A Geometric Approach to Electromagnetism in Media

Jordi Gaset

CUNEF University

Workshop on Geometrical aspects of material modelling

August 21, 2024

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$A_\mu = \left(\frac{\phi}{c}, \mathbf{A}\right)$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$A_\mu = \left(\frac{\phi}{c}, \mathbf{A}\right)$$

Four-dimensional space-time	M
Principle bundle with structure group $U(1)$	$P \to M$
Bundle of connections	$\pi : C \to M$
First jet bundle:	$\mathcal{P} = J^1\pi$

Local coordinates $(x^\mu, A_\mu, A_{\mu,\nu})$

$$L_{EM} = -\frac{1}{4\mu_0}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} \, ,$$

$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the curvature of A_μ called electromagnetic tensor field
$\eta^{\mu\nu}$ is the Minkowski metric.

Multisymplectic form of L_{EM}

$$\Omega_{EM} = \mathrm{d}E_L \wedge \mathrm{d}^4 x - \mathrm{d}\frac{\partial L_{EM}}{\partial A_{\mu,\nu}} \wedge \mathrm{d}A_\mu \wedge \mathrm{d}^3 x_\nu$$

where E_L is the Lagrangian Energy:

$$E_L = A_{\mu,\nu}\frac{\partial L_{EM}}{\partial A_{\mu,\nu}} - L$$

Multisymplectic form of L_{EM}

$$\Omega_{EM} = \mathrm{d}E_L \wedge \mathrm{d}^4x - \mathrm{d}\frac{\partial L_{EM}}{\partial A_{\mu,\nu}} \wedge \mathrm{d}A_\mu \wedge \mathrm{d}^3x_\nu$$

where E_L is the Lagrangian Energy:

$$E_L = A_{\mu,\nu}\frac{\partial L_{EM}}{\partial A_{\mu,\nu}} - L$$

A holonomic section $\phi : M \to J^1\pi$ is a solution if

$$\phi^*\iota_Y\Omega_{EM} = 0 \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P}).$$

$$\frac{\partial F^{\alpha\mu}}{\partial x^\mu} = 0 \Leftrightarrow \begin{cases} \nabla\cdot\mathbf{E} = 0 \\ \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{cases}$$

Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{cases}$$

Constitutive equations

$$\begin{cases} \nabla \cdot \mathbf{D} = 0 \\ \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \end{cases}$$

Constitutive equations

$$\begin{cases} \nabla \cdot \mathbf{D} = 0 \\ \nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{cases}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

Linear materials and sources

$$\eta^{\mu\nu} \to g^{\mu\nu} = \frac{1}{\sqrt{1+\chi_m}}\text{diag}\Big(\frac{1}{c^2}(1+\chi_e)(1+\chi_m), -1, -1, -1\Big)$$

Source: $J^\alpha = (c\rho, \mathbf{j}) \in \mathcal{C}^\infty(M)$

$$L_{Lin} = -\frac{1}{4\mu_0} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta} - A_\alpha J^\alpha\,,$$

Linear materials and sources

$$\eta^{\mu\nu} \to g^{\mu\nu} = \frac{1}{\sqrt{1+\chi_m}}\text{diag}\Big(\frac{1}{c^2}(1+\chi_e)(1+\chi_m), -1, -1, -1\Big)$$

Source: $J^\alpha = (c\rho, \mathbf{j}) \in \mathcal{C}^\infty(M)$

$$L_{Lin} = -\frac{1}{4\mu_0} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta} - A_\alpha J^\alpha \,,$$

$$\frac{\partial F^{\alpha\mu}}{\partial x^\mu} = \mu_0 J^\alpha \Leftrightarrow \begin{cases} (1+\chi_e)\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho \\ (1+\chi_m)\nabla\times\mathbf{B} = \frac{1+\chi_e}{c}\frac{\partial\mathbf{E}}{\partial t} - \mu_0\mathbf{j} \end{cases}$$

Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{cases}$$

Non-linear materials: Proca

$$L_{Proca} = -\frac{1}{4\mu_0}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} - \frac{1}{2}m^2\eta^{\alpha\beta}A_\alpha A_\beta \,,$$

Non-linear materials: Proca

$$L_{Proca} = -\frac{1}{4\mu_0}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} - \frac{1}{2}m^2\eta^{\alpha\beta}A_\alpha A_\beta \ ,$$

$$\frac{\partial F^{\alpha\mu}}{\partial x^\mu} = \mu_0 m^2 A^\alpha \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{E} = -\mu_0 m^2 \phi \\ \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \mu_0 m^2 \mathbf{A} \end{cases}$$

Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{cases}$$

Non-linear materials: Proca

$$L_{Proca} = -\frac{1}{4\mu_0}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} - \frac{1}{2}m^2\eta^{\alpha\beta}A_\alpha A_\beta\ ,$$

$$\frac{\partial F^{\alpha\mu}}{\partial x^\mu} = \mu_0 m^2 A^\alpha \Leftrightarrow \begin{cases} \nabla\cdot\mathbf{E} = -\mu_0 m^2\phi \\ \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} - \mu_0 m^2\mathbf{A} \end{cases}$$

Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \end{cases}$$

Metamaterials

$$\varepsilon(k,\omega) = 1 + \frac{m^2c^2}{\omega^2}\left[1 - \frac{\lambda_{crit}}{\lambda^2}\right]$$

The Herglotz variational principle

Gustav Herglotz (left) and Steffi (right)

Definition (The Herglotz variational principle)

A curve $\gamma : [0,1] \to Q$ is a solution of the Herglotz variational principle if it is extremal for the action:

$$\mathcal{A} = \int_0^1 L(t, \gamma, \dot{\gamma}, z) = z(1) - z(0) \ ,$$

where $z : [0,1] \to \mathbb{R}$ satisfies

$$\dot{z} = L(t, \gamma, \dot{\gamma}, z) \ .$$

The Herglotz variational principle for fields

The action of a field $\phi(x)$ is an integral over a domain in space-time, $D \subset M$,

$$\mathcal{A}(\phi) = \int_D L(\phi) \mathrm{d}^m x \ .$$

The density of action is the Lagrangian density $L \, \mathrm{d}^m x$.

The Herglotz variational principle for fields

The action of a field $\phi(x)$ is an integral over a domain in space-time, $D \subset M$,

$$\mathcal{A}(\phi) = \int_D L(\phi) \mathrm{d}^m x \, .$$

The density of action is the Lagrangian density $L \, \mathrm{d}^m x$.
Consider an $(m-1)$-form over M: $Z = s^\mu(x^\nu) \mathrm{d}^{m-1} x_\mu$,

$$\mathrm{d}Z = \mathrm{d}\left(s^\mu \mathrm{d}^{m-1} x_\mu \right) = \frac{\partial s^\mu}{\partial x^\mu} \mathrm{d}^m x = L(\phi) \mathrm{d}^m x \, ,$$

(where $\mathrm{d}^{m-1} x_\mu = (-1)^{\mu-1} \mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^\mu} \wedge \ldots \wedge \mathrm{d}x^m$). Therefore, Z is like the potential of the density of action. The components s^μ are new fields.

The Herglotz variational principle for fields

A couple $(\phi(x), s^\mu(x))$ is a solution to the Herglotz variational principle if they are critical for the action

$$\mathcal{A}(\phi, s^\mu) = \int_D L(x^\mu, \phi, \partial_\mu \phi, s^\mu) \mathrm{d}^m x \,,$$

under the constraint

$$\frac{\partial s^\mu}{\partial x^\mu} = L(x^\mu, \phi, \partial_\mu \phi, s^\mu) \,.$$

The Herglotz variational principle for fields

A couple $(\phi(x), s^\mu(x))$ is a solution to the Herglotz variational principle if they are critical for the action

$$\mathcal{A}(\phi, s^\mu) = \int_D L(x^\mu, \phi, \partial_\mu \phi, s^\mu) \mathrm{d}^m x \,,$$

under the constraint

$$\frac{\partial s^\mu}{\partial x^\mu} = L(x^\mu, \phi, \partial_\mu \phi, s^\mu) \,.$$

$$\mathcal{A}(\phi, s^\mu) = \int_D L(\phi, s^\mu) \mathrm{d}^m x \; = \int_D \frac{\partial s^\mu}{\partial x^\mu} \mathrm{d}^m x = \int_{\partial D} s^\mu \mathrm{d}\tau_\mu \;.$$

Field equations for an action-dependent Lagrangian

Given a Lagrangian $L(x^\mu, y^i, y^i_\mu, s^\mu)$, the functions $(y^i(x), s^\mu(x))$ are a solution to the Herglotz variational principle for fields if, and only if,

$$\frac{\partial L}{\partial y^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial L}{\partial y^i_\mu} + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu} = 0\,, \qquad \frac{\partial s^\mu}{\partial x^\mu} = L\,.$$

These are the *Herglotz–Euler–Lagrange equations*. New terms:

$$\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\;; \qquad \frac{\partial^2 L}{\partial s^\nu y^i_\mu}\frac{\partial s^\nu}{\partial x^\mu}$$

Field equations for an action-dependent Lagrangian

Given a Lagrangian $L(x^\mu, y^i, y^i_\mu, s^\mu)$, the functions $(y^i(x), s^\mu(x))$ are a solution to the Herglotz variational principle for fields if, and only if,

$$\frac{\partial L}{\partial y^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\partial L}{\partial y^i_\mu} + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu} = 0\,, \qquad \frac{\partial s^\mu}{\partial x^\mu} = L\,.$$

These are the *Herglotz–Euler–Lagrange equations*. New terms:

$$\frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_\mu}\;; \qquad \frac{\partial^2 L}{\partial s^\nu y^i_\mu}\frac{\partial s^\nu}{\partial x^\mu}$$

- Not linear on the Lagrangian.
- The sum of a total derivative does not leave to an equivalent Lagrangian.
- Symmetries are related to dissipated quantities.
- The s^μ variables can appear in the equations.
- The background geometry is multicontact.

