A lagrangian theory of volumetric growth including nonholonomic constraints

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In this work, we merge the Analytical Mechanics of nonholonomic systems [1] with the Biomechanics of the volumetric growth of biological tissues [2]. For our purposes, some fundamental laws of growth must be formulated as nonholonomic constraints, and the mechanics of nonholonomic systems must be adapted to the continuum biomechanical framework [3-5]. To proceed in this direction, the first step consists of rewriting the mass balance law of a growing medium as a scalar differential equation in the medium's *growth tensor*, which represents the inelastic factor of the Bilby-Kröner-Lee decomposition of the medium's deformation gradient tensor. Then, regarding the source of mass characterizing the growth of the medium as known, the mass balance is viewed as a nonholonomic constraint, which the growth tensor must comply with in the course of its dynamics. The main reason for this approach is that it is often convenient to relate phenomenologically the mass source with the mechanical and chemical variables that are biologically relevant for growth models respectful of experiments. Another pillar of this modeling perspective is that, as suggested in [6], the growth tensor is considered as the descriptor of the medium's structural kinematics.
With this background, we construct a theory of growth based on a Lagrangian function that "recruits" the nonholonomic constraint on the growth tensor, and handles it variationally even though it is non-integrable [5]. To this end, we expand a methodology developed in [7], and lately studied in [8], that modifies Kozlov's "Vakonomic dynamics" to make it compatible with formulations of mechanical problems that are known, or assumed, to be correct. The fundamental tools for pursuing our scopes are the Hamilton-Suslov variational principle [7], and the generalization of the kinematic descriptors referred to as *quasi-velocities* to the context of growth mechanics. Our main result is that the methodology outlined in [7] needs to be modified further to make it applicable to the nonholonomic constraint on the growth tensor [5]. We show how this is done and analyze the consequences of the proposed adjustments.

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