Review of Geometry of Differential Spaces

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Abstract

Differential space is a topological space S endowed with a differential structure $C^\infty(S)$, defined below:

Definition 1 *A differential structure on a topological space S is a family $C^\infty(S)$ of functions on S such that:*
1. $\{f^{-1}(a,b) \mid f \in C^\infty(S) \text{ and } a < b \in \mathbb{R}\}$ is a subbasis for the topology of S.
2. If $f_1, ..., f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$ then $F(f_1, ..., f_n) \in C^\infty(S)$.
3. For $f : S \to \mathbb{R}$ such that, for every $x \in S$,there exist an open neighbourhood open U of x in S and $f_x \in C^\infty(S)$ such that $f_{x|U} = f_{|U}$.

The first condition of Definition 1 relates the topology of S to its differential structure. The remaining conditions ensure that if S is a topological manifold then S endowed with a differential structure $C^\infty(S)$ is a smooth manifold in the usual sense.

Example 2 *If S is an arbitrary subset of a Euclidean space $\mathbb{R}^n$, it inherits from $\mathbb{R}^n$ a differential structure $C^\infty(S)$ generated by restrictions to S of smooth functions on $\mathbb{R}^n$.*

The intrinsic differential structure of a differential space is completely encoded in its differential structure $C^\infty(S)$. One gets understanding of geometry of differential spaces by using standard definitions in differential geometry of manifolds, formulated in terms of properties of smooth functions, and investigating their consequences.

Definition 3 *A map $\varphi : S \to R$ between differential spaces $(S, C^\infty(S))$ and $(R, C^\infty(R))$ is smooth (of class C^∞) if, for every $f \in C^\infty(R)$, its pull-back $\varphi^* f = f \circ \varphi$ is in $C^\infty(S)$. A smooth map $\varphi : S \to R$ is a diffeomorphism if its invertible, and its inverse $\varphi^{-1} : R \to S$ is smooth.*

In this lecture, I will discuss geometry of differential spaces, including their tangent and cotangent bundles, integration of vector fields and distributions, under the assumption that differential spaces under consideration are locally diffeomorphic to arbitrary subsets of Euclidean spaces.