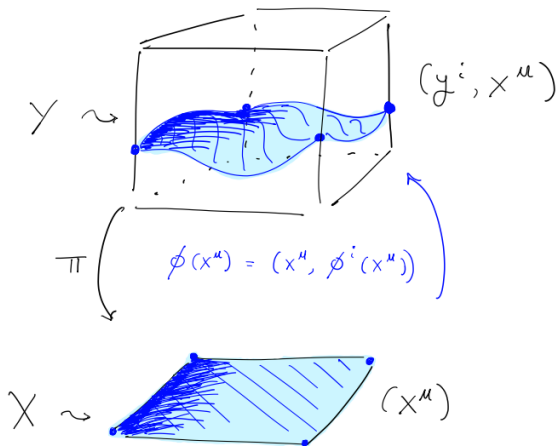
$$J^1\pi_{YX} \text{ with coordinates } (x^\mu, y^i, z_\mu^i).$$

We call it **the Lagrangian density**

$$\mathcal{L} = L(z^\mu, y^i, z_\mu^i) d^n x.$$

We can rewrite the action as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$



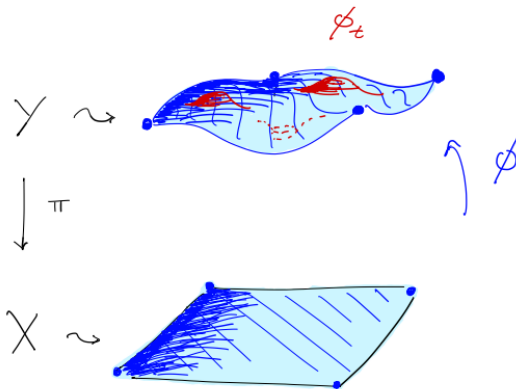
$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L},$$

where $\mathcal{L} \in \Omega^n(J^1\pi_{YX})$ is the Lagrangian density.

The Euler-Lagrange equations I

If ϕ is a minimizer/maximizer (more generally, **stationary section**),

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{J}[\phi_t] = 0, \forall \text{ variation } \phi_t.$$



The Euler-Lagrange equations II

Equivalently,

$$0 = \int_X \frac{d}{dt} \Big|_{t=0} (j^1 \phi_t)^* \mathcal{L}.$$

Locally, we get

$$\frac{\partial L}{\partial y^i} = \frac{d}{dx^\mu} \left(\frac{\partial L}{\partial z_\mu^i} \right).$$

What about intrinsic Euler-Lagrange equations?

If we define

$$\xi := \frac{d}{dt} \Big|_{t=0} \phi_t = \xi^i \frac{\partial}{\partial y^i} \in \mathfrak{X}(Y),$$

$$\xi^{(1)} := \frac{d}{dt} \Big|_{t=0} j^1 \phi_t = \xi^i \frac{\partial}{\partial y^i} + \left(\frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial x^i}{\partial y^j} z_\mu^j \right) \frac{\partial}{\partial z_\mu^i} \in \mathfrak{X}(J^1 \pi_{YX}).$$

The Euler-Lagrange equations III

If we define

$$\xi := \left. \frac{d}{dt} \right|_{t=0} \phi_t = \xi^i \frac{\partial}{\partial y^i} \in \mathfrak{X}(Y),$$

$$\xi^{(1)} := \left. \frac{d}{dt} \right|_{t=0} j^1 \phi_t = \xi^i \frac{\partial}{\partial y^i} + \left(\frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial x^i}{\partial y^j} z_\mu^j \right) \frac{\partial}{\partial z_\mu^j} \in \mathfrak{X}(J^1 \pi_{YX}),$$

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mathcal{J}[\phi_t] = \int_X (j^1 \phi)^* \mathcal{E}_{\xi^{(1)}} \mathcal{L}, \text{ for every vertical } \xi \in \mathfrak{X}(Y)$$

Applying Stokes' Theorem

$$0 = \int_X (j^1 \phi)^* \iota_{\xi^{(1)}} d\mathcal{L} + \int_X d\iota_{\xi^{(1)}} \mathcal{L} = \int_X (j^1 \phi)^* \iota_{\xi^{(1)}} d\mathcal{L}.$$

The Euler-Lagrange equations IV

$$0 = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} \text{ for every vertical } \xi \in \mathfrak{X}(Y).$$

Does not yield equations.