Changing the geometry of the system: multicontact

New manifold: $\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)$
Coordinates: $(x^\mu, A_\mu, A_{\mu,\nu}, s^\mu)$

Changing the geometry of the system: multicontact

New manifold: $\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)$
Coordinates: $(x^\mu, A_\mu, A_{\mu,\nu}, s^\mu)$

$$L_{MC} = -\frac{1}{4\mu_0}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} - \gamma_\alpha s^\alpha \,,$$

Multicontact form

$$\Theta_{MC} = \mathrm{d}E_L \wedge \mathrm{d}^4x - \mathrm{d}\frac{\partial L_{MC}}{\partial A_{\mu,\nu}} \wedge \mathrm{d}A_\mu \wedge \mathrm{d}^3x_\nu + \mathrm{d}s^\mu \wedge \mathrm{d}^3x_\mu$$

$$\sigma_{\Theta_{MC}} = \gamma_\alpha \mathrm{d}x^\alpha$$

Changing the geometry of the system: multicontact

New manifold: $\mathcal{P} = J^1\pi \times_M \Lambda^{m-1}(\mathrm{T}^*M)$
Coordinates: $(x^\mu, A_\mu, A_{\mu,\nu}, s^\mu)$

$$L_{MC} = -\frac{1}{4\mu_0}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}F_{\alpha\beta} - \gamma_\alpha s^\alpha\,,$$

Multicontact form

$$\Theta_{MC} = \mathrm{d}E_L \wedge \mathrm{d}^4x - \mathrm{d}\frac{\partial L_{MC}}{\partial A_{\mu,\nu}} \wedge \mathrm{d}A_\mu \wedge \mathrm{d}^3x_\nu + \mathrm{d}s^\mu \wedge \mathrm{d}^3x_\mu$$

$$\sigma_{\Theta_{MC}} = \gamma_\alpha \mathrm{d}x^\alpha$$

A holonomic section $\phi : M \to \mathcal{P}$ is a solution if

$$\phi^*\iota_Y(\mathrm{d}\Theta_{MC} + \sigma_{\Theta_{MC}} \wedge \Theta_{MC}) = 0\,, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P})$$
$$\phi^*\Theta_{MC} = 0$$

$$\frac{\partial s^\mu}{\partial x^\mu} = L$$

Changing the geometry of the system: multicontact

$$\frac{\partial F^{\alpha\mu}}{\partial x^{\mu}} = -\mu_0 \gamma_\mu F^{\alpha\mu} \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{E} = -\mu_0 \boldsymbol{\gamma} \cdot \mathbf{E} \\ \nabla \times \mathbf{B} + \boldsymbol{\gamma} \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + \gamma \mathbf{E}\right) \end{cases}$$

$\gamma = (\gamma_0, \boldsymbol{\gamma})$. Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{cases}$$

Changing the geometry of the system: multicontact

$$\frac{\partial F^{\alpha\mu}}{\partial x^{\mu}} = -\mu_0 \gamma_\mu F^{\alpha\mu} \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{E} = -\mu_0 \boldsymbol{\gamma} \cdot \mathbf{E} \\ \nabla \times \mathbf{B} + \boldsymbol{\gamma} \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + \gamma \mathbf{E}\right) \end{cases}$$

$\gamma = (\gamma_0, \boldsymbol{\gamma})$. Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{cases}$$

Generalised Poynting's theorem:

$$\int_V \frac{\partial u}{\partial t} dV + \int_{S=\partial V} \mathbf{S} \cdot \hat{\mathbf{n}} dS = -\int_V \mathbf{E} \cdot \mathbf{j} dV - \int_V \mathbf{E} \cdot \left(\epsilon_0 \gamma \mathbf{E} - \boldsymbol{\gamma} \times \frac{\mathbf{B}}{\mu_0}\right) dV,$$

$$u = \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B}, \quad \mathbf{S} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0}$$

Changing the geometry of the system: multicontact

$$\frac{\partial F^{\alpha\mu}}{\partial x^{\mu}} = -\mu_0 \gamma_\mu F^{\alpha\mu} \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{E} = -\mu_0 \boldsymbol{\gamma} \cdot \mathbf{E} \\ \nabla \times \mathbf{B} + \boldsymbol{\gamma} \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + \gamma \mathbf{E}\right) \end{cases}$$

$\gamma = (\gamma_0, \boldsymbol{\gamma})$. Moreover

$$\mathrm{d}F = 0 \Leftrightarrow \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{cases}$$

Generalised Poynting's theorem:

$$\int_V \frac{\partial u}{\partial t} dV + \int_{S=\partial V} \mathbf{S} \cdot \hat{\mathbf{n}} dS = -\int_V \mathbf{E} \cdot \mathbf{j} dV - \int_V \mathbf{E} \cdot \left(\epsilon_0 \gamma \mathbf{E} - \boldsymbol{\gamma} \times \frac{\mathbf{B}}{\mu_0}\right) dV,$$

$$u = \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B}, \quad \mathbf{S} = \mathbf{E} \times \frac{\mathbf{B}}{\mu_0}$$

Examples:Lorentz Dipole Model, Highly Resistive Dielectric