Incidence of Geometric Measure Theory in Continuum Mechanics

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The occurrence of singularities and/or other peculiarities concerning the application of direct methods in calculus of variations hinders proving existence of energy minimizers directly in a class of regular functions. Thus, it appears necessary to enlarge the space of competitors and consider functions whose differentials are represented by measures. Consequent is a choice to look at geometric properties of functions and sets in a measure theoretic sense. Pertinent tools are encoded in what we commonly call a *Geometric Measure Theory*. Among them, k-currents and varifolds play a nontrivial role in modeling and analyzing issues pertaining to continuum mechanics.

For example, in nonlinear elasticity, Ball's 1976/1977 classical existence result foresees the (constitutive) choice of deformations in $W^{1,p}$ with $p > 3$ because for lower values of p the proof would not exclude occurrence of singularities, say holes or fractures, which are offspring of an elastic-brittle behavior. To describe an ideal elastic way of behaving by deformations in $W^{1,p}$, with $p > 1$, we need to constraint competitors. A way to do this, namely to avoid undesired singularities, which can be viewed as (internal) boundaries in the deformation mapping graph, is to consider integer rectifiable currents and to find minimizers in terms of them, as shown in 1989/1990 by Giaquinta, Modica, and Souček.

A k-current is a linear functional on the space of compactly supported forms. Among them, those admitting an integral representation are associated with a class of generalized oriented surfaces with well-defined notions of boundary and area (so-called mass), a class large enough to have compactness properties with respect to a topology that makes the mass a lower semicontinuous functional. In turn, since every oriented k-dimensional surface defines by integration a linear functional on forms, currents can be regarded as generalized oriented surfaces.

Discontinuities in energy minimizers are desirable, instead, when we look at equilibrium states foreseeing in large strain setting plastic slips or crack nucleation. In the first case, currents play still a role because the Burgers tensor can be naturally interpreted as the curl of a measure supported over a $2D$-rectifiable set. In the second case, curvature varifolds come into play when we aim at only determining the final state of a cracking process implying at the penultimate step partial contact among crack margins without local deformation jump but with loss of molecular bonds.

With Ω an open subset of $\mathbb{R}^n$, consider the Grassmannian G_m of m-dimensional subspaces of $\mathbb{R}^n$. A varifold is a Radon measure over $\Omega \times G_m$. In particular, pertinent to the description of cracks is the case in which the set Ω, where the varifold is concentrated, is rectifiable, and the Grassmannian is associated with the (approximate) tangent space to the crack path (so it describes potential evolution directions when restricted to the crack tip). A generalized notion of curvature can be associated with varifolds: it enters the energy attributed to a crack path. This choice is a way to account for the natural roughness of the crack surface at low spatial scales.

There are many issues in continuum mechanics where tools from geometric measure theory may play a significant modeling and analytical role. So, without claiming completeness in any way, if anything burdened by the limits of my knowledge, in my talk I will review some interactions between geometric measure theory and continuum mechanics, focusing primarily on some results obtained especially with Domenico Mucci in recent years.