Idea: modify $\mathcal{L}$

We want to find an n -form $\Theta_{\mathcal{L}}$ satisfying

$$(j^1\phi)^* \mathcal{L} = (j^1\phi)^* \Theta_{\mathcal{L}}$$

such that ϕ is an stationary field of the action if and only if

$$0 = \int_X (j^1\phi)^* \iota_{\eta} d\Theta_{\mathcal{L}} \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

The Euler-Lagrange equations V

Proposition

There is such $\Theta_{\mathcal{L}}$, and can be intrinsically defined (using the geometry of $J^1\pi_{YX}$).

Locally,

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z_{\mu}^i} dy^i \wedge d^{n-1}x_{\mu} - \left(\frac{\partial L}{\partial z_{\mu}^i} z_{\mu}^i - L \right) d^n x$$

and it is called **the Poincaré-Cartan form**.

Corollary (Intrinsic Euler-Lagrange equations)

A field $\phi : X \rightarrow Y$ is stationary if and only if it satisfies

$$(j^1\phi)^* \iota_{\eta} d\Theta_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

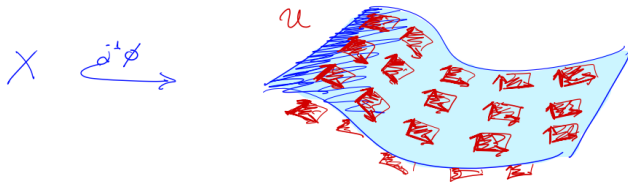
Looking for solutions

To find solutions, we can look for distributions on $J^1\pi_{YX} \rightarrow$ such that an integral section of such this distribution $\sigma : X \rightarrow J^1\pi_{YX}$ satisfies

$$\sigma^* \iota_\eta \Omega_{\mathcal{L}} = 0, \forall \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

We can define such distributions via decomposable n -multivector fields

$$U = X_1 \wedge \cdots \wedge X_n.$$



Then, being stationary is characterized by $\iota_U \Omega_{\mathcal{L}} = 0$.

Giving such a multivector field U does not immediately give a solution:

- We need to make sure that the corresponding distribution is integrable.
- Even if it is integrable, it may not be holonomic. That is, that the corresponding integral section $\sigma : X \rightarrow J^1\pi_{YX}$ could fail to be the jet lift of some section

$$\phi : X \rightarrow Y.$$

When $\mathcal{L}$ is **regular**, this is not an issue.

- Even if it satisfies the previous conditions, there may not exist global sections of $Y \xrightarrow{\pi_{YX}} X$.

Summary

- Fields, denoted by ϕ , are sections of a fibered manifold $Y \xrightarrow{\pi_{YX}} X$.
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n -form on X at each point), and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$

- If we define the **multisymplectic form** as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

stationary fields are characterized by

$$(j^1\phi)^* \iota_{\eta} \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

In particular, we can look for decomposable horizontal n -multivector fields U satisfying

$$\iota_U \Omega_{\mathcal{L}} = 0.$$

Multisymplectic geometry

Basic definitions I

Definition

A **multisymplectic manifold** of order n is a pair (M, ω) , where M is a smooth manifold, and ω is a closed $(n+1)$ -form.

An immediate example is the bundle of n -forms on a manifold Q .

$$M := \bigwedge^n T^*Q \xrightarrow{\tau} Q$$

has a canonical n -form,

$$\Theta|_{\alpha}(v_1, \dots, v_n) := \alpha(\tau_* v_1, \dots, \tau_* v_n)$$

and

$$\Omega := -d\Theta$$

defines a multisymplectic structure on M .

Basic definitions II

Definition

Let (M, ω) be a multisymplectic manifold of order n . A q -multivector field U on M ($q \leq n$) is called **Hamiltonian** if

$$\iota_U \omega = d\alpha,$$

for certain $(n - q)$ -form α , which will also be called **Hamiltonian**.

- Top degree Hamiltonian multivector fields (n -multivector fields) represent **solutions** to the variational problem,

$$\iota_U \omega = dH, H \in C^\infty(M).$$

- Hamiltonian vector fields $X \in \mathfrak{X}(M)$ are **symmetries**, $\mathcal{L}_X \omega = 0$ and the corresponding $(n - 1)$ -form can be thought of as the **Noether current** of the symmetry.

Noether's Theorem I

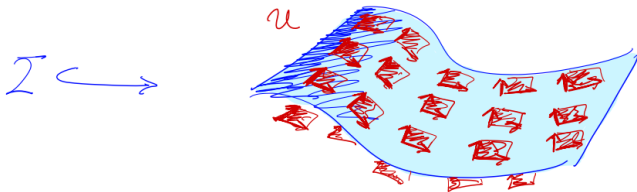
Given a Hamiltonian function H , and a top degree decomposable multivector field U such that

$$\iota_U \omega = dH,$$

let

$$j : \Sigma \hookrightarrow M$$

be an n -dimensional integral submanifold of U (which can be thought of as a distribution).



Noether's Theorem II

Theorem

Let X be a Hamiltonian vector field and α the corresponding $(n-1)$ -form (the current),

$$\iota_X \omega = d\alpha.$$

Then, if X is a symmetry of H , that is,

$$X(H) = 0,$$

α is a **conserved current** on Σ , this means $d(j^* \alpha) = 0$.

Proof.

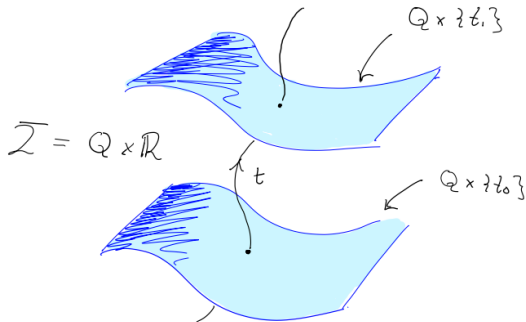
Indeed,

$$(d\alpha)(U) = \iota_U \iota_X \Omega = (-1)^n \iota_X \iota_U \Omega = (-1)^n X(H) = 0.$$



Noether's Theorem III

What does **conserved** means here?



$$\int_{\mathcal{Q} \times \{t_0\}} \alpha = \int_{\mathcal{Q} \times \{t_1\}} \alpha$$

Noether's Theorem IV

How can we obtain the corresponding current from the symmetry defined by X ?

Theorem

Let (M, ω) be an *exact* multisymplectic manifold, that is, there exists a *multisymplectic potential*

$$\omega = -d\theta.$$

Then, if X is a symmetry of θ , $\mathcal{L}_X\theta = 0$ (and hence of ω),

$$\alpha := -\iota_X\theta$$

is a current for X .

Proof.

Indeed,

$$d\alpha = -d\iota_X\theta = -\iota_Xd\theta = \iota_X\omega.$$



Proposition

Let (M, ω) be a multisymplectic manifold and α, β be Hamiltonian forms, with Hamiltonian multivector fields, X, Y , respectively. Then

$$\{\alpha, \beta\} := \iota_Y \iota_X \omega$$

is a Hamiltonian form. Its Hamiltonian multivector field is $-[X, Y]$ (the Schouten-Nijenhuis bracket).

Definition

Define the **Poisson bracket** of two Hamiltonian forms by

$$\{\alpha, \beta\} := -(-1)^{(k-1-\text{ord } \alpha)} \iota_Y \iota_X \omega,$$

which is again Hamiltonian by the previous proposition.

What are the properties that $\{\cdot, \cdot\}$ satisfies?

If we define a new degree:

$$\deg \alpha := k - 1 - \text{ord} \alpha,$$

- It is *graded-skew-symmetric*, that is,

$$\{\alpha, \beta\} = (-1)^{\deg \alpha \deg \beta} \{\beta, \alpha\}.$$

- Its satisfies *graded-Jacobi identity* (up to an exact form)

$$(-1)^{\deg \alpha \deg \gamma} \{\{\alpha, \beta\}, \gamma\} + \text{cycl.} = \text{exact term}$$

Brackets III

Theorem

Let (M, ω) be a multisymplectic manifold. Then, the space of all Hamiltonian forms *modulo exact forms* is a graded Lie algebra.

Some remarks:

- When restricted to the subspace of forms of $\deg \alpha = 0$, that is, $\alpha = k - 1$, we have a Lie algebra, *the Lie algebra of currents*.
- Some brackets are zero just by degree considerations, more particularly, when

$$\deg \alpha + \deg \beta > k - 1,$$

that is, the bracket is trivial when

$$\text{ord } \alpha + \text{ord } \beta < k - 1.$$

- Dynamics can be characterized by this Poisson bracket. Indeed, fixed a Hamiltonian, an n -multivector field U is a solution ($\iota_U \omega = dH$) if

$$\{\alpha, H\} = (d\alpha)(U).$$

Multisymplectic geometry is a very active area of research, and there has been a lot of interest in generalizing classical results from symplectic geometry to the multisymplectic setting.

- Reduction by symmetries.
- Coisotropic reduction.
- Constraint analysis.
- Darboux-like Theorems.
- Is everything a Lagrangian submanifold? (Weinstein's creed)
- A dialogue to Poisson geometry and Dirac geometry (work in progress...).

Summary

- Multisymplectic geometry gives an abstract formulation of field theories (calculus of variations).
- We can talk about the dynamics and conserved quantities with Hamiltonian multivector fields and forms.
- We can prove Noether's Theorem in this formalism.
- Hamiltonian forms are endowed with a graded Lie algebra structure (when quotiented by exact forms) which yields a Lie algebra when restricted to currents, $(n - 1)$ -forms.

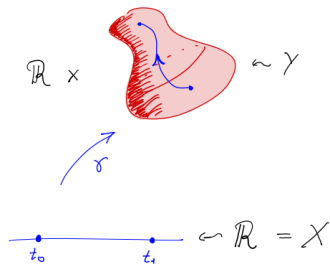
Examples

Classical Mechanics I

We recover Classical Mechanics by taking the bundle

$$\mathbb{R} \times Q \xrightarrow{\pi} \mathbb{R}.$$

Then, a section is just a curve $\gamma : \mathbb{R} \rightarrow Q$.



$$\mathcal{Q}[\gamma] = \int_{t_0}^{t_1} \mathcal{L}(t, \gamma(t), \dot{\gamma}(t)) dt$$

The jet bundle:

$$J^1\pi = \mathbb{R} \times TQ,$$

and the **Poincaré-Cartan** form fixed a Lagrangian (which will be identified with a function)

$$L : \mathbb{R} \times TQ \rightarrow \mathbb{R}$$

is

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i - \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right) dt.$$

Dynamics are vector fields X satisfying

- *Stationary condition*, $\iota_X d\theta_L = 0$.
- *Normalization*, $dt(X) = 1$.

We recover **cosymplectic** geometry!

What about Noether Theorem? Suppose L t -invariant. Then, time translations $\frac{\partial}{\partial t}$ define a symmetry of the corresponding multisymplectic form. Hence, by previous considerations, $\iota_{\frac{\partial}{\partial t}} \theta_L$ is an **conserved current**, that is, a **conserved quantity**. Locally,

$$\iota_{\frac{\partial}{\partial t}} \theta_L = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L = H,$$

obtaining conservation of energy.

Hyperelastic materials I

We will give an example of hyperelastic dynamics in a fixed background.

- Fix a **body** (B, G) , ρ , where B is a smooth manifold, G is a Riemannian metric on B , and ρ is the mass density.
- Fix a background manifold (M, g) , with M a smooth manifold and g a Riemannian metric on B .

Dynamics of B on M are time-dependent embeddings

$$\phi_t : B \rightarrow M.$$

We model these embeddings as **fields**

$$Y \xrightarrow{\pi} X.$$

Hyperelastic materials II

- $X := \mathbb{R} \times B$.
- $Y := \mathbb{R} \times B \times M$.
- The **projection** π is the trivial choice.

There is an **issue**:

- Arbitrary fields of the previous bundle $\phi : X \rightarrow Y$ *do not* necessarily correspond to time-dependent embeddings.

But not for long...

- Nevertheless, we can still apply the theory developed because embeddings are stable under local perturbations (variations).

What is the Lagrangian?

Notation:

- Coordinates on B are denoted by $(x^i) = x^1, \dots, x^{n-1}$.
- When adding the time coordinate $t = x^0$, we get coordinates (x^μ) on X .
- Coordinates on M are denoted by (y^a) .

Then, the Lagrangian is:

$$\begin{aligned}\mathcal{L} &= \mathbb{K} - \mathbb{P} \\ &= \frac{1}{2} \sqrt{\det G} \rho g_{ab} z_0^a z_0^b d^{n+1}x - \sqrt{\det G} \rho W(x^\mu, G, g, z_i^a) d^{n+1}x,\end{aligned}$$

where W is the stored energy.

Hyperelastic materials IV

The Poincaré-Cartan form is

$$\begin{aligned}\Theta_{\mathcal{L}} = & \rho g_{ab} z_0^b \sqrt{\det G} dy^a \wedge d^n x_0 - \rho \frac{\partial V}{\partial z_i^a} \sqrt{\det G} dy^a \wedge d^n x_i \\ & - \left(-\frac{\partial W}{\partial z_i^a} z_i^a + \frac{1}{2} g_{ab} z_0^a z_0^b + W \right) \rho \sqrt{\det G} d^{n+1} x\end{aligned}$$

How can we apply Noether's Theorem?

- There is a clear symmetry, **time-invariance**.
- Then, the current obtained through the theory is

$$\begin{aligned}\alpha = & -\rho \frac{\partial W}{\partial z_i^a} \sqrt{\det G} dy^i d^n x_{i0} \\ & + \left(\frac{1}{2} g_{ab} z_0^a z_0^b + W - \frac{\partial W}{\partial z_i^a} z_i^a \right) \rho \sqrt{\det G} d^{n+1} x_0.\end{aligned}$$

Hyperelastic materials V

on holonomic sections it takes the expression

$$\alpha = \left(\frac{1}{2} g_{ab} z_0^a z_0^b + W \right) \rho \sqrt{\det G} d^{n+1} x_0 + \rho \frac{\partial W}{\partial z_i^a} z_0^a \sqrt{\det G} d^n x_i$$

Since

$$e = \left(\frac{1}{2} g_{ab} z_0^a z_0^b + W \right) \rho \sqrt{\det G} d^{n+1} x_0$$

can be thought of as the **energy density**. This gives us a conservation law, where

$$\rho \frac{\partial W}{\partial z_i^a} z_0^a \sqrt{\det G} d^n x_i$$

is the energy flux.

- Multisymplectic geometry is a tool that allows us to study variational problems (field theory, Classical Mechanics, some problems in material modelling...)
- It is a very active area of research, both from the mathematical and the physical point of view.
- Applications to material modelling seem interesting, have been practically unexplored.

References

- [1] Mark J. Gotay et al. ***Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory.*** 2004. arXiv: physics/9801019 [math-ph].
- [2] M. de León and R. Izquierdo-López. ***Coisotropic reduction in Multisymplectic Geometry.*** Tech. rep. arXiv:2405.12898 [math] type: article. arXiv, June 2024. DOI: 10.48550/arXiv.2405.12898.
- [3] Jerrold E. Marsden et al. **“Variational methods, multisymplectic geometry and continuum mechanics”**. In: *Journal of Geometry and Physics* 38.3 (2001), pp. 253–284. ISSN: 0393-0440. DOI: [https://doi.org/10.1016/S0393-0440\(00\)00066-8](https://doi.org/10.1016/S0393-0440(00)00066-8).

Thank you for your attention